From Hamiltonians and Lagrangians to Legendrians

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Abstract

The aim of the paper is to define Legendrians and their dual objects, Legendriens*, as generalizations of Lagrangians and affine Hamiltonians. The structure of Legendrians and some of their properties are also studied.

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Finsler and Lagrange spaces of higher order were studied in [9, 4] using the bundles of accelerations $T^kM \to M$. Related to T^kM, we consider in this paper the affine bundle $T^kM \to T^{k-1}M$. A dual theory of higher order Hamilton spaces was recently studied in [5, 6] using the dual bundles $T^{k-1}M \times_M T^*M = T^{k*}M \to M$. Related to T^kM and $T^{k*}M$, we consider in this paper the affine bundle $T^kM \to T^{k-1}M$ and the vector bundle $T^{k*}M \to T^{k-1}M$ respectively.

The aim of this paper is to define and to investigate some basic properties of a class of new geometrical objects, called *Legendrians*, that extends Lagrangians and Hamiltonians. There are two kind of Legendrians. The first one, simply called *Legendrians,* are 1-forms on T^kM. The differential dL of a Lagrangian L of order k on M is an *exact Legendrian* of order k on M. The second type, called a *Legendrian** is a section $\chi : \widetilde{T^{k*}M} \to J^1\Pi$ of the first jet bundle of the affine bundle $\Pi : T^kM^\dagger \to T^{k*}M$, the affine dual of the affine bundle $T^kM \to T^{k-1}M$. In this case, the class J^1h of an affine Hamiltonian $h : T^{k*}M \to T^kM^\dagger$ (that is a section of the affine bundle Π, not necessary affine) is an *exact Legendrian** of order k on M. The forms of closed Lagrangians and Lagrangians* of order $k \geq 1$ are given by Propositions 1.1 and 1.5 respectively. The top components of a Legendrian or of a Legendrian* are particular cases of a *top Legendrian* and of a *top Legendrian** respectively. If these are non-degenerated, the Legendrian, respectively the Legendrian* is called *regular*. We prove that the regular Legendrians and Legendrians* are in duality by Legendre transformations (Theorem 1.1). The analogy with Lagrangians is pointed out in Proposition 1.3, where a semispray is canonical associated with a regular Legendrian. Some concrete examples are also given.

The basic ideas used here will be further investigated in forthcoming papers.

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1 Legendrians

A *Legendrian* of order k on M is an 1-form $\Omega \in \mathcal{X}^*(\widetilde{T^kM})$. In local coordinates, $\Omega = \Omega_{(0)i}dx^i + \Omega_{(1)i}dy^{(1)i} + \cdots + + \Omega_{(k)i}dy^{(k)i}$. The change rules of the local components of Ω are:

$$\begin{aligned}
\Omega_{(0)i} &= \Omega_{(0)i'}\frac{\partial x^{i'}}{\partial x^i} + \Omega_{(1)i'}\frac{\partial y^{(1)i'}}{\partial x^i} + \cdots + \Omega_{(k)i'}\frac{\partial y^{(k)i'}}{\partial x^i}, \\
\Omega_{(1)i} &= \Omega_{(1)i'}\frac{\partial y^{(1)i'}}{\partial y^{(1)i}} + \cdots + \Omega_{(k)i'}\frac{\partial y^{(k)i'}}{\partial y^{(1)i}}, \\
&\vdots \\
\Omega_{(k)i} &= \Omega_{(k)i'}\frac{\partial y^{(k)i'}}{\partial y^{(k)i}}.
\end{aligned} \tag{1.1}$$

Considering the vertical vector bundle $V^k_{k-1}M \to T^kM$ of the affine bundle $(T^kM, p_k, T^{k-1}M)$ and $V^k_{k-1}M \to T^kM$ its dual, a *top Legendrian* of order k on M is a section of the fibered manifold $\left(\widetilde{V^k_{k-1}M}\right)^* \to \widetilde{T^kM}$. The tensors defined on the fibers of the vector bundle $V^k_{k-1}M \to T^kM$ (even less the null section) are called *d-tensors of order* k. Thus a top Legendrian is a 1-covariant d-tensor of order k. If Ω is a Legendrian of order k on M, then $(\Omega_{(k)i})$ defines a top Legendrian. It is easy to see that in general a top Legendrian is not a Legendrian.

The *Legendre map* defined by a top Legendrian Ω of order k on M is the fibered manifolds map over the base $T^{k-1}M$, $\mathcal{L}_\Omega : T^kM \to T^{*k}M$, having the local form $\mathcal{L}_\Omega(x^i, y^{(1)i}, \ldots, y^{(k)i}) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \Omega_{(k)i}(x^i, y^{(1)i}, \ldots, y^{(k)i}))$.

It is easy to see that $g_{ij} = \dfrac{\partial \Omega_{(k)i}}{\partial y^{(k)j}}$ are the components of a 2-covariant d-tensor, i.e. the change rule of its local components are $g_{ij}(x^i,\ y^j) = g_{i'j'}(x^{i'},\ y^{j'})\cdot \dfrac{\partial x^{i'}}{\partial x^i}\ \dfrac{\partial x^{j'}}{\partial x^j}$. It is easy to see that this d-tensor is symmetric iff there is a local function L such that $\Omega_{(k)i} = \dfrac{\partial L}{\partial y^{(k)i}}$; we say that Ω is top-closed. In particular, if $\Omega = dL$, where $L \in \mathcal{F}(\widetilde{T^kM})$ (i.e. Ω is exact), then L is usually called a Lagrangian of order k on M. We say that the Legendrian Ω is *top-regular* if the tensor g is non-degenerate, i.e. the matrix $(g_{ij})_{i,j=\overline{1,m}}$ has the rank m. In this case the Legendre map $\mathcal{L}_\Omega : T^kM \to T^{*k}M$ is a local diffeomorphism and the inverse $\mathcal{L}_\Omega^{-1} : T^{*k}M \to T^kM$ has the local form $\mathcal{L}_\Omega^{-1}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \xi^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$. We suppose also that $\mathcal{L}_\Omega$ is a global diffeomorphism. Since $\xi^i(x^i, y^{(1)i}, \ldots,\ y^{(k-1)i}, \Omega_{(k)i}(x^i, y^{(1)i}, \ldots,\ y^{(k)i})) = y^{(k)i}$, we have $\dfrac{\partial \xi^i}{\partial p_u}(x^i, y^{(1)i}, \ldots, y^{(k-1)i},\ \Omega_{(k)i}(x^i, y^{(1)i}, \ldots, y^{(k)i}))\cdot \dfrac{\partial \Omega_{(k)u}}{\partial y^{(k)j}}(x^i, y^{(1)i}, \ldots, y^{(k)i}) = \delta^i_j$. It follows that denoting $\dfrac{\partial \xi^i}{\partial p_j} = g^{ij}$, then $\left(g^{ij}\right)_{i,j=\overline{1,m}} = (g_{ij})^{-1}_{i,j=\overline{1,m}}$.

In the case when $\Omega = dL$ is regular, the Lagrangian L is called *regular* and $\mathcal{L}_\Omega$ is the usual *Legendre transformation.*

We consider now some examples.

Consider some local canonical coordinates $(t^1, t^2) \in (0, 2\pi) \times (0, 2\pi)$ on the torus $\mathcal{T}^2$. The tangent space $T\mathcal{T}^2$ has as coordinates (t^1, t^2, T^1, T^2). Consider the following 1-forms, which are global defined on $\mathcal{X}(T\mathcal{T}^2)$: $\Omega_1 = T^1 dT^1 + T^2 dT^2$, $\Omega_2 = T^2 dT^1 + T^1 dT^2$, $\Omega_3 = T^2 dT^1 - T^1 dT^2$, $\Omega_4 = dt^1 + dt^2 + T^1 dT^1 + T^2 dT^2$, $\Omega_5 = dt^1 + dt^2 + T^2 dT^1 + T^1 dT^2$, $\Omega_6 = dt^1 + dt^2 + T^2 dT^1 - T^1 dT^2$. The forms Ω_1, Ω_2, Ω_4 and Ω_5 are all closed, but only the forms Ω_1 and Ω_2 are exact: $\Omega_1 = dL_1$, $L_1(t^1, t^2, T^1, T^2) = \frac{1}{2}\left(\left(T^1\right)^2 + \left(T^2\right)^2\right)$ and $\Omega_2 = dL_2$, $L_2(t^1, t^2, T^1, T^2) = T^1 T^2$. Since the global 1-forms dt^1 and dt^2 are not exact, the 1-forms Ω_4 and Ω_5 are not exact. All the above 1-forms define Legendrians on $\mathcal{T}^2$. The forms Ω_1, Ω_2 and Ω_3 are also the top Legendrians of the Legendrians Ω_4, Ω_5 and Ω_6 respectively. But, in general, the top Legendrians are not Legendrians.

The following 1-forms are globally defined on $\mathcal{X}(T^{(k)}\mathcal{T}^2)$, for $k \geq 1$: $\Omega_1 = T^{(k)1} dT^{(k)1} + T^{(k)2} dT^{(k)2}$, $\Omega_2 = T^{(k)2} dT^{(k)1} + T^{(k)1} dT^{(k)2}$, $\Omega_3 = T^{(k)2} dT^{(k)1} - T^{(k)1} dT^{(k)2}$, $\Omega_4 = dt^1 + dt^2 + T^{(k)1} d^{(k)} T^1 + T^{(k)2} dT^2$, $\Omega_5 = dt^1 + dt^2 + T^{(k)2} dT^{(k)1} + T^{(k)1} dT^{(k)2}$, $\Omega_6 = dt^1 + dt^2 + T^{(k)2} dT^{(k)1} - T^{(k)1} dT^{(k)2}$. They have similar properties as in the case $k = 1$.

In general, if $L : \widetilde{T^k M} \to I\!R$ is a Lagrangian of order k on M and ω_0 is a differentiable form on the manifold $T^{k-1}M$, then the 1-form $dL + \pi^*\omega_0$ is a Legendrian of order k on M, where $\pi : T^k M \to T^{k-1} M$ is the canonical projection. Notice that in a similar way we can take ω_0 a differentiable form on M and $\pi : T^k M \to M$ the canonical projection.

In the case $k = 1$, if $L : \widetilde{TM} \to I\!R$ is a Lagrangian (of order 1) on M and $\omega_0 \in \mathcal{X}^*(M)$ is a 1-form on M, then $dL + \pi^*\omega_0$ is a 1-Legendrian on M. It is the case of the above examples of Legendrians.

Let $\Omega \in \mathcal{X}(\widetilde{TM})$, $d\Omega = 0$. Then, according to the Poincaré lemma, Ω is locally closed, thus there is an open cover $\mathcal{U} = \{U\}$ of $\widetilde{TM}$ such that the restriction $\Omega_{|U}$ is exact, i.e. $\Omega_{|U} = df_U$, $f_U \in \mathcal{F}(U)$. We can take U such that $U' = \pi(U) \subset M$ is open and $\{U' = \pi(U)\}$ is an open cover of M, where $\pi : \widetilde{TM} \to M$ is the canonical projection. Let $\{\varphi'_{U'}\}$ be a partition of unity on M, which is subordinated to this cover. The family $\{\varphi_U = (\pi^*\varphi'_{U'})_{|U}\} \subset \mathcal{F}(\widetilde{TM})$ is a partition of unity on $\widetilde{TM}$, subordinate to the cover $\mathcal{U}$. Using local coordinates, it is easy to see that the Lagrangian $L_0 = \sum_{U\in\mathcal{U}} \varphi_U f_U \in \mathcal{F}(\widetilde{TM})$ has the same top Legendrian as Ω, thus the Legendrian $\Omega_0 = \Omega - dL_0$ has a null top Legendrian and it is also closed. It has the local form $\Omega_0 = \Omega_i(x^j, y^j)dx^i$. Since $d\Omega_0 = 0$, it easy follows that $\Omega_i = \omega_i(x^j)dx^j$, thus $\Omega_0 = \pi^*\omega_0$, $\omega_0 \in \mathcal{X}^*(M)$, $d\omega_0 = 0$. Thus $\Omega = dL_0 + \pi^*\omega_0$.

If $\tilde{\Omega}$ is a top Legendrian which is closed, then the local Lagrangians L_U glue together to a Lagrangian $L_0 = \sum_{U\in\mathcal{U}} \varphi_U L_U \in \mathcal{F}(\widetilde{TM})$, and $\tilde{\Omega}$ is the top Legendrian of dL_0.

For $k \geq 1$, a similar argument can be used to prove the following result.

Proposition 1.1. *1. If $\tilde{\Omega}$ is a top Legendrian of order k which is closed, then there is a Lagrangian $L_0 \in \mathcal{F}(\widetilde{T^k M})$ such that $\tilde{\Omega}$ is the top Legendrian of dL_0.*

2. If $\Omega \in \mathcal{X}^*(\widetilde{T^kM})$ is a closed Legendrian, then there is a Lagrangian $L_0 \in \mathcal{F}(\widetilde{T^kM})$ and a closed 1-form $\omega_0 \in \mathcal{X}^*(M)$ such that $\Omega = dL_0 + \pi_k^*\omega_0$, where $\pi_k : \widetilde{T^kM} \to M$ is the canonical projection.

Proof. If $\tilde{\Omega}$ is a top Legendrian which is closed, then the local Lagrangians L_U glue together to a Lagrangian $L_0 = \sum_{U \in \mathcal{U}} \varphi_U L_U \in \mathcal{F}(\widetilde{TM})$, and $\tilde{\Omega}$ is the top Legendrian of dL_0. Thus 1. follows.

Let $\Omega \in \mathcal{X}(\widetilde{T^kM})$, $d\Omega = 0$. We use similar argument as in the case $k = 1$. Thus, using Poincaré lemma, Ω is locally closed, thus there is an open cover $\mathcal{U} = \{U\}$ of $\widetilde{T^kM}$ such that the restriction $\Omega_{|U} = df_U$, $f_U \in \mathcal{F}(U)$. We can take U such that $U' = \pi_k(U) \subset M$ is open and $\{U' = \pi_k(U)\}$ is an open cover of M, where $\pi_k : \widetilde{TM} \to M$ is the canonical projection. If we consider a partition of unity $\{\varphi'_{U'}\}$ on M, which is subordinated to this cover, then the family $\{\varphi_U = \pi_k^*\varphi'_{U'}\} \subset \mathcal{F}(\widetilde{T^kM})$ is a partition of unity on $\widetilde{T^kM}$, subordinate to the cover $\mathcal{U}$. Using local coordinates, it is easy to see that considering the Lagrangian $L_0 = \sum_{U \in \mathcal{U}} \varphi_U f_U \in \mathcal{F}(\widetilde{T^kM})$, then the Legendrian $\Omega_0 = \Omega - dL_0$ is closed and it has the form $\Omega_0 = \pi^*\omega_0$, $\omega_0 \in \mathcal{X}^*(M)$, $d\omega_0 = 0$. Indeed, it has the local form $\Omega_0 = \Omega_{(0)i}(x^j, y^{(1)j}, \ldots, y^{(k)i})dx^i$, since $\varphi_U = \pi_k^*\varphi'_{U'}$. Using that $d\Omega_0 = 0$, it easy follows that $\Omega_{(0)i} = \omega_i(x^j)dx^j$, thus $\Omega_0 = \pi_k^*\omega_0$, $\omega_0 \in \mathcal{X}^*(M)$, $d\omega_0 = 0$. Thus the conclusion of 2. follows. $\square$

The first statement can be reformulated as: *a top Legendrian is locally exact iff it is globally exact.*

Examples of Legendrians which are not closed are those in the form $\Omega = \alpha \cdot dL_0$, when α, $L_0 \in \mathcal{F}(\widetilde{T^kM})$. Since $d\Omega = d\alpha \wedge dL_0$, these Legendrians do not come, in general, from a Lagrangian, also from a local or a global viewpoint. Let us suppose that the Lagrangian L_0 is top-regular and $\alpha \in \mathcal{F}(\widetilde{T^kM})$ has the property that $\alpha > 0$. Using local coordinates, let us denote by $\gamma = \left(\gamma_{ij} = \frac{\partial L_0}{\partial y^{(k)i} \partial y^{(k)j}}\right)_{i,j=\overline{1,m}}$, $\gamma^{-1} = \left(\gamma^{ij}\right)_{i,j=\overline{1,m}}$, $\alpha_i = \frac{\partial \alpha}{\partial y^{(k)i}}$, $\alpha^i = \gamma^{ij}\alpha_j$, $\Omega_i = \alpha \frac{\partial L_0}{\partial y^{(k)i}} = \alpha\gamma_i$ and $\Omega^i = \gamma^{ij}\Omega_j$; notice that (Ω_i) is the top component of Ω. Then $\frac{\partial \Omega_i}{\partial y^{(k)j}} = \alpha_i\gamma_i + \alpha\gamma_{ij}$. Let us denote by $\beta = \alpha + \alpha_i\gamma_j\gamma^{ij} = \alpha + \alpha_i\gamma^i = \alpha + \alpha^j\gamma_j$..

Proposition 1.2. *Let us suppose that the Lagrangian L_0 is top-regular and $\alpha \in \mathcal{F}(\widetilde{T^kM})$ has the property that $\alpha > 0$ and $\beta = \alpha + \alpha_i\gamma_j\gamma^{ij} \neq 0$ on $\widetilde{T^kM}$. Then $\Omega = \alpha \cdot dL_0$ is a top-regular Legendrian on $\widetilde{T^kM}$.*

Proof. The condition $\beta \neq 0$ reads $\beta = \alpha + \alpha_i\gamma_j\gamma^{ij} \neq 0$, thus there exists the inverse $\left(\frac{\partial \Omega_i}{\partial y^{(k)j}}\right)^{-1} = \left(\frac{1}{\alpha}\gamma^{ij} - \frac{1}{\alpha\beta}\gamma^i\alpha^j\right)$. $\square$

Notice that we can consider instead $\widetilde{T^kM}$ the points where $\beta = 0$.

Let us take $k = 1$. We are going to consider some particular cases, which leads also to other applications.

First, let L_0 be a Finsler metric on TM, i.e. $L_0 : TM \to I\!R$ and $\alpha \in \mathcal{F}(\widetilde{TM})$. Since L_0 is 2-homogenous, we have that $\gamma^i = y^i = y^{(1)i}$, thus $\beta = \alpha + y^i \dfrac{\partial \alpha}{\partial y^{(1)i}} = \alpha + \overset{(1)}{\Gamma}(\alpha)$. Thus the condition $\beta \neq 0$ means that $\overset{(1)}{\Gamma}(\alpha) \neq -\alpha$. It is obvious that this condition is independent of the Finsler metric. If we take $\alpha = e^{2\sigma}$, the condition is $y^i \dfrac{\partial \sigma}{\partial y^i} \neq -\dfrac{1}{2}$. In particular, if σ is 0-homogeneous, this condition is satisfied on $\widetilde{TM}$ (for example, in the case of Antonelly metric, according to [10, XI.6]).

An other case comes from a 2-covariant d-tensor of the form $g_{ij} = \gamma_{ij} + \delta y_i y_j$, where $\gamma_{ij} = \dfrac{\partial^2 L_0}{\partial y^i \partial y^j}$ comes from a Finsler metric $L_0 : \widetilde{TM} \to I\!R$, $\delta \geq 0$ is a real function on $\widetilde{TM}$ and $y_i = \gamma_{ij} y^j = \dfrac{\partial L_0}{\partial y^i}$. Let $\Omega_i = y^j g_{ij} = (1 + \delta L_0)\dfrac{\partial L_0}{\partial y^i}$. Taking $\alpha = 1 + \delta L_0$, we get to the case discussed previously. In the particular case when $\delta = 1 - \dfrac{1}{n^2(x,y)} \geq 0$ is a positive function on $\widetilde{TM}$, the 2-covariant d-tensor g is used by R. Miron [10, XI.6] to construct a model in optics.

The following result can be proved by a straightforward verification.

Proposition 1.3. *If Ω is a regular Legendrian, then the local formula*

$$S^i = g^{ij}\left(\Gamma^{(k)}\left(\Omega_{(k)j}\right) - \Omega_{(k-1)j}\right) \tag{1.2}$$

defines a k-semi-spray on M.

In the case when $\Omega = dL$ is the differential of a regular Lagrangian L of order k on M, one obtain the Miron-Atanasiu semi-spray, canonically associated with L [8, 9]:

Corollary 1.1. *If L is a regular Lagrangian of order k on M, then the local formula*

$$S^i = g^{ij}\left(\Gamma^{(k)}\left(\frac{\partial L}{\partial y^{(k)j}}\right) - \frac{\partial L}{\partial y^{(k-1)j}}\right) \tag{1.3}$$

defines a k-semi-spray on M.

Taking into account of Proposition 1.1, we obtain the following result.

Proposition 1.4. *If Ω is a regular Legendrian of order $k \geq 2$ on M and $\Omega = dL_0 + \pi_k^* \omega_0$ is a decomposition of Ω given by Proposition 1.1, then the Lagrangian L_0 is also regular and it has the same semispray as Ω.*

A *top Legendrian** of order k on M is a continuous bundle map over $T^{k-1}M$, $\mathcal{H} : T^{k*}M \to T^kM$, differentiable on $\widetilde{T^{k*}M}$. We say that $\mathcal{H}$ is *regular* if it is a local diffeomorphism. Using local coordinates, $\mathcal{H}$ has the local form $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i},\ H^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$. The condition that $\mathcal{H}$ is *regular* reads that the matrix $\left(g^{ij} = \dfrac{\partial H^i}{\partial p_j}\right)_{i,j=\overline{1,n}}$ is non-degenerate. The local form $(\Omega_i)_{i=\overline{1,m}}$ of the local inverse $\mathcal{H}^{-1}$defines obviously a top Legendrian Ω_1; if a regular Legendrian Ω has Ω_1 as its top Legendrian, we say that $\mathcal{H}$ is the top Legendrian* of Ω.

We say that a top Legendrian $\mathcal{H}$ is *exact* if there is a section (as a fibered manifold) $h : \widetilde{T^{k*}M} \to \widetilde{T^kM^\dagger}$ (called an *affine k-Hamiltonian*) having the local form $h(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, -H_0(x^i,\ y^{(1)i}, \ldots,\ y^{(k-1)i},\ p_i))$ such that $H^i = -\dfrac{\partial H_0}{\partial p_i}$.

Using a suitable partition of unity, it can be proved that a locally exact Legendrian* is (globally), exactly as in the case of Legendrians.

Let us suppose that $E \stackrel{\pi}{\to} M$ is an affine bundle with fiber the affine line $\mathbb{R}$, such that there is an affine atlas on E whose affine maps induces the identity of the associate vector line $\mathbb{R}$. It means that if (x^i) and (x^i, Ω) are the local coordinates on M and on E respectively, then the local change of coordinates are $x^{i''} = x^{i''}(x^i)$ and $\Omega' = \Omega + f(x^i)$. Using [13, Proposition 4.1.7] it can be easily proved that there is an affine bundle $E_\pi \to M$, such that the induced affine bundle $\pi^* E_\pi \stackrel{\pi_2}{\to} E$ is canonically isomorphic with the first jet bundle $J^1E \to E$. Every section $s \in \Gamma(E)$ lifts to a section $S \in \Gamma(E_\pi)$; using local coordinates, if s has the local form $s = s(x^i)$, then S has as local coordinates $S^i = \dfrac{\partial s}{\partial x^i}$. It is also easy to see that the corresponding section on $\pi^* E_\pi$ corresponds to the first prolongation $j^1 s \in \Gamma(J^1 E)$ ([13, Definition 4.2.1]). Let us consider a section $\xi \in \Gamma(E_\pi)$, that have local components (ξ_i). On the intersection of two charts one has $\xi_{i'} = \left(\dfrac{\partial f}{\partial x^i}(x^i) + \xi_i\right)\dfrac{\partial x^i}{\partial x^{i'}}$. It is easy to see that $\eta_{ij} = \dfrac{\partial \xi_i}{\partial x^j} - \dfrac{\partial \xi_j}{\partial x^i}$ is an antisymmetric tensor, which we call the *curvature* of ξ; the section ξ is locally the lift of a section $s \in \Gamma(E)$ iff it has a vanishing curvature. The name ,,curvature" has an explicit meaning, since every section $S \in \Gamma(E_\pi \stackrel{\pi_1}{\to} M)$ corresponds to a jet field $S' \in \Gamma(\pi^* E_\pi \stackrel{\pi_2}{\to} E) = \Gamma(J^1 E)$, thus to a connection on E (according to [13, Proposition 4.6.3]); the curvature of this connection is just the induced tensor from M to E.

An affine Hamiltonian h is a section in the affine bundle $\Pi : T^kM^\dagger \to T^{k*}M$ with canonical fiber $\mathbb{R}$. Since its associated vector bundle is the trivial vector bundle over $T^{k*}M$ with fiber $\mathbb{R}$, we can consider the affine bundle $E_\Pi \stackrel{\pi_1}{\to} T^{k*}M$ constructed above in the general case. A *Legendrian** of order k on M is a section $\chi : \widetilde{T^{k*}M} \to E_\Pi$ (excluding the image of the null section). We say that the Legendrian* χ is *exact* if there is a section h of the affine bundle defined by Π such that the corresponding section on $\Gamma(\pi^* E_\pi)$ corresponds to the first prolongation $j^1 s$. Using local coordinates, χ has the local form $(\bar{x}) \stackrel{\chi}{\to} (\bar{x}, \chi(\bar{x}), \chi_{(0)i}(\bar{x}), \ldots, \chi_{(k-1)i}(\bar{x}), \chi^i(\bar{x}))$, where $\bar{x} = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)$. The exactness of χ means that there is $h : T^{k*}M \to T^kM^\dagger$, $h(\bar{x}) = (\bar{x}, -H_0(\bar{x}))$ such that $\chi_{(0)i} = -\dfrac{\partial H_0}{\partial x^i}, \ldots,\ \chi_{(k-1)i}(\bar{x}) = -\dfrac{\partial H_0}{\partial y^{(k-1)i}}$, $\chi^i(\bar{x}) = -\dfrac{\partial H_0}{\partial p_i}$. The machinery of the differentials used in the case of a Legendrian (which is a 1-form) is replaced in the case of a Legendrian* by its curvature, as defined above.

As in the case of Legendrians, the following result can be proved.

Proposition 1.5. *If a k-Legendrian** χ *has a vanishing curvature, then there is an affine k-Hamiltonian* h *and a closed* 1*-form* ω *on* M *such that* $\chi = J^1 h + \pi^*\omega$*, where* $\pi : \widetilde{T^{k*}M} \to M$ *is the canonical projection.*

Notice that the sum $J^1 h + \pi^*\omega$ must be read in the sense that on the first local components $\chi_{(0)i} = -\dfrac{\partial H_0}{\partial x^i}$ of χ are added $\omega_i(x^j)$, the components of ω, i.e. $J^1 h + \pi^*\omega$ has as components $-\dfrac{\partial H_0}{\partial x^i} + \omega_i$, $-\dfrac{\partial H_0}{\partial y^{(1)i}} \ldots$, $-\dfrac{\partial H_0}{\partial y^{(k-1)i}}$, $-\dfrac{\partial H_0}{\partial p_i}$.

Let $\Omega \in \mathcal{X}^*(\widetilde{T^k M})$ be a Legendrian of order k that has $(\Omega_{(0)i}, \ldots, \Omega_{(k)i})$ as local components. If Ω is regular (i.e. the Legendre map is a global diffeomorphism), the inverse $\mathcal{L}_\Omega^{-1} : T^{*k}M \to T^k M$ has the local form $\mathcal{L}_\Omega^{-1}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \xi^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$. Let us consider the local real functions on $T^{k*}M$:

$\chi_{(0)i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = -\Omega_{(0)i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \xi^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i))$, …, $\chi_{(k-1)i} = -\Omega_{(k-1)i}$, $\chi^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = \xi^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)$. Then the local map $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, \chi_{(0)i}, \ldots, \chi_{(k-1)i}, \chi^i)$ defines a global section $\chi : \widetilde{T^{k*}M} \to J^1\Pi$ in the first jet bundle of Π, thus a Legendrian* χ.

We say that a Legendrian* χ, having the local form $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i, \chi_{(0)i}, \ldots, \chi_{(k-1)i}, \chi^i)$, is *top regular* if the matrix $\left(\dfrac{\partial \chi^i}{\partial p_j}\right)$ is non-singular. A top-Legendrian χ defines a map $\mathcal{H} : \widetilde{T^{k*}M} \to \widetilde{T^k M}$, using the local formulas $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \overset{\mathcal{L}_\chi^*}{\to} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \chi^i)$, called the *Legendre** *map* of χ. If the Legendrian* χ is top regular, then the Legendre* map is a local diffeomorphism; if it is a global diffeomorphism, we say that χ is *regular*. Let us assume that χ is regular. The inverse $\mathcal{L}_\chi^{*-1} : T^k M \to T^{k*}M$ has the local form $\mathcal{L}_\Omega^{-1}(x^i, y^{(1)i}, \ldots, y^{(k)i}) = (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \omega_i(x^i, y^{(1)i}, \ldots, y^{(k)i})$. Let us also consider the local real functions on $T^k M$: $\Omega_{(0)i}(x^i, y^{(1)i}, \ldots, y^{(k)i}) = -\chi_{(0)i}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \omega_i(x^i, y^{(1)i}, \ldots, y^{(k)i}))$, …, $\Omega_{(k-1)i} = -\chi_{(k-1)i}$, $\Omega_{(k)i}(x^i, y^{(1)i}, \ldots, y^{(k)i}) = \chi^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \Omega_i)$. Then the local map $(x^i, y^{(1)i}, \ldots, y^{(k)i}) \to (x^i, y^{(1)i}, \ldots, y^{(k)i}, \Omega_{(0)i}, \ldots, \Omega_{(1)i})$ defines a global 1-form $\Omega : \widetilde{T^k M} \to T^*\widetilde{T^k M}$, i.e. a Legendrian Ω. The regular Legendrian Ω and the regular Legendrian* χ are called *dual* each to the other.

Theorem 1.1. *The regular Legendrians correspond one to one to their dual regular Legendrians** *via Legendre and Legendre** *maps respectively.*

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