Measuring how far from fibrations are certain pairs of manifolds

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Abstract

Recall that *the φ-category* of a pair (M, N) of differentiable manifolds is defined as $\varphi(M, N) := \min\{\#(C(f) \mid f \in C^\infty(M, N)\}$, where $C(f)$ is the critical set of f. In this paper we provide, by using the main results of [2], new pairs of manifolds with infinite φ-category. For the equivariant case we also provide new pairs of manifolds with infinitely many critical orbits.

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1 Introduction

In this section we just recall some results that have been proved in [2] as well as the definitions of the Stiefel and the Brieskorn manifolds.

Theorem 1.1 ([2])

1. *Assume that M^{n+1}, N^n are compact connected differentiable manifolds.*
   - *(a) If $n \geq 4$, $\pi_1(M)$ is a torsion group and $\pi_2(N) \simeq 0$, then $\varphi(M, N) = \infty$;*
   - *(b) If $n \geq 5$ and $\pi_q(M) \not\simeq \pi_q(N)$ for some $q \in \{3, \ldots, n-2\}$, then $\varphi(M, N) = \infty$;*
2. *Let M^{n+2}, N^n be compact connected differentiable manifolds. If $n \geq 4$, $\pi_2(N) \simeq 0$, $\pi_1(M)$ is a torsion group such that $\#\pi_1(M) \geq 3$ and also $Hom\big(\pi_1(M), \pi_1(N)\big) = \{0\}$, then $\varphi(M, N) = \infty$;*
3. *Assume that M^{n+2}, N^n are compact connected differentiable manifolds such that $n \geq 5$, $\pi_2(M) \simeq 0$ and $\pi_3(N)$ is a torsion group..*
   - *(a) If $\pi_2(N) \simeq 0$ and $\pi_1(M)$ is a torsion group, then $\varphi(M, N) = \infty$;*
   - *(b) If $\pi_q(M) \not\simeq \pi_q(N)$ for some $q \in \{3, \ldots, n-2\}$, then $\varphi(M, N) = \infty$.*

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4. *Assume that M^{n+k}, N^n are compact differentiable manifolds such that $\pi_1(M) \simeq 0$ and N is (k+1)-connected. If $n \geq 4$, $1 \leq k \leq n-3$ and $H_k(M) \simeq 0$, then $\varphi(M,N) = \infty$.*

Let $\bar{G}$, G be two Lie groups acting on the manifolds M and N respectively. If $\rho : \bar{G} \to G$ is a Lie group homomorphism, recall that a mapping $f : M \to N$ is said to be *ρ-equivariant* if $f(\bar{g}x) = \rho(\bar{g})f(x)$ for all $\bar{g} \in \bar{G}$ and all $x \in M$. If $\bar{G} = G$, then any id_G-equivariant mapping $f : M \to N$ is simply called *equivariant*.

Theorem 1.2 ([2]) *Let $\bar{G}$, G be compact Lie groups acting freely on the compact connected manifolds M^m, N^n respectively and $\psi : \bar{G} \to G$ be a Lie group homomorphism. Assume that $\bar{G}$ is either G or its universal covering Lie group $\tilde{G}$ and that ψ is either id_G or the covering Lie group homomorphism $\rho : \tilde{G} \to G$ and denote by k the common dimension of $\tilde{G}$ and G.*

1. *If $m = n+1 \geq k+6$ and $\pi_q(M) \not\simeq \pi_q(N)$ for some $q \in \{3, \dots, n-k-2\}$, then any ψ-equivariant smooth mapping $f : M \to N$ has infinitely many critical orbits.*

2. *If $m = n+2 \geq k+7$, $\pi_2(M/\bar{G}) \simeq 0$, $\pi_3(N/G)$ is a torsion group and $\pi_q(M) \not\simeq \pi_q(N)$ for some $q \in \{3, \dots, n-k-2\}$, then any ψ-equivariant smooth mapping $f : M \to N$ has infinitely many critical orbits.*

3. *Assume that $\bar{G} = G = S^1$.*

   (a) *If $m = n+2 \geq 8$, N is a homotopy n-sphere and $\pi_q(S^2) \not\simeq \pi_q(M)$, $\pi_q(M) \not\simeq 0$ for some $q \in \{3, \dots, n-3\}$, then any equivariant smooth mapping $f : M \to N$ has infinitely many critical orbits.*

   (b) *If $m = n+2 \geq 8$, M is a homotopy m-sphere and $\pi_{q-1}(S^2) \not\simeq \pi_q(N)$, $\pi_q(N) \not\simeq 0$ for some $q \in \{3, \dots, n-3\}$, then any equivariant smooth mapping $f : M \to N$ has infinitely many critical orbits.*

The *Stiefel* manifold $V_{m+n,m}$ consists of all m-frames in $\mathbf{R}^{m+n}$. The compact orthogonal group $O(m+n)$ is obviously acting transitively on V_{m+n} and the isotropy group of any point of $V_{m+n,m}$ is a subgroup of $O(m+n)$ isomorphic with $O(n)$. Therefore the Stiefel manifold $V_{m+n,m}$ is diffeomorphic with the compact homogeneous space $O(m+n)/O(n)$, its dimension being $mn + \frac{m(m-1)}{2}$.

The *Brieskorn manifold* W_d^{2n-1}, where $n \geq 2$ and $d \geq 1$ are integers, is defined as the $(2n-1)$-dimensional real algebraic submanifolds of $\mathbf{C}^{n+1}$ defined by the equations

$$z_0^d + z_1^2 + \cdots + z_n^2 = 0 \text{ and } |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1.$$

If $n = 2m$ is even, then W_d^{4m-1} is a rational homology sphere whose only nontrivial integral homology groups are, according to [1, Corollary 9.3, pp. 275], given by

$$H_{2m-1}(W_d^{4m-1}) \simeq \mathbf{Z}_d \text{ and } H_0(W_d^{4m-1}) \simeq H_{4m-1}(W_d^{4m-1}) \simeq \mathbf{Z}.$$

All manifolds W_d^{2n-1} are invariant under the standard linear action of $O(n)$ on the $(z_1,\dots,z_n)$-coordinates. If $n=2m$ is even there is a free circle action on W_d^{4m-1} given by the action of the circle group $S^1=Z(U(m))\subset O(2m)$ where Z denotes the center. Moreover if $n=4m$, then $Sp(1)$ realized as subgroup of $O(4m)$ by the scalar multiplication on $\mathbf{R}^{4m}\simeq\mathbf{H}^m$, acts also freely on W_d^{8m-1}. The quotient manifolds $N_d^{4m-2}:=W_d^{4m-1}/S^1$ and $\tilde{N}_d^{8m-4}:=W_d^{8m-1}/Sp(1)$ are simply connected. For more details see also [3].

2 New pairs of manifolds with infinite φ-category

In this section we apply theorem 1.1 in order to provide some new pairs of manifolds with infinite φ-category and theorem 1.2 to provide some new pairs of G-manifolds having the property that all equivariant mappings between them have infinitely many critical orbits.

Proposition 2.1 *If $m,n\geq 2$ and $d:=\dim(V_{m+n,m})=p+q$, where $p=mn$, $q=\frac{m(m-1)}{2}$, then we have the following infinite φ-categories:*

1. (a) $\varphi\big(V_{m+n,m},S^{d-1}\big)=\infty$ *and* $\varphi\big(S^{d+1},V_{m+n,m}\big)=\infty$ *for* $n\geq 3$;

   (b) $\begin{cases}\varphi\big(V_{m+n,m},P^{d-1}(\mathbf{R})\big)=\infty;\\ \varphi\big(P^{d+1}(\mathbf{R}),V_{m+n,m}\big)=\infty\ \ for\ \ n\geq 3;\end{cases}$

   (c) $\begin{cases}\varphi\big(V_{m+n,m},S^{p-1}\times S^{q}\big)=\infty;\\ \varphi\big(S^{p+1}\times S^{q},V_{m+n,m}\big)=\infty\ \ for\ \ n\geq 3;\end{cases}$

   (d) $\begin{cases}\varphi\big(V_{m+n,m},P^{p-1}(\mathbf{R})\times P^{q}(\mathbf{R})\big)=\infty;\\ \varphi\big(P^{p+1}(\mathbf{R})\times P^{q}(\mathbf{R}),V_{m+n,m}\big)=\infty\ \ for\ \ n\geq 3;\end{cases}$

   (e) $\begin{cases}\varphi\big(S^{p+1}\times SO(m),V_{m+n,m}\big)=\infty;\\ \varphi\big(P^{p+1}(\mathbf{R})\times SO(m),V_{m+n,m}\big)=\infty\ \ for\ \ n\geq 3;\end{cases}$

   (f) $\begin{cases}\varphi\big(V_{m+n,m},S^{p-1}\times SO(m)\big)=\infty\ for\ m\geq 5;\\ \varphi\big(V_{m+n,m},P^{p-1}(\mathbf{R})\times SO(m)\big)=\infty\ for\ m\geq 5;\end{cases}$

   (g) $\varphi\big(V_{2n+2,2},L_p^{2n+1}\times L_p^{2n-1}\big)=\infty$ *and* $\varphi\big(L_p^{2n+1}\times L_p^{2n+1},V_{2n+2,2}\big)=\infty$.

2. (a) $\begin{cases}\varphi\big(P^{p+2}(\mathbf{R})\times P^{q}(\mathbf{R}),V_{m+n,m}\big)=\infty;\\ \varphi\big(P^{p+2}(\mathbf{R})\times SO(m),V_{m+n,m}\big)=\infty\ \ for\ \ m\geq 5;\end{cases}$

   (b) $\varphi\big(L_p^{2n+1},L_q^{2n-1}\big)=\infty$ *for* $n,p\geq 3$ *and* $p\neq q$.

3. (a) $\begin{cases}\varphi\big(V_{m+n,m},S^{d-2}\big)=\infty;\\ \varphi\big(S^{d+2},V_{m+n,m}\big)=\infty\ \ for\ \ m\geq 3\ \ and\ \ n\geq 4;\end{cases}$

   (b) $\varphi\big(V_{m+n,m},P^{d-2}(\mathbf{R})\big)=\infty$ *for* $m\geq 3$ *and* $n\geq 4$;

   (c) $\begin{cases}\varphi\big(V_{m+n,m},S^{p-2}\times S^{q}\big)=\infty\ \ for\ \ m\geq 3;\\ \varphi\big(S^{p+2}\times S^{q},V_{m+n,m}\big)=\infty;\end{cases}$

   (d) $\varphi\big(V_{n+2,2},W_d^{2n-1}\big)=\infty,$ *and* $\varphi\big(W_d^{2n+3},V_{n+2,2}\big)=\infty$.

Proof. Indeed (1a), (1b), (1c), (1d), (1e), (1f), (1g) follows by using theorem 1.1 (1a) taking into account that $V_{m+n,m}$ is, $(n-1)$-connected [5, pp. 203], the homotopy groups

$$\pi_2\left(S^{d-1}\right),\ \pi_2\left(P^{d-1}(\mathbf{R})\right), \pi_2\left(S^{p-1}\times S^q\right), \pi_2\left(P^{p-1}(\mathbf{R})\times P^q(\mathbf{R})\right),$$
$$\pi_2\left(S^{p-1}\times SO(m)\right), \pi_2\left(P^{p-1}(\mathbf{R})\times SO(m)\right), \pi_2\left(L_p^{2n-1}\right)$$

are obviously trivial, while the fundamental groups

$$\pi_1\left(S^{d+1}\right), \pi_1\left(P^{d+1}(\mathbf{R})\right), \pi_1\left(S^{p+1}\times S^q\right), \pi_1\left(P^{p+1}(\mathbf{R})\times P^q(\mathbf{R})\right),$$
$$\pi_1\left(S^{p+1}\times SO(m)\right), \pi_1\left(P^{p+1}(\mathbf{R})\times SO(m)\right), \pi_1\left(L_p^{2n+1}\times L_p^{2n+1}\right)$$

are obviously finite. Similarly (2a), (2b) follows by using theorem 1.1 (2) while (3a), (3b), (3c) and (3d) follows by using theorem 1.1 (4). □

Proposition 2.2 *Consider the previously defined S^1 free action on W_d^{4m-1}, $m\geq 3$ and the $Sp(1)=S^3$ free action on W_d^{8m-1}, $m\geq 3$ as well as the usual free actions of S^1 on S^{4m+1} and on S^{4m-3} and that of $Sp(1)=S^3$ on S^{4m-6} and on S^{4m+3}.*

1. *Any S^1-equivariant mapping $f: S^{4m+1}\to W_d^{4m-1}$ has infinitely many critical orbits whenever d is an odd number. Also any equivariant mapping $g: W_d^{4m-1}\to S^{4m-3}$ has infinitely many critical orbits.*
2. *Considering the S^3 free action on $S^{4m-6}\times S^{4m+3}$*

$$S^3\times\left(S^{4m-6}\times S^{4m+3}\right)\to S^{4m-6}\times S^{4m+3},\ (q,(z_1,z_2))\mapsto q(z_1,z_2)=(qz_1,qz_2),$$

any S^3-equivariant mapping $f: W_d^{8m-1}\to S^{4m-6}\times S^{4m+3}$, $m\geq 3$ has infinitely many critical orbits.

Proof. (1) Indeed W_d^{4m-1} is a homotopy sphere whenever d is an odd number and $\pi_{2m-1}(W_d^{4m-1})\not\simeq 0$ since $H_{2m-1}(W_d^{4m-1})\simeq \mathbf{Z}_d\not\simeq 0$ and W_d^{4m-1} is $(2m-2)$-connected. On the other hand $\pi_3(S^2)\simeq\mathbf{Z}$ while $\pi_4(W_d^{4m-1})\simeq 0\simeq\pi_3(W_d^{4m-1})$ for $m\geq 3$. Therefore f has, according to theorem 1.2 (3b), infinitely many critical orbits and g has, according to theorem 1.2 (3a), infinitely many critical orbits as well.

(2) By using the exact homotopy sequence of the fibration

$$S^3\hookrightarrow W_d^{8m-1}\to W_d^{8m-1}/S^3=\tilde{N}_d^{8m-4}$$

we get the exact sequence

$$\pi_3(W_d^{8m-1})\to\pi_3(\tilde{N}_d^{8m-4})\to\pi_2(S^3)\to\pi_2(W_d^{8m-1})\to\pi_2(\tilde{N}_d^{8m-4})\to\pi_1(S^3)$$

which ensures us that $\pi_3(\tilde{N}_d^{8m-4})\simeq 0\simeq\pi_2(\tilde{N}_d^{8m-4})$ taking into account that S^3 is 2-connected and W_d^{8m-1} is, according to [1, Corollary 9.3, pp. 275] and the Hurewicz theorem, $(4m-2)$-connected. In a completely similar way, by considering the exact homotopy sequence of the fibration

$$S^3\hookrightarrow S^{4m-6}\times S^{4m+3}\longrightarrow\frac{S^{4m-6}\times S^{4m+3}}{S^3}$$

and by taking into account that S^3 is 2-connected while $S^{4m-6} \times S^{4m+3}$ is $(4m-7)$-connected, one can immediately deduce that

$$\pi_2\Big(\frac{S^{4m-6} \times S^{4m+3}}{S^3}\Big) \simeq 0 \simeq \pi_3\Big(\frac{S^{4m-6} \times S^{4m+3}}{S^3}\Big).$$

On the other hand we obviously have that

$$\pi_{_{4m-6}}\Big(\frac{S^{4m-6} \times S^{4m+3}}{S^3}\Big) \simeq \mathbf{Z} \not\simeq 0 \simeq \pi_{_{4m-6}}(W_d^{8m-1}),$$

namely the quotient manifolds $\tilde{N}_d^{8m-4}$, $\big(S^{4m-6} \times S^{4m+3}\big)\big/S^3$ satisfy the conditions of theorem 1.2 (2) such that the proof of (2) is now finished. □

References

[1] G.E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972.

[2] C. Pintea, *A measure of the deviation from there being fibrations between a pair of compact manifolds*, to be published in *Differential Geometry and its Applications*.

[3] L.J. Schwachhöfer, and W. Tuchmann, *Metrics of positive Ricci Curvature on quotient spaces*, Preprintreihe des SFB 478-Geometrische Strukturen in der Mathematik, 2003.

[4] E.H. Spanier, *Algebraic Topology*, McGraw-Hill Book Company, 1966.

[5] G.W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, 1978.

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