Einstein and Maxwell equations for a couple of remarkable generalized Lagrange spaces of type

$GL^{2(n)}\left(M, g_{ij}(x, y^{(1)}, y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})}\gamma_{ij}(x)\right)$

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Abstract

The generalized Lagrange space $GL^{2(n)} = (Osc^2M, g_{ij}(x, y^{(1)}, y^{(2)}))$ provides a convenient relativistic model. The purpose of this paper is to study the Einstein and the Maxwell equations in the case $g_{ij}(x, y^{(1)}, y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})}\gamma_{ij}(x)$ giving the complete calculation for two remarkable metric tensors. On the base manifold the metric tensor $g_{ij}(x) = e^{2\sigma(x)}\gamma_{ij}(x)$ was introduced by Watanabe, Ikeda S. and Ikeda F. Einstein and Maxwell equations for the space $GL^n(M, g_{ij}(x, y) = e^{2\sigma(x,y)}\gamma_{ij}(x))$ were studied by Miron and Tavakol. Generalized Einstein Yang Mills equations for the same space were studied by V. Balan.

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1. Preliminaries

The theory developed in this paper relies on the ideas from the generalized second order Lagrange theory of physical fields that naturally generalize the Finsler and Lagrage one. The geometrical background is formed by the notion of generalized Lagrange space $GL^{2(n)} = (Osc^2M, g_{ij}(x, y^{(1)}, y^{(2)}))$ which consist of the Osc^2M space whose coordinates are induced by the coordinates $x = (x^i)_{i=1..n}$ of a real n-dimensional C^∞ differentiable manifold M and a fundamental metric d-tensor $g_{ij}(x, y^{(1)}, y^{(2)})$ globally defined on $Osc^2M \setminus \{0\}$, symmetric, of rank n and having a constant signature.

The field theory developed on $GL^{2(n)}$ uses an "a priori" defined fixed nonlinear connection N whose coefficients are:

$$\Gamma = \left(\underset{(1)}{N^i_{\ j}}\ (x, y^{(1)}, y^{(2)}),\ \ \underset{(2)}{N^i_{\ j}}\ (x, y^{(1)}, y^{(2)}) \right)$$

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This nonlinear connection plays the role of mapping operators of "'internal"' fields $y^{(1)},\ y^{(2)}$ to the "'external"' x field and "'prescribes"' the interaction between x and y fields. It allows the construction of an adapted basis

$$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}}\right\} \qquad (i=1,...,n) \tag{1.1}$$

where:

$$\begin{cases} \dfrac{\delta}{\delta x^i} & = & \dfrac{\partial}{\partial x^i} - \underset{(1)}{N^j{}_i} \dfrac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N^j{}_i} \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\delta}{\delta y^{(1)i}} & = & \dfrac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N^j{}_i} \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\delta}{\delta y^{(2)i}} & = & \dfrac{\partial}{\partial y^{(2)i}} \end{cases} \tag{1.2}$$

and of the dual basis:

$$\left\{\delta x^i,\ \delta y^{(1)i},\ \delta y^{(2)i}\right\} \qquad (i=1,...,n) \tag{1.3}$$

where:

$$\begin{cases} \delta x^i & = & dx^i \\ \delta y^{(1)i} & = & dy^{(1)i} + \underset{(1)}{M^i{}_j}\ dx^j \\ \delta y^{(2)i} & = & dy^{(2)i} + \underset{(1)}{M^i{}_j}\ dy^{(1)j} + \underset{(2)}{M^i{}_j}\ dx^j \end{cases} \tag{1.4}$$

We have the relations:

$$\underset{(1)}{N^i{}_j} = \underset{(1)}{M^i{}_j}\ ; \qquad \underset{(2)}{N^i{}_j} = \underset{(2)}{M^i{}_j} - \underset{(1)}{M^i{}_k}\ \underset{(1)}{M^k{}_j} \tag{1.5}$$

From the geometrical point of view it is important to determine the canonical metrical connection (the Cartan connection) which respects Matsumuto's axioms:

$$g_{ij|m} = 0; \quad g_{ij} \overset{(A)}{|}_m = 0; \quad T^m{}_{ij} = 0; \quad \underset{(A)}{S}{}^m_{ij} = 0 \quad A = 1,2$$

This canonical connection plays an important role in the generalized Lagrange theory of physical fields. Concerning the generalized Lagrange gravitational theory we point out that the Einstein equations of the gravitational potentials $g_{ij}(x, y^{(1)}, y^{(2)})$ are postulated as being the abstract geometrical Einstein equations attached to $C\Gamma(N)$ and G, whose local expression are given by Miron and Atanasiu in [5] with the corresponding conservation laws. As for the metric we have to say that on the base manifold M, this type of metric was introduced and studied by Watanabe S., Ikeda F., Ikeda S. in 1983 [9], Einstein and Maxwell equations on TM for the metric $g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x)$ make the object of study of Miron R. and Tavakol R. First approach of this study was presented in Brasov at Seminarul National de spatii Finsler, Lagrange si Hamilton, feb. 1992 by prof. V. Balan. We remark that if $\frac{\delta\sigma}{\delta y^{(1)i} \neq 0}$ then the space $GL^{2(n)}$ is not reducible to a Lagrange space or to a Finsler space.

2. The coefficients of the canonical metrical N-linear connection

Let M be a n dimensional C^∞ differentiable manifold endowed with the metric tensor $g_{ij}(x,y^{(1)},y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})}\gamma_{ij}(x)$ defined on $Osc^2M \setminus \{0\}$ where $\sigma \in \mathcal{F}(Osc^2M)$ is a given function and $\gamma_{ij}(x)$ is a Riemannian metric tensor field. Consider also N the canonical nonlinear connection with the coefficients:

$$\text{(2.1)} \qquad \begin{cases} \underset{(1)}{N}{}^i{}_j &= \gamma^i_{kj} y^{(1)k} \\ \underset{(2)}{N}{}^i{}_j &= \frac{1}{2}\left(\frac{\partial\gamma^i_{kj}}{\partial x^r} - \gamma^i_{rh}\gamma^h_{kj}\right) y^{(1)k}y^{(2)r} + \gamma^i_{kj}y^{(2)k} \end{cases},$$

Proposition 2.1. *The nonlinear connection N is integrable iff the Riemannian metric is flat.*

Proposition 2.2. *The Berwald connection of the space $GL^{2(n)}$ has the coefficients*

$$B\Gamma = (\gamma^i_{jk},\ 0,\ 0)$$

Proposition 2.3. *If*

$$\text{(2.2)} \qquad \frac{\delta\sigma}{\delta y^{(2)i}} \neq 0$$

then there exists no Lagrangian $L : Osc^2M \to R$ of class at least 3 on $Osc^2M \setminus \{0\}$, continuous on the null section of Osc^2M such that

$$g_{ij}(x,y^{(1)},y^{(2)}) = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(2)i}\partial y^{(2)j}}.$$

Consequently the pair $GL^{2(n)}$ previously defined is a generalized Lagrange space and is not reducible to a Lagrange space or a Finsler space if condition (2.2) is fulfilled.

We know that on the total space E exists a single N-linear connection depending only on g_{ij} and which satisfies Matsumoto's axioms. This is the N-linear canonical metrical connection (Cartan connection) $C\Gamma(N)$ and it has the coefficients $C\Gamma(N) = \left(L^i{}_{jk},\ \underset{(1)}{C}{}^i_{jk}\ ,\ \underset{(2)}{C}{}^i_{jk}\right)$ with the expressions:

(2.3)
$$L^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\delta g_{ks}}{\delta x^j} + \frac{\delta g_{sj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} \right)$$

$$\underset{(\alpha)}{C^i_{jk}} = \frac{1}{2} g^{is} \left(\frac{\delta g_{ks}}{\delta y^{(\alpha)j}} + \frac{\delta g_{sj}}{\delta y^{(\alpha)k}} - \frac{\delta g_{jk}}{\delta y^{(\alpha)s}} \right) \quad (\alpha = 1, 2)$$

Proposition 2.4. *For the considered metric the coefficients of $C\Gamma(N)$ are:*

(2.4)
$$L^i_{jk} = \gamma^i_{jk} + \underset{0}{\Lambda^i_{jk}} \ ; \ \underset{(\alpha)}{C^i_{jk}} = \underset{\alpha}{\Lambda^i_{jk}} \qquad (\alpha = 1, 2)$$

where $\underset{\beta}{\Lambda^i_{jk}} = \delta^i_k \overset{\beta}{\sigma_j} + \delta^i_j \overset{\beta}{\sigma_k} - \gamma_{jk} \overset{\beta}{\sigma^i}$ $\quad (\beta = 0, 1, 2)$ *and*

$\overset{\beta}{\sigma_j} = \dfrac{\delta \sigma}{\delta y^{(\beta)j}}$, $\quad \overset{\beta}{\sigma^i} = \gamma^{is} \overset{\beta}{\sigma_s}$;, $y^{(0)} = x$; δ^i_j *is the Kronecker symbol.*

Corollary 2.1. $\underset{\alpha}{\Lambda^i_{jk}} = 0$ $\alpha = 0, 1, 2$ *if and only if σ is constant.*

Proposition 2.5. *With respect to $C\Gamma(N)$ we have:*
- the h-paths are given by the equations:

(2.5)
$$\frac{d^2 x^i}{dt^2} + \left(\gamma^i_{jk} + \underset{0}{\Lambda^i_{jk}} \right) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \ ;$$

$$\frac{dy^{(1)i}}{dt} + \gamma^i_{jk} y^{(1)j} \frac{dx^k}{dt} = 0 \ ;$$

$$\frac{dy^{(2)i}}{dt} + \gamma^i_{jk} y^{(1)k} \left(\frac{dy^{(1)j}}{dt} + \gamma^j_{ms} y^{(1)m} \frac{dx^s}{dt} \right) +$$

$$+ \left[\frac{1}{2} \left(\frac{\partial \gamma^i_{jk}}{\partial x^r} - \gamma^i_{rh} \gamma^h_{kj} \right) y^{(1)k} y^{(1)r} + \gamma^i_{kj} y^{(2)k} \right] \frac{dx^j}{dt} = 0 \ ;$$

- the v_1-paths are given by:

(2.6)
$$x^i(t) = x^i_0; \quad \frac{dx^i}{dt} = 0 \ ;$$

$$\frac{d^2y^{(1)i}}{dt^2} + \underset{1}{\Lambda^i_{jk}} \frac{dy^{(1)j}}{dt}\frac{dy^{(1)k}}{dt} = 0 \ ;$$

$$\frac{d^2y^{(2)i}}{dt^2} + \gamma^i_{jk} y^{(1)j}\frac{dy^{(1)k}}{dt} = 0 \ ;$$

- the v_2-paths are given by the following equations:

$$x^i(t) = x^i_0; \quad \frac{dx^i}{dt} = 0 \ ; \tag{2.7}$$

$$y^{(1)i}(t) = y^{(1)i}_0 \ ; \quad \frac{dy^{(1)i}}{dt} = 0 \ ;$$

$$\frac{d^2y^{(2)i}}{dt^2} + \underset{2}{\Lambda^i_{jk}} \frac{dy^{(2)j}}{dt}\frac{dy^{(2)k}}{dt} = 0 \ .$$

3 Torsios and curvatures

Let $\mathcal{T}$ be the tensor of torsion of an N-linear connection D. For any vector fields $X, Y \in \chi(E)$ we have:

$$\mathcal{T}(X,Y) = D_X Y - D_Y X - [X,Y] \tag{3.1}$$

This tensor can be evaluated for the pairs of tensor fields (X^H, Y^H), (X^H, Y^{V_α}), $(X^{V_\alpha}, Y^{V_\beta})$ and for the canonical metrical linear N-connection $C\Gamma(N)$. By direct calculations we obtain:

Theorem 3.1. *The d-tensor field of torsion for $C\Gamma(N)$ has the local components given by:*

$$T^i{}_{jk} = 0, \quad \underset{(0)}{\overset{1}{T}}{}^i{}_{jk} \ = \ r_m{}^i{}_{jk} \ y^{(1)m} \tag{3.2}$$

$$\underset{(0)}{\overset{2}{T}}{}^i{}_{jk} \ = \frac{1}{2}\left(r_q{}^i{}_{mj}\gamma^m_{kp} - r_q{}^i{}_{mk}\gamma^m_{jp}\right) y^{(1)q} y^{(1)p} + r_m{}^i{}_{jk} y^{(1)m}$$

$$\underset{(\alpha)}{\overset{0}{P}}{}^i{}_{jk} \ = \ \underset{\alpha}{\Lambda^i_{jk}} \quad (\alpha = 1,2) \quad \underset{(1)}{\overset{1}{P}}{}^i{}_{jk} \ = - \ \underset{0}{\Lambda^i_{jk}} \ ; \quad \underset{(1)}{\overset{2}{P}}{}^m_{ij} \ = \gamma^m_{kr}\gamma^r_{ij} y^{(1)k} - \gamma^m_{ij};$$

$$(3.3)\qquad \underset{(1)}{\overset{1}{P}}{}^{i}{}_{jk} = -\underset{0}{\Lambda}{}^{i}_{jk}\ ;\qquad \underset{(1)}{\overset{2}{P}}{}^{m}_{ij} = \gamma^m_{kr}\gamma^r_{ij}y^{(1)k} - \gamma^m_{ij};$$

$$\underset{(2)}{\overset{1}{P}}{}^{i}{}_{jk} = 0;\qquad \underset{(2)}{\overset{2}{P}}{}^{i}{}_{jk} = -\underset{0}{\Lambda}{}^{i}_{jk}\ ;\ \underset{(12)}{\overset{0}{P}}{}^{i}{}_{jk} = 0;\qquad \underset{(12)}{\overset{\alpha}{P}}{}^{i}{}_{jk} = \underset{(\alpha)}{C}{}^{i}_{jk}\ ;$$

$$(3.4)\qquad \underset{(\alpha)}{\overset{\beta}{S}}{}^{i}{}_{jk} = 0 \qquad (\alpha = 1,2; \beta = 0,1,2)$$

Theorem 3.2. *The d-tensor field of curvature of the N-linear canonical metrical connection $C\Gamma(N)$ are locally expressed in the shape:*

$$(3.5)\qquad R_b{}^a{}_{pq} = r_b{}^a{}_{pq} + \delta^a_p \overset{0}{\sigma}_{bq} - \delta^a_q \overset{0}{\sigma}_{bp} - \gamma^{as}\gamma_{bp}\overset{0}{\sigma}_{sq} + \gamma^{as}\gamma_{bq}\overset{0}{\sigma}_{sp} +$$

$$+\delta^a_b(\overset{0}{\sigma}_{pq} - \overset{0}{\sigma}_{qp}) + \gamma_{bt}(r_s{}^a{}_{pq}\overset{1}{\sigma}{}^t - r_s{}^t{}_{pq}\overset{1}{\sigma}{}^a)y^{(1)s} +$$

$$+\gamma_{bt}(r_s{}^a{}_{pq}\overset{2}{\sigma}{}^t - r_s{}^t{}_{pq}\overset{2}{\sigma}{}^a)y^{(2)s}$$

$$(3.6)\qquad \underset{(1)}{P}{}_b{}^a{}_{pq} = \delta^a_p\overset{01}{\sigma}_{bq} - \delta^a_q\overset{10}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp}\overset{01}{\sigma}_{sq} - \gamma_{bq}\overset{10}{\sigma}_{sp}) +$$

$$+\gamma^{as}(\gamma_{bp}\overset{0}{\sigma}_q\overset{1}{\sigma}_s - \gamma_{bq}\overset{0}{\sigma}_s\overset{1}{\sigma}_p) + \delta^a_b(\overset{01}{\sigma}_{pq} - \overset{10}{\sigma}_{qp}) - 2\gamma^{as}\gamma_{pq}(\overset{0}{\sigma}_b\overset{1}{\sigma}_s - \overset{0}{\sigma}_s\overset{1}{\sigma}_b) -$$

$$-\delta^a_p\overset{0}{\sigma}_p\overset{1}{\sigma}_b - \delta^a_q\overset{0}{\sigma}_p\overset{1}{\sigma}_b$$

$$(3.7)\qquad \underset{(2)}{P}{}_b{}^a{}_{pq} = \delta^a_p\overset{01}{\sigma}_{bq} - \delta^a_q\overset{20}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp}\overset{02}{\sigma}_{sq} - \gamma_{bq}\overset{20}{\sigma}_{sp}) +$$

$$+\gamma^{as}(\gamma_{bp}\overset{0}{\sigma}_q\overset{2}{\sigma}_s - \gamma_{bq}\overset{0}{\sigma}_s\overset{2}{\sigma}_p) + \delta^a_b(\overset{02}{\sigma}_{pq} - \overset{20}{\sigma}_{qp}) - 2\gamma^{as}\gamma_{pq}(\overset{0}{\sigma}_b\overset{2}{\sigma}_s - \overset{0}{\sigma}_s\overset{2}{\sigma}_b) -$$

$$-\delta^a_p\overset{0}{\sigma}_p\overset{2}{\sigma}_b - \delta^a_q\overset{0}{\sigma}_p\overset{2}{\sigma}_b$$

(3.8) $$\underset{(12)}{P}{}_b{}^a{}_{pq} = \delta^a_p\overset{12}{\sigma}_{bq} - \delta^a_q\overset{21}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp}\overset{12}{\sigma}_{sq} - \gamma_{bq}\overset{21}{\sigma}_{sp}) +$$

$$+\gamma^{as}(\gamma_{qp}\overset{1}{\sigma}_s\overset{2}{\sigma}_b - \gamma_{qp}\overset{1}{\sigma}_b\overset{2}{\sigma}_s) + \delta^a_b(\overset{12}{\sigma}_{pq} - \overset{21}{\sigma}_{qp})$$

(3.9) $$\underset{(1)}{S}{}_b{}^a{}_{pq} = \delta^a_p\overset{1}{\sigma}_{bq} - \delta^a_q\overset{1}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp}\overset{1}{\sigma}_{sq} - \gamma_{bq}\overset{1}{\sigma}_{sp}) +$$

$$+\delta^a_b(\overset{12}{\sigma}_{pt}\gamma^t_{lq} - \overset{12}{\sigma}_{qt}\gamma^t_{lp})y^{(1)l}$$

(3.10) $$\underset{(2)}{S}{}_b{}^a{}_{pq} = \delta^a_p\overset{2}{\sigma}_{bq} - \delta^a_q\overset{2}{\sigma}_{bp} - \gamma^{as}(\gamma_{bp}\overset{2}{\sigma}_{sq} - \gamma_{bq}\overset{2}{\sigma}_{sp})$$

where

$$\overset{\alpha}{\sigma}_{cd} = \left(\frac{\delta\overset{\alpha}{\sigma}_c}{\delta y^{(\alpha)d}} - \overset{\alpha}{\sigma}_c\overset{\alpha}{\sigma}_d + \frac{1}{2}\gamma_{cd}\overset{\alpha}{\sigma}{}^t\overset{\alpha}{\sigma}_t\right) \qquad (\alpha = 1, 2)$$

$$\overset{\alpha\beta}{\sigma}_{cd} = \left(\frac{\delta\overset{\alpha}{\sigma}_c}{\delta y^{(\beta)d}} - \overset{\beta}{\sigma}_c\overset{\alpha}{\sigma}_d + \frac{1}{2}\gamma_{cd}\overset{\beta}{\sigma}{}^t\overset{\alpha}{\sigma}_t\right) \qquad (\alpha, \beta = 1, 2; \quad \alpha \neq \beta)$$

Considering this shape of Bianchi identities:

$$
(3.11)\qquad \begin{cases} \sum\limits_{X,Y,Z} \{D_X \mathcal{T}(Y,Z) - \Re(X,Y)Z + \mathcal{T}(\mathcal{T}(X,Y),Z)\} = 0 \\[2ex] \sum\limits_{X,Y,Z} \{D_X \Re(U,Y,Z) + \Re(\mathcal{T}(X,Y),Z)U\} = 0 \end{cases}
$$

we can rewrite them, using the projectors h, v_1, v_2 and taking the vector fields X, Y, Z as in the table:

X	A	A	A	A	A	A	B	B	B	C
Y	A	A	A	B	B	C	B	B	C	C
Z	A	B	C	B	C	C	B	C	C	C

where $A = \dfrac{\delta}{\delta x^i}$, $B = \dfrac{\delta}{\delta y^{(1)i}}$, $C = \dfrac{\delta}{\delta y^{(2)i}}$. For U we take, one by one, $\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta y^{(1)i}}, \dfrac{\delta}{\delta y^{(2)i}}$. Hence, we can obtain all Bianchi identities. Those we use in this paper, we can group in:

Proposition 3.1. *For the space $GL^{2(n)}$, endowed with the considered metric tensor, for the N-linear canonical metrical connection, these Bianchi identities hold:*

$$
(3.12)\qquad \sum_{cicl\ p,q,r} \left(R_r{}^a{}_{qp} - \overset{1}{T}{}^b_{qp} \underset{(1)}{C}{}^a{}_{rb} - \overset{2}{T}{}^b_{qp} \underset{(2)}{C}{}^a{}_{rb} \right) = 0\ ;
$$

$$
\sum_{cicl\ p,q,r} \underset{(\alpha)}{S}{}_r{}^a{}_{qp} = 0\ ;
$$

$$
\sum_{cicl\ p,q,r} \underset{(\alpha)}{S}{}_r{}^a{}_{qp} \overset{(\alpha)}{|}_b = 0\ ;
$$

$$
\sum_{cicl\ p,q,r} \left(R_r{}^a{}_{qp|b} - \overset{1}{T}{}^i_{pr} \underset{(1)}{P}{}_b{}^a{}_{qi} - \overset{2}{T}{}^i_{pr} \underset{(2)}{P}{}_b{}^a{}_{qi} \right) = 0\ ;
$$

$$
\underset{(1)}{P}{}_q{}^a{}_{pr} - \underset{(1)}{P}{}_r{}^a{}_{pq} + \underset{(1)}{\overset{1}{P}}{}^a_{pq} \overset{(1)}{|}_r - \underset{(1)}{\overset{1}{P}}{}^a_{pr} \overset{(1)}{|}_q + \underset{(1)}{\overset{2}{P}}{}^b_{pq} \underset{(2)}{C}{}^a{}_{rb} - \underset{(1)}{\overset{2}{P}}{}^b_{pr} \underset{(2)}{C}{}^a{}_{qb} +
$$

$$
\underset{(1)}{\overset{1}{P}}{}^a_{lq} \underset{(1)}{C}{}^l{}_{pr} - \underset{(1)}{\overset{1}{P}}{}^a_{lr} \underset{(1)}{C}{}^l{}_{pq} = 0\ ;
$$

$$\underset{(2)}{P}{}_{q}{}^{a}{}_{pr} - \underset{(2)}{P}{}_{r}{}^{a}{}_{pq} + \overset{2}{\underset{(2)}{P}}{}^{a}{}_{pq}\overset{(2)}{|}_{r} - \overset{2}{\underset{(2)}{P}}{}^{a}{}_{pr}\overset{(2)}{|}_{q} + \overset{2}{\underset{(2)}{P}}{}^{b}{}_{pq}\underset{(2)}{C}{}^{a}{}_{rb} - \overset{2}{\underset{(2)}{P}}{}^{b}{}_{pr}\underset{(2)}{C}{}^{a}{}_{qb} = 0\ ;$$

$$\underset{(12)}{P}{}_{q}{}^{a}{}_{pr} - \underset{(12)}{P}{}_{r}{}^{a}{}_{pq} + \overset{2}{\underset{(12)}{P}}{}^{a}{}_{pq}\overset{(2)}{|}_{r} - \overset{2}{\underset{(12)}{P}}{}^{a}{}_{pr}\overset{(2)}{|}_{q} + \overset{2}{\underset{(12)}{P}}{}^{b}{}_{pq}\underset{(2)}{C}{}^{a}{}_{rb} - \overset{2}{\underset{(12)}{P}}{}^{b}{}_{pr}\underset{(2)}{C}{}^{a}{}_{qb} = 0\ .$$

By a straightforward computation we obtain the Ricci tensors and the scalars of curvature and by consequence we can write the Einstein equations.

Theorem 3.3. *With respect to the N-linear canonical metrical connection the Einstein equations for the space GL^{2n} endowed with the metric tensor $g_{ij}(x, y^{(1)}, y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})}\gamma_{ij}(x)$ are given by:*

$$R_{bp} - \frac{1}{2}\gamma_{bp}R = \chi\ \overset{0}{T}{}_{bp} \tag{3.13}$$

$$\underset{(1)}{S}{}_{bp} - \frac{1}{2}\gamma_{bp}R = \chi\ \overset{1}{T}{}_{bp}\ ; \qquad \underset{(2)}{S}{}_{bp} - \frac{1}{2}\gamma_{bp}R = \chi\ \overset{2}{T}{}_{bp}$$

$$\overset{1}{\underset{(\alpha)}{P}}{}_{bp} = \chi\ \overset{01}{\underset{(\alpha)}{T}}{}_{bp}\ ; \qquad \overset{2}{\underset{(\alpha)}{P}}{}_{bp} = \chi\ \overset{02}{\underset{(\alpha)}{T}}{}_{bp} \qquad (\alpha = 1, 2) \tag{3.14}$$

$$\overset{1}{\underset{(12)}{P}}{}_{bp} = \chi\ \overset{12}{\underset{(1)}{T}}{}_{bp}\ ; \qquad \overset{2}{\underset{(12)}{P}}{}_{bp} = \chi\ \overset{12}{\underset{(2)}{T}}{}_{bp}$$

Theorem 3.4. *We have the following conservation law:*

$$\left(R^{b}_{p} - \frac{1}{2}R\delta^{b}_{p}\right)_{|b} + \overset{1}{\underset{(\alpha)}{P}}{}^{b}{}_{p}\overset{(\alpha)}{|}_{b} + \overset{2}{\underset{(\alpha)}{P}}{}^{b}{}_{p}\overset{(\alpha)}{|}_{b} = 0 \tag{3.15}$$

summation on α, $\alpha = 1, 2$

$$\left(\underset{(1)}{S}{}^{b}{}_{p} - \frac{1}{2}R\delta^{b}_{p}\right)\overset{(1)}{|}_{b} - \overset{2}{\underset{(1)}{P}}{}^{b}{}_{p|b} + \overset{2}{\underset{(12)}{P}}{}^{b}{}_{p}\overset{(2)}{|}_{b} = 0$$

$$\left(\underset{(2)}{S}{}^{b}{}_{p} - \frac{1}{2}R\delta^{b}_{p}\right)\overset{(2)}{|}_{b} - \overset{2}{\underset{(2)}{P}}{}^{b}{}_{p|b} + \overset{2}{\underset{(12)}{P}}{}^{b}{}_{p}\overset{(1)}{|}_{b} = 0$$

Corollary 3.1. *If the nonlinear connection N is integrable, then the conservation law is:*

$$R^b_{p|b} - \frac{1}{2}R_{|p} = 0 \quad ; \quad \underset{(\alpha)}{S^b{}_p}\overset{(\alpha)}{|_b} - \frac{1}{2}R\overset{(\alpha)}{|_p} = 0 \quad (\alpha = 1, 2) \tag{3.16}$$

4 Maxwell equations

Let us consider the Liouville vector fields $z^{(1)i}, z^{(2)i}$ given by

$z^{(1)i} = y^{(1)i},\ z^{(2)i} = y^{(2)i} + \frac{1}{2}\underset{(1)}{M^i{}_j}\ y^{(1)j}$.

With respect to the canonical nonlinear connection N, this fields depend only on the Riemannian metric γ_{ij} and are defined by:

$$z^{(1)i} = y^{(1)i}, \quad z^{(2)i} = y^{(2)i} + \frac{1}{2}\gamma^i_{jk}y^{(1)j}y^{(1)k} \tag{4.1}$$

The deflection tensors of $C\Gamma(N)$ are given by:

$$\overset{\alpha}{D}{}^i{}_j = z^{(\alpha)i}{}_{|j}; \qquad \overset{\alpha\beta}{d}{}^i{}_j = z^{(\alpha)i}\overset{(\beta)}{|_j} \qquad (\alpha, \beta = 1, 2) \tag{4.2}$$

and the covariant deflection tensors are given by the formulas:

$$\overset{\alpha}{D}_{ij} = g_{ik}\overset{\alpha}{D}{}^k{}_j \ : \ \overset{\alpha\beta}{d}_{ij} = g_{ik}\overset{\alpha\beta}{d}{}^k{}_j \qquad (\alpha, \beta = 1, 2) \tag{4.3}$$

Their local components can be obtained by direct computations.

Proposition 4.1. *The deflection tensor fields satisfy the following identities:*

$$\overset{\alpha}{D}_{ij|k} - \overset{\alpha}{D}_{ik|j} = R_{0ijk} - T^m_{jk}\overset{\alpha}{D}_{im} - \underset{(0)}{\overset{1}{T}}{}^m_{jk}\overset{\alpha 1}{d}_{im} - \underset{(0)}{\overset{2}{T}}{}^m_{jk}\overset{\alpha 2}{d}_{im} \tag{4.4}$$

$$\overset{\alpha}{D}_{ij}\overset{(\beta)}{|_k} - \overset{\alpha\beta}{d}_{ik|j} = \underset{(\beta)}{P}{}_{0ijk} - \underset{(\beta)}{C}{}^m_{jk}\overset{\alpha}{D}_{im} - \underset{(\beta)}{\overset{1}{P}}{}^m_{jk}\overset{\alpha 1}{d}_{im} - \underset{(\beta)}{\overset{2}{P}}{}^m_{jk}\overset{\alpha 2}{d}_{im}$$

$$\overset{\alpha 1}{d}_{ij}\overset{(\alpha)}{|_k} - \overset{\alpha 2}{d}_{ik}\overset{(1)}{|_j} = \underset{(2)}{\overset{1}{S}}{}_{0ijk} - \underset{(2)}{C}{}^m_{jk}\overset{\alpha 1}{d}_{im} - \underset{(12)}{\overset{1}{P}}{}^m_{jk}\overset{\alpha 2}{d}_{im}$$

$$\overset{\beta\alpha}{d}_{ij}\overset{(\alpha)}{|}_k - \overset{\beta\alpha}{d}_{ik}\overset{(\alpha)}{|}_j = \underset{(\alpha)}{\overset{\alpha}{S}}{}_{0ijk} - \underset{(\alpha)}{S}{}^m{}_{jk}\,\overset{\beta1}{d}_{im} - \underset{(\alpha)}{R}{}^m{}_{jk}\,\overset{\beta2}{d}_{im} \qquad (\alpha,\beta=1,2)$$

where index 0 means the contraction by $z^{(\alpha)m}$.

Definition 4.1. *The h- and* $v_\alpha-$ *components of the electromagnetic tensor fields are:*

$$F^{(\alpha)}_{ij} = \frac{1}{2}\left(\overset{\alpha}{D}_{ij} - \overset{\alpha}{D}_{ji}\right); \tag{4.5}$$

$$f^{(\alpha\beta)}_{ij} = \frac{1}{2}\left(\overset{\alpha\beta}{d}_{ij} - \overset{\alpha\beta}{d}_{ji}\right).$$

By straight calculation we obtain the h- and $v_\alpha-$ components of the electromagnetic tensors.

Theorem 4.1. *The electromagnetic tensor fields of the space satisfy the following generalized Maxwell equations:*

$$\sum_{cicl\ (i,j,k)} F^{(1)}_{ij|k} = \frac{e^{2\sigma}}{2}\sum_{cicl\ (i,j,k)}\Big[z^{(1)l}\gamma_{jt}\,\underset{(1)}{C}{}^t{}_{ls}\,\underset{(0)}{\overset{1}{R}}{}^s{}_{ik} + \tag{4.6}$$

$$+z^{(1)l}\gamma_{jt}\,\underset{(2)}{C}{}^t{}_{ls}\,\underset{(0)}{\overset{2}{R}}{}^s{}_{ik} + \overset{11}{d}_{js}\,\underset{(0)}{\overset{1}{R}}{}^s{}_{ik} + \overset{12}{d}_{js}\,\underset{(0)}{\overset{2}{R}}{}^s{}_{ik}\Big]$$

$$\sum_{cicl\ (i,j,k)} F^{(2)}_{ij|k} = \frac{e^{2\sigma}}{2}\sum_{cicl\ (i,j,k)}\Big[z^{(2)l}\gamma_{jt}\,\underset{(1)}{C}{}^t{}_{ls}\,\underset{(0)}{\overset{1}{R}}{}^s{}_{ik} +$$

$$+z^{(2)l}\gamma_{jt}\,\underset{(2)}{C}{}^t{}_{ls}\,\underset{(0)}{\overset{2}{R}}{}^s{}_{ik} + \overset{21}{d}_{js}\,\underset{(0)}{\overset{1}{R}}{}^s{}_{ik} + \overset{22}{d}_{js}\,\underset{(0)}{\overset{2}{R}}{}^s{}_{ik}\Big]$$

$$\sum_{cicl\ (i,j,k)} F^{(\alpha)}_{ij}\overset{(1)}{|}_k + \sum_{cicl\ (i,j,k)} f^{(\alpha1)}_{ij|k} = \frac{e^{2\sigma}}{2}\sum_{cicl\ (i,j,k)}\left(z^{(\alpha)l}\gamma_{jt}\,\underset{(2)}{C}{}^t{}_{ls} + \overset{\alpha2}{d}_{is}\right) r_p{}^s{}_{jk} z^{(1)p}$$

$$\sum_{cicl\ (i,j,k)} F^{(\alpha)}_{ij}\overset{(2)}{|}_k + \sum_{cicl\ (i,j,k)} f^{(\alpha1)}_{ij|k} = 0 \qquad (\alpha=1,2)$$

$$\sum_{cicl\ (i,j,k)} f_{ij}^{(\alpha\beta)} \overset{(\beta)}{|_k} = 0 \qquad (\alpha, \beta = 1, 2)$$

$$\sum_{cicl\ (i,j,k)} f_{ij}^{(\alpha 1)} \overset{(2)}{|_k} + \sum_{cicl\ (i,j,k)} f_{ij}^{(\alpha 2)} \overset{(1)}{|_k} = 0 \qquad (\alpha = 1, 2)$$

5 Applications

We present a complete calculations for two remarkable cases.

First let us consider the function σ given by:

(5.1) $$\sigma(x^i, y^{(1)i}, y^{(2)i}) = \frac{1}{2}\|y^{(1)i}\|^2 = \frac{1}{2}\gamma_{ij} z_i^{(1)} z_j^{(1)}$$

The following results holds:

Proposition 5.1.

(5.2) $$\overset{0}{\bar{\sigma}_k} = 0; \quad \overset{1}{\bar{\sigma}_k} = z_k^{(1)}; \quad \overset{2}{\bar{\sigma}_k} = 0$$

$$\bar{L}^i_{jk} = \gamma^i_{jk}; \quad \underset{(1)}{\bar{C}^i_{jk}} = \delta^i_k z_j^{(1)} + \delta^i_j z_k^{(1)} - \gamma_{jk} z^{(1)i}; \quad \underset{(2)}{\bar{C}^i_{jk}} = 0$$

$$\overset{0}{\bar{\sigma}_{ij}} = 0; \quad \overset{01}{\bar{\sigma}_{ij}} = 0; \quad \overset{02}{\bar{\sigma}_{ij}} = 0; \quad \overset{0}{\bar{\sigma}^t} \ \overset{\alpha}{\bar{\sigma}_t} = \overset{0}{\bar{\sigma}_t} \ \overset{\alpha}{\bar{\sigma}^t} = 0 \qquad (\alpha = 0, 1, 2)$$

$$\overset{1}{\bar{\sigma}_{ij}} = \delta^i_j + \frac{1}{2}\gamma_{ij}\|y^{(1)}\|^2 - z_i^{(1)} z_j^{(1)}; \quad \overset{1}{\bar{\sigma}^t} \ \overset{1}{\bar{\sigma}_t} = \|y^{(1)}\|^2; \quad \overset{1}{\bar{\sigma}^t} \ \overset{\alpha}{\bar{\sigma}_t} = 0 \quad (\alpha = 0, 2)$$

$$\overset{1\alpha}{\bar{\sigma}_{ij}} = 0; \quad \alpha = 1, 2; \quad \overset{2}{\bar{\sigma}_{ij}} = 0$$

Proposition 5.2. *The torsion d-vector fields are:*

(5.3) $$\underset{(0)}{\overset{\alpha}{\bar{T}^i}_{jk}} = \underset{(0)}{\overset{\alpha}{T^i}_{jk}}; \quad \underset{(1)}{\overset{0}{\bar{P}^i}_{jk}} = \underset{(1)}{C^i_{jk}}; \quad \underset{(2)}{\overset{0}{\bar{P}^i}_{jk}} = 0$$

$$\underset{(1)}{\overset{1}{\bar{P}}}{}^{i}{}_{jk} = 0; \quad \underset{(1)}{\overset{2}{\bar{P}}}{}^{i}{}_{jk} = \underset{(1)}{\overset{2}{P}}{}^{i}{}_{jk}\ ; \quad \underset{(21)}{\overset{1}{\bar{P}}}{}^{i}{}_{jk} = \underset{(1)}{C}{}^{i}_{jk} \tag{5.4}$$

$$\underset{(2)}{\overset{1}{\bar{P}}}{}^{i}{}_{jk} = \underset{(2)}{\overset{2}{\bar{P}}}{}^{i}{}_{jk} = \underset{(21)}{\overset{0}{\bar{P}}}{}^{i}{}_{jk} = \underset{(21)}{\overset{2}{\bar{P}}}{}^{i}{}_{jk} = 0$$

$$\underset{(\alpha)}{\overset{\beta}{\bar{S}}}{}^{i}{}_{jk} = 0; \quad (\alpha, \beta = 0, 1, 2) \tag{5.5}$$

Proposition 5.3. *The curvature vector fields are:*

$$\bar{R}_{b}{}^{a}{}_{pq} = r_{b}{}^{a}{}_{pq} + g_{bt}\left(r_{s}{}^{a}{}_{pq} y^{(1)t} y^{(1)s} - r_{s}{}^{t}{}_{pq} y^{(1)a} y^{(1)s}\right) \tag{5.6}$$

$$\underset{(\alpha)}{\bar{P}_{b}{}^{a}{}_{pq}} = 0 \qquad (\alpha = 1, 2); \qquad \underset{(12)}{\bar{P}_{b}{}^{a}{}_{pq}} = 0 \tag{5.7}$$

$$\underset{(1)}{\bar{S}_{b}{}^{a}{}_{pq}} = \delta^{a}_{p} \overset{1}{\bar{\sigma}}_{bq} - \delta^{a}_{q} \overset{1}{\bar{\sigma}}_{bp} + g^{as}\left(g_{bq} \overset{1}{\bar{\sigma}}_{sp} - g_{bp} \overset{1}{\bar{\sigma}}_{sq}\right) \tag{5.8}$$

$$\underset{(2)}{\bar{S}_{b}{}^{a}{}_{pq}} = 0$$

Proposition 5.4. *Ricci tensors and curvature scalars have the following expressions:*

$$\bar{R}_{bp} = r_{bp} + \gamma_{bt}\left(r_{0}{}^{a}{}_{pa} y^{(1)t} - r_{0}{}^{t}{}_{pa} y^{(1)a}\right) \tag{5.9}$$

$$\underset{(\alpha)}{\overset{1}{\bar{P}}}{}_{bp} = \underset{(\alpha)}{\overset{2}{\bar{P}}}{}_{bp} = 0 \quad (\alpha = 1, 2); \qquad \underset{(12)}{\overset{1}{\bar{P}}}{}_{bp} = \underset{(12)}{\overset{2}{\bar{P}}}{}_{bp} = 0; \tag{5.10}$$

$$\underset{(1)}{\bar{S}}{}_{bp} = (1 - n) \overset{1}{\bar{\sigma}}_{bp} + \gamma^{as}\left(\gamma_{bp} \overset{1}{\bar{\sigma}}_{sa} - \gamma_{ba} \overset{1}{\bar{\sigma}}_{sp}\right); \quad \underset{(2)}{\bar{S}}{}_{bp} = 0 \tag{5.11}$$

and:

$$(5.12)\qquad \bar{R} = \left(r + 2r_{st}y^{(1)s}y^{(1)t} + 2(1-n)\gamma^{bp} \overset{1}{\sigma}_{bp} \right) e^{-2\sigma}$$

$$(5.13)\qquad \underset{(\alpha)}{\overset{1}{\bar{P}}} = \underset{(\alpha)}{\overset{2}{\bar{P}}} = 0; \quad \underset{(12)}{\overset{1}{\bar{P}}} = \underset{(12)}{\overset{1}{\bar{P}}} = 0; \quad (\alpha = 1, 2)$$

$$(5.14)\qquad \underset{(1)}{\bar{S}} = 2(1-n)\gamma^{bp} \overset{1}{\bar{\sigma}}_{bp}\ e^{-2\sigma}; \qquad \underset{(2)}{\bar{S}} = 0$$

Proposition 5.5. *Einstein equations for the space endowed with this metric are:*

(5.15)

$$\frac{1}{2} r\gamma_{bp} + \gamma_{bp} \left[(n-1)\gamma^{ij} \overset{1}{\sigma}_{ij} - r_{st}y^{(1)s}y^{(1)t} \right] = \chi \underset{()}{\overset{0}{T}}{}_{bp} - \gamma_{bt} \left(r_0{}^{a}{}_{pa} y^{(1)t} - r_0{}^{t}{}_{pa} y^{(1)a} \right)$$

$$\underset{(1)}{\bar{S}}{}_{bp} - \frac{1}{2} g_{bp} R = \chi \underset{()}{\overset{1}{T}}{}_{bp}$$

$$\frac{1}{2} g_{bp} R = -\chi \underset{()}{\overset{1}{T}}{}_{bp}$$

We write the first equation in this shape in order to emphasise the relation between Einstein equations of the space and Einstein equations of the riemannian space $V^n = (M, \gamma_{ij}(x))$.

Proposition 5.6. *$h-$, and $v_\alpha-$ components of the electromagnetic tensor fields have the expressions:*

$$(5.16)\qquad F_{ij}^{(1)} = 0; \quad F_{ij}^{(2)} = e^{2\sigma} \left(\frac{\partial \gamma_{ipj}}{\partial x^q} - \frac{\partial \gamma_{ipq}}{\partial x^j} \right) y^{(1)p} y^{(1)q}$$

$$f_{ij}^{(11)} = f_{ij}^{(21)} = f_{ij}^{(12)} = 0$$

$$f_{ij}^{(22)} = e^{2\sigma} (z_i^{(2)} z_j^{(1)} - z_j^{(2)} z_i^{(1)})$$

and the generalized Maxwell equations are satisfied.

For the second application we consider that σ is given by:

$$\sigma(x^i,\ y^{(1)i},\ y^{(2)i}) = A_k(x)z^{(1)k} \tag{5.17}$$

where $A_k(x)$ can be regarded as a potential.

Proposition 5.7. *In this context we obtain:*

$$\overset{0}{\tilde{\sigma}_k} = y^{(1)s}A_{s|k}; \quad \overset{1}{\tilde{\sigma}_k} = A_k; \quad \overset{2}{\tilde{\sigma}_k} = 0 \tag{5.18}$$

$$\tilde{L}^i_{jk} = \gamma^i_{jk} + \left(\delta^i_j A_{p|k} + \delta^i_k A_{p|j} - \gamma_{jk}\gamma^{si}A_{p|s}\right) y^{(1)p}$$

$$\underset{(1)}{\tilde{C}^i_{jk}} = \delta^i_j A_k + \delta^i_k A_j - \gamma_{jk}\gamma^{si}A_s, \quad \underset{(2)}{\tilde{C}^i_{jk}} = 0$$

Proposition 5.8. *If the vector $A_k(x)$ is parallel with respect to Berwald connection then we have:*

$$\overset{0}{\tilde{\sigma}_k} = 0, \quad \overset{1}{\tilde{\sigma}_k} = A_k, \quad \overset{2}{\tilde{\sigma}_k} = 0 \tag{5.19}$$

$$\tilde{L}^i_{jk} = \gamma^i_{jk}; \quad \underset{(1)}{\tilde{C}^i_{jk}} = \delta^i_k A_j + \delta^i_j A_k - \gamma_{jk}\gamma^{is}A_s; \quad \underset{(2)}{\tilde{C}^i_{jk}} = 0$$

$$\overset{0}{\tilde{\sigma}_{ij}} = 0: \quad \overset{1\alpha}{\tilde{\sigma}_{ij}} = 0 \quad (\alpha = 1,2); \quad \overset{0}{\tilde{\sigma}^t}\ \overset{\beta}{\tilde{\sigma}_t} = 0 \quad (\beta = 0,1,2)$$

$$\overset{1}{\sigma_{ij}} = \frac{1}{2}\gamma_{ij}\gamma_{ts}A_sA_t - A_iA_j; \quad \overset{1}{\tilde{\sigma}^t}\ \overset{1}{\tilde{\sigma}_t} = \|A\|^2; \quad \overset{1}{\tilde{\sigma}^t}\ \overset{\alpha}{\tilde{\sigma}_t} = 0 \quad (\alpha = 0,2)$$

$$\overset{2}{\tilde{\sigma}_{ij}} = 0$$

Observe that in this case the coefficients of the N-linear canonical metrical connection does not depend on the directional vectors $y^{(1)},\ y^{(2)}$ and $A_{i|j} = 0$ implies that the covector $A_k(x)$ is a gradient.

Proposition 5.9. *If the vector $A_k(x)$ is parallel with respect to the Berwald connection then the following relations hold:*
i) the d-vector fields of torsion are:

$$\underset{(0)}{\overset{\alpha}{\tilde{T}}}{}^{i}{}_{jk} = \underset{(0)}{\overset{\alpha}{T}}{}^{i}{}_{jk} \;;\quad \underset{(1)}{\overset{0}{\tilde{P}}}{}^{i}{}_{jk} = \underset{(1)}{\tilde{C}}{}^{i}_{jk} \;;\quad \underset{(2)}{\overset{0}{\tilde{P}}}{}^{i}{}_{jk} = \underset{(2)}{\tilde{C}}{}^{i}_{jk} \tag{5.20}$$

$$\underset{(1)}{\overset{1}{\tilde{P}}}{}^{i}{}_{jk} = 0;\quad \underset{(1)}{\overset{2}{\tilde{P}}}{}^{i}{}_{jk} = \underset{(1)}{\overset{2}{P}}{}^{i}{}_{jk} \;;\quad \underset{(21)}{\overset{1}{\tilde{P}}}{}^{i}{}_{jk} = \underset{(1)}{\tilde{C}}{}^{i}_{jk}$$

$$\underset{(2)}{\overset{1}{\tilde{P}}}{}^{i}{}_{jk} = \underset{(2)}{\overset{2}{\tilde{P}}}{}^{i}{}_{jk} = \underset{(21)}{\overset{0}{\tilde{P}}}{}^{i}{}_{jk} = \underset{(21)}{\overset{2}{\tilde{P}}}{}^{i}{}_{jk} = 0$$

$$\underset{(\alpha)}{\overset{\beta}{\tilde{S}}}{}^{i}{}_{jk} = 0;\quad (\alpha, \beta = 0, 1, 2) \tag{5.21}$$

ii) the d-vector fields of curvature are given by:

$$\underset{()}{\tilde{R}}{}_{b}{}^{a}{}_{pq} = r_{b}{}^{a}{}_{pq} + e^{2\sigma}\left(r_{s}{}^{a}{}_{pq}A_b - \gamma_{bt}\gamma^{al}r_{s}{}^{t}{}_{pq}A_l\right)y^{(1)s} \tag{5.22}$$

$$\underset{(\alpha)}{\tilde{P}}{}_{b}{}^{a}{}_{pq} = 0 \qquad (\alpha = 1, 2);\qquad \underset{(12)}{\tilde{P}}{}_{b}{}^{a}{}_{pq} = 0 \tag{5.23}$$

$$\underset{(1)}{\tilde{S}}{}_{b}{}^{a}{}_{pq} = \delta^a_p \overset{1}{\tilde{\sigma}}_{bq} - \delta^a_q \overset{1}{\tilde{\sigma}}_{bp} + g^{as}\left(g_{bq}\overset{1}{\tilde{\sigma}}_{sp} - g_{bp}\overset{1}{\tilde{\sigma}}_{sq}\right) \tag{5.24}$$

$$\underset{(2)}{\tilde{S}}{}_{b}{}^{a}{}_{pq} = 0$$

iii) Ricci tensor fields are:

$$\tilde{R}_{bp} = r_{bp} + r_{0}{}^{a}{}_{pa}A_b - \gamma_{bt}\gamma^{al}r_{0}{}^{t}{}_{pa}A_l \tag{5.25}$$

$$\underset{(\alpha)}{\overset{1}{\tilde{P}}}{}_{bp} = \underset{(\alpha)}{\overset{2}{\tilde{P}}}{}_{bp} = 0 \quad (\alpha = 1,2); \qquad \underset{(12)}{\overset{1}{\tilde{P}}}{}_{bp} = \underset{(12)}{\overset{2}{\tilde{P}}}{}_{bp} = 0;$$

$$\underset{(1)}{\tilde{S}}{}_{bp} = (1-n) \overset{1}{\tilde{\sigma}}_{bp} + \gamma^{as}\left(\gamma_{bp} \overset{1}{\tilde{\sigma}}_{sa} - \gamma_{ba} \overset{1}{\tilde{\sigma}}_{sp}\right); \qquad \underset{(2)}{\tilde{S}}{}_{bp} = 0$$

iv) the curvature scalars have the following expressions:

$$\tilde{R} = \left(r + 2r_{st}y^{(1)s}\gamma^{tp}A_p + 2(1-n)\gamma^{bp} \overset{1}{\tilde{\sigma}}_{bp}\right)e^{-2\sigma} \tag{5.26}$$

$$\underset{(\alpha)}{\overset{1}{\tilde{P}}} = \underset{(\alpha)}{\overset{2}{\tilde{P}}} = 0; \qquad \underset{(12)}{\overset{1}{\tilde{P}}} = \underset{(12)}{\overset{1}{\tilde{P}}} = 0; \quad (\alpha = 1,2)$$

$$\underset{(1)}{\tilde{S}} = 2(1-n)\gamma^{bp} \overset{1}{\bar{\sigma}}_{bp}\ e^{-2\sigma}; \qquad \underset{(2)}{\tilde{S}} = 0$$

v) Einstein equations are:

$$\tilde{r}_{bp} - \tilde{r}\gamma_{bp} + \tilde{t}_{bp} = \chi\tilde{T}_{bp} \tag{5.27}$$

$$\underset{(1)}{\tilde{S}}{}_{bp} - \frac{1}{2}g_{bp}\tilde{R} = \chi \overset{1}{T}{}_{bp}$$

$$\frac{1}{2}g_{bp}\tilde{R} = -\chi \overset{2}{T}{}_{bp}$$

where

$$\tilde{t}_{bp} = \gamma_{bp}\left[(n-1)\gamma^{ij} \overset{1}{\tilde{\sigma}}_{ij} - \tilde{r}_{st}\gamma^{tl}A_l y^{(1)s}\right] \tag{5.28}$$

$$\tilde{T}_{bp} = \overset{0}{T}{}_{bp} - \frac{1}{\chi}\gamma_{bt}\left(r_0{}^a{}_{pa}\gamma^{lt}A_l - r_0{}^t{}_{pa}\gamma^{al}A_l\right)$$

and the following conservation laws are verified:

$$\overset{0}{T}{}^i{}_{j|k} = 0; \qquad \overset{1}{T}{}^i{}_j \overset{(1)}{|}_k = 0; \qquad \overset{2}{T}{}^i{}_j \overset{(2)}{|}_k = 0 \tag{5.29}$$

Because $A_k(x)$ is a gradient we obtain that:

Proposition 5.10. *The components of the electromagnetic tensor field are expressed by:*

$$F_{ij}^{(1)} = 0; \quad F_{ij}^{(2)} = 0; \quad f_{ij}^{(11)} = f_{ij}^{(12)} = f_{ij}^{(21)} = 0 \tag{5.30}$$

$$f_{ij}^{(22)} = (g_{is}A_j - g_{js}A_i)y^{(1)s}$$

and the generalized Maxwell equations are trivially satisfied.

References

[1] M. Anastasiei, *Vector bundles. Einstein Equations,* An. St. Univ. Al.I.Cuza Iaşi, 32 (1986), 17-24.

[2] V. Balan, *Generalized Einstein Yang Mills equations for the space $GL^n = (M, g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x)$,* Tensor N.S. 52 (1993), 199-203.

[3] R. Miron, *The geometry of higher order Lagrange spaces. Applications to Mechanics and Physics,* Kluwer Acad.Publ. FTPH, 1996.

[4] R. Miron, Gh. Atanasiu, *Compendium sur les espaces Lagrange d'ordre superior,* Sem. Mec. Univ. Timişoara, nr. 40 (1994), 1-27.

[5] R. Miron, Gh. Atanasiu, *Geometrical Theory of Gravitational and Electromagnetic Fields in Higher Order Lagrange Spaces,* Tsukuba J. Math. 20, 1 (1996), 137-149.

[6] R. Miron, M. Anastasiei, *The geometry of Lagrange spaces. Theory and Applications,* Kluwer Acad. Pub. FTPH 59, 1994.

[7] R. Miron, R.K. Tavakol, V. Balan, I. Roxburgh,, *Geometry of space-time and generalized Lagrange gauge theory,* Pub. Mathematicae, Debrecen, 42, 3-4 (1993), 215-224.

[8] Păun, M., *Study of the generalized Lagrange space of order two* $GL^{2n}(M, g_{ij}(x, y^{(1)}, y^{(2)})) = e^{2\sigma}\gamma_{ij}(x))$ Novi Sad Jour. Math. 35, 1 (2005), 79-94.

[9] Watanabe, S., Ikeda, F., Ikeda S., *On a metrical Finsler connection of a generalized Finsler metric $g_{ij} = e^{2\sigma(x)}\gamma_{ij}(x)$,* Tensor, N.S. 40 (1983), 97-102.

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