Hopf bifurcation for the rigid body with time delay

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Abstract

In this paper we define the differential equations with time delay for the rigid body by using eight 2-covariant tensor fields on $\mathbb{R}^3 \times \mathbb{R}^3$. We prove the existence of the revised system and investigate the existence of a Hopf bifurcation in the neighborhood of the equilibrium point $M_1(m,0,0)$. The normal form is described.

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1. The differential equations for rigid body with time delay

The differential equations for the rigid body in $\mathbb{R}^3$ are described by a 2-contravariant tensor field P_0 and by the Hamiltonian function h_0 given by:

$$P_0 = (P_0^{ij}(x)) = \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix} \tag{1}$$

$$h_0(x) = \frac{1}{2}a_1(x_1)^2 + \frac{1}{2}a_2(x^2)^2 + \frac{1}{2}a_3(x^3)^2$$

where $(x^1, x^2, x^3) \in \mathbb{R}^3$, $a_1, a_2, a_3 \in \mathbb{R}$, $a_1 > a_2 > a_3$. These differential equations are

$$\dot{x}(t) = P_0(x(t)) \bigtriangledown_x h_0(x(t)) \tag{2}$$

where $\dot{x}(t) = (\dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t))^T$ and $\bigtriangledown h_0$ is the gradient of h_0 with respect to the canonical metric on $\mathbb{R}^3$. The differential equations (2) have the equilibrium points $M_1(m,0,0)$, $M_2(0,m,0)$, $M_3(0,0,m)$, $m \in \mathbb{R}^*$ and have been studied in [5].

The revised differential equations for the rigid body have been studied in [6] and [3]. They are described by: two 2-contravariant tensor fields P_0, g_0, the Hamiltonian

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function h_0 and the Casimir function l_0, where P_0 and h_0 were given by (1) and g_0 is defined by:

$$\begin{aligned}
&g_0(x) = (g_0^{ij}(x)), l_0(x) = \frac{1}{2}(x^1)^2 + \frac{1}{2}(x^2)^2 + \frac{1}{2}(x^3)^2\\
&g_0^{ij}(x) = \frac{\partial h_0(x)}{\partial x^i}\frac{\partial h_0(x)}{\partial x^j}, i \neq j\\
&g_0^{ij}(x) = -\sum_{k=1}^{3}\left(\frac{\partial h_0(x)}{\partial x^k}\right)^2, i,j = 1,2,3.
\end{aligned}$$

The revised differential equations for the rigid body are:

$$\dot{x}(t) = P_0(x(t))\nabla_x h_0(x) + g_0(x(t))\nabla_x l_0(x(t)). \tag{3}$$

The equations (3) has the same equilibrium points $M_1(m,0,0)$, $M_2(0,m,0)$, $M_3(0,0,m)$, $m \in \mathbb{R}^*$.

The differential equations with time delay are described with the vector field $X \in \mathcal{X}(\mathbb{R}^3 \times \mathbb{R}^3)$ that satisfy the property $X(\pi_1^* f) = 0$, for all $f \in C^\infty(\mathbb{R}^3)$, where π_1 is the canonical projection on the first argument $\pi_1 : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and we denote by $C^\infty(\mathbb{R}^3)$ the ring of smooth real valued functions on $\mathbb{R}^3$ and by $\mathcal{X}(\mathbb{R}^3 \times \mathbb{R}^3)$ the Lie algebra of all smooth vector fields on $\mathbb{R}^3 \times \mathbb{R}^3$. The differential equations with time delay associated to vector field $X(\widetilde{x}, x)$ are

$$\dot{x}(t) = X(x(t-\tau), x(t)) \tag{4}$$

where $(\widetilde{x}, x) \in \mathbb{R}^3 \times \mathbb{R}^3$, $\tilde{x}(t) = x(t-\tau), \tau \geq 0$, with initial value $x(\theta) = \varphi(\theta)$, $\theta \in [-\tau, 0]$, where $\varphi : [-\tau, 0] \to \mathbb{R}^3$, $\varphi \in C^\infty(\mathbb{R}^3)$.

The definition of differential equations with time delay on differential manifolds was given in [4] and it was studied by [1,2].

The differential equations with time delay for the rigid body are generated by an antisymetric 2-contravariant tensor field P on the manifold $\mathbb{R}^3 \times \mathbb{R}^3$ that satisfies the following relations:

$$P(\pi_1^* df_1, \pi_1^* df_2) = 0, \quad P(\pi_1^* df_1, \pi_2^* dh) = 0$$

for all $f_1, f_2 \in C^\infty(\mathbb{R}^3)$, where $\pi_1, \pi_2 : M \times M \to M$ are canonical projections and $h \in C^\infty(M \times M)$. The 2-contravariant tensor field P is:

$$P(\tilde{x}, x) = \sum_{i=o}^{7} \varepsilon_i P_i(\tilde{x}, x) \tag{5}$$

where

$$P_0(x) = \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix}, \quad P_1(\tilde{x}, x) = \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & \tilde{x}^1 \\ x^2 & -\tilde{x}^1 & 0 \end{pmatrix},$$

$$P_2(\tilde{x},x)=\begin{pmatrix}0 & x^3 & -\tilde{x}^2\\ -x^3 & 0 & x^1\\ \tilde{x}^2 & -x^1 & 0\end{pmatrix},\quad P_3(\tilde{x},x)=\begin{pmatrix}0 & \tilde{x}^3 & -x^2\\ -\tilde{x}^3 & 0 & x^1\\ x^2 & -x^1 & 0\end{pmatrix},\qquad (6)$$

$$P_4(\tilde{x},x)=\begin{pmatrix}0 & x^3 & -\tilde{x}^2\\ -x^3 & 0 & \tilde{x}^1\\ \tilde{x}^2 & -\tilde{x}^1 & 0\end{pmatrix},\quad P_5(\tilde{x},x)=\begin{pmatrix}0 & \tilde{x}^3 & -x^2\\ -\tilde{x}^3 & 0 & \tilde{x}^1\\ x^2 & -\tilde{x}^1 & 0\end{pmatrix},$$

$$P_6(\tilde{x},x)=\begin{pmatrix}0 & \tilde{x}^3 & -\tilde{x}^2\\ -\tilde{x}^3 & 0 & x^1\\ \tilde{x}^2 & -x^1 & 0\end{pmatrix},\quad P_7(\tilde{x},x)=\begin{pmatrix}0 & \tilde{x}^3 & -\tilde{x}^2\\ -\tilde{x}^3 & 0 & \tilde{x}^1\\ \tilde{x}^2 & -\tilde{x}^1 & 0\end{pmatrix},$$

and $\varepsilon=(\varepsilon_i)\in[0,1]^8$, $i=0,1,2,...,7$ with $\sum\limits_{i=0}^{7}\varepsilon_i=1$.

The function $h(\tilde{x},x)$ is called the Hamiltonian function and it has the following form:

$$h(\tilde{x},x)=\sum_{i=0}^{7}\delta_i h_i(\tilde{x},x)\qquad (7)$$

where

$$\begin{aligned}
h_0(x)&=\frac{1}{2}a_1(x^1)^2+\frac{1}{2}a_2(x^2)^2+\frac{1}{2}a_3(x^3)^2,\\
h_1(\tilde{x},x)&=\frac{1}{2}a_1x^1\tilde{x}^1+\frac{1}{2}a_2(x^2)^2+\frac{1}{2}a_3(x^3)^2,\\
h_2(\tilde{x},x)&=\frac{1}{2}a_1(x^1)^2+a_2x^2\tilde{x}^2+\frac{1}{2}a_3(x^3)^2,\\
h_3(\tilde{x},x)&=\frac{1}{2}a_1(x^1)^2+\frac{1}{2}a_2(x^2)^2+a_3x^3\tilde{x}^3,\\
h_4(\tilde{x},x)&=a_1x^1\tilde{x}^1+a_2x^2\tilde{x}^2+\frac{1}{2}a_3(x^3)^2,\\
h_5(\tilde{x},x)&=a_1x^1\tilde{x}^1+\frac{1}{2}a_2(x^2)^2+a_3x^3\tilde{x}^3,\\
h_6(\tilde{x},x)&=\frac{1}{2}a_1(x^1)^2+a_2x^2\tilde{x}^2+a_3x^3\tilde{x}^3,\\
h_7(\tilde{x},x)&=a_1x^1\tilde{x}^1+a_2x^2\tilde{x}^2+a_3x^3\tilde{x}^3,
\end{aligned}$$

and $\delta=(\delta_i)\in[0,1]^8$, with $\sum\limits_{i=o}^{7}\delta_i=1$.

The differential equations with time delay for the rigid body are given by:

$$\dot{x}(t)=P(\tilde{x}(t),x(t))\nabla_x h(\tilde{x}(t),x(t))\qquad (8)$$

where $\tilde{x}(t)=x(t-\tau)$, $\tau\geq 0$ with initial value $x(\theta)=\phi(\theta)$, $\theta\in[-\tau,0]$, where $\phi:[-\tau,0]\to\mathbb{R}^3$, $\phi\in C^\infty(\mathbb{R}^3)$. The differential equations (8) have the equilibrium points $M_1(m,0,0)$, $M_2(0,m,0)$, $M_3(0,0,m)$, $m\in\mathbb{R}^*$.

In what follows, we consider the function $l \in C^{\infty}(\mathbb{R}^3 \times \mathbb{R}^3)$ given by:

$$l(\tilde{x}, x) = \sum_{i=o}^{7} \varepsilon_i h_i(\tilde{x}, x) \tag{9}$$

where

$$\begin{aligned}
&l_0(\tilde{x}, x) = \frac{1}{2}(x^1)^2 + \frac{1}{2}(x^2)^2 + \frac{1}{2}(x^3)^2, l_1(\tilde{x}, x) = x^1\tilde{x}^1 + \frac{1}{2}(x^2)^2 + \frac{1}{2}(x^3)^2,\\
&l_2(\tilde{x}, x) = \frac{1}{2}(x^1)^2 + x^2\tilde{x}^2 + \frac{1}{2}(x^3)^2, l_3(\tilde{x}, x) = \frac{1}{2}(x^1)^2 + \frac{1}{2}(x^2)^2 + x^3\tilde{x}^3,\\
&l_4(\tilde{x}, x) = x^1\tilde{x}^1 + x^2\tilde{x}^2 + \frac{1}{2}(x^3)^2, l_5(\tilde{x}, x) = x^1\tilde{x}^1 + \frac{1}{2}(x^2)^2 + x^3\tilde{x}^3,\\
&l_6(\tilde{x}, x) = \frac{1}{2}(x^1)^2 + x^2\tilde{x}^2 + x^3\tilde{x}^3, l_7(\tilde{x}, x) = x^1\tilde{x}^1 + x^2\tilde{x}^2 + x^3\tilde{x}^3
\end{aligned}$$

Proposition 1. *(i) The function $l(\tilde{x}, x)$ given by (9) satisfies the following relation*

$$\nabla_x l(\tilde{x}, x) P(\tilde{x}, x) \nabla_x f(\tilde{x}, x) = 0 \tag{10}$$

for all $f \in C^{\infty}(M \times M)$.

(ii) The revised differential equations with time delay have the form:

$$\dot{x}(t) = P(\tilde{x}(t), x(t)) \nabla_x h(\tilde{x}(t), x(t)) + g(\tilde{x}(t), x(t)) \nabla_{\tilde{x}} l(\tilde{x}(t), x(t)) \tag{11}$$

where $\tilde{x}(t) = x(t - \tau)$, $\tau \geq 0$ and $g(\tilde{x}, x)$ is a 2-contravariant tensor field given by:

$$\begin{aligned}
&g(\tilde{x}, x) = (g^{ij}(\tilde{x}, x)),\\
&g^{ij}(\tilde{x}, x) = \frac{\partial h(\tilde{x}, x)}{\partial x^i} \frac{\partial h(\tilde{x}, x)}{\partial x^j}, i \neq j\\
&g^{ii}(\tilde{x}, x) = -\sum_{h=1, k\neq i}^{3} \left(\frac{\partial h(\tilde{x}, x)}{\partial x^k}\right)^2.
\end{aligned}$$

The system (11) has the equilibrium points $M_1(m, 0, 0)$, $M_2(0, m, 0)$, $M_3(0, 0, m)$, $m \in \mathbb{R}^*$.

In order to analyze system (11) in the neighborhood of equilibrium point $M_1(m, 0, 0)$ we will consider $\varepsilon_1 = \varepsilon_4 = \varepsilon_5 = \varepsilon_7 = 0$, $\varepsilon_0, \varepsilon_2, \varepsilon_3, \varepsilon_6 \in [0, 1]$ with $\alpha_1 = \varepsilon_0 + \varepsilon_2 + \varepsilon_3 + \varepsilon_6 = 1$ and $\delta_i \in [0, 1]$, $i = 0, ...7$ so that

$$\delta_0 + \delta_1 + \delta_2 + \delta_4 = \frac{a_1}{a_3}(\varepsilon_0 + \varepsilon_2), \delta_0 + \delta_1 + \delta_3 + \delta_5 = \frac{a_1}{a_2}(\varepsilon_0 + \varepsilon_3), \sum_{i=0}^{7} \delta_i = 1. \tag{12}$$

From the above conditions we obtain:

$$\begin{aligned}
&\alpha_1 = 1, \quad \beta_1 = 0, \quad \alpha_2 = \varepsilon_0 + \varepsilon_3, \quad \beta_2 = \varepsilon_2 + \varepsilon_6, \quad \alpha_3 = \varepsilon_0 + \varepsilon_2,\\
&\beta_3 = \varepsilon_3 + \varepsilon_6, a_3\mu_3 = a_1\alpha_3, \quad a_2\mu_2 = a_1\alpha_2, \quad \mu_3 = \delta_0 + \delta_1 + \delta_2 + \delta_4,\\
&\eta_3 = 1 - \mu_3, \quad \mu_2 = \delta_0 + \delta_1 + \delta_3 + \delta_5, \quad \eta_2 = 1 - \mu_2,\\
&\mu_1 = \delta_0 + \delta_2 + \delta_3 + \delta_6, \quad \mu_1 = \delta_1 + \delta_4 + \delta_5 + \delta_7.
\end{aligned} \tag{13}$$

The Hopf bifurcation in $M_1(m, 0, 0)$

Fist we consider the linear system associated to the differential equations with time delay (11) in $M_1(m, 0, 0)$:

$$\dot{u}(t) = A_1 u(t) + B_1 u(t - \tau) \tag{14}$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{23} \\ 0 & b_{32} & 0 \end{pmatrix}, \tag{15}$$

$u(t) = (u_1(t), u_2(t), u_3(t))^T$ and

$$a_{22} = -\beta_2 a_1^2 m^2, \quad a_{33} = -\beta_3 a_1^2 m^2, \quad b_{23} = m(a_3 - a_1), \quad b_{32} = m(a_1 - a_2).$$

The characteristic equation corresponding to (14) is given by

$$det(\lambda I - A_1 - B_1 e^{-\lambda\tau}) = -\lambda \Delta_1(\lambda, \tau) = 0 \tag{16}$$

where

$$\Delta_1(\lambda, \tau) = \lambda^2 + a\lambda + b + ce^{-2\lambda\tau} \tag{17}$$

and

$$a = (\beta_2 + \beta_3) a_1^2 m^2, \quad b = \beta_2 \beta_3 a_1^4 m^4, \quad c = m^2 (a_1 - a_2)(a_1 - a_3).$$

Proposition 2. *(i) If $\tau = 0$ then the equation $\Delta_1(\lambda, 0) = 0$ has the roots with negative real parts.*

(ii) If $m \in \left(\frac{a_1 - a_3}{\beta_2 a_1^2}, \frac{a_1 - a_2}{\beta_3 a_1^2}\right)$ then the equation $\Delta_1(\lambda, \tau) = 0$ has the roots with negative real parts for any $\tau > 0$.

(iii) If $m \in (0, p)$ then there exists τ_0 and ω_0 such that $\lambda_1 = i\omega_0, \lambda_2 = \bar{\lambda}_1$ are simple roots for $\Delta_1(\lambda, \tau_0) = 0$ and the other roots have negative real part, where

$$p = \frac{1}{a_1} \sqrt{\frac{(a_1 - a_2)(a_1 - a_3)}{\beta_2 \beta_3}}, \quad \tau_0 = \frac{1}{2\omega_0} arctan \frac{a\omega_0}{\omega_0^2 - b} \tag{18}$$

and ω_0 is one positive root of the following equation:

$$\omega^4 + (a^2 - 2b)\omega^2 + b^2 - c^2 = 0.$$

Proposition 3. *For $m_0 \in (0, p)$ and $\tau_0 = \tau_0(\omega_0)$ given by (18), τ is a Hopf bifurcation for system (11).*

Proof. The roots of equation $\Delta_1(\lambda, \tau) = 0$ are continuously dependent on τ so that we consider $\lambda = \lambda(\tau)$. Deriving (21) with respect to τ and computing the derivative in $\tau = \tau_0$, $\lambda_1 = i\omega$, it results that:

$$M = Re\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_0, \lambda=\lambda_1} = \frac{2\omega_0 \tau_0 [a(a + b - 2\omega_0^2) + 2\omega_0(a + 1)(\omega_0^2 - b)]}{(a + b - 2\omega_0^2)^2 + 4\omega_0^2 (a + 1)^2}$$

$$N = Im\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_0,\lambda=\lambda_1} = \frac{2\omega_0\tau_0[(\omega_0^2-b)(a+b-2\omega_0^2)^2-2a(a+1)\omega_0^2]}{(a+b-2\omega_0^2)^2+4\omega_0^2(a+1)^2}$$

Because M is a positive number then τ_0 is a Hopf bifurcation.

Remark 1. *Characteristic equation (16) has one zero eigenvalue corresponding to the eigenvector with the direction along the line of equilibria. Thus the equilibria on this line will not be asymtotically stable. However, applying the results from [4] they will be orbitally asymptotically stable if all the other roots of* $\Delta_1(\lambda,\tau)$ *have negative real parts. Using Propositions 2 and 3 we obtain that if* $\tau \in [0,\tau_o)$ *and* $m \in (0,p)$*, the point* $M_1(m,0,0)$ *is orbitally asymptotically stable.*

The normal form for the system (11)

From (12) and (13) and by using the translation $x^1 = y^1 + m$, $x^2 = y^2$, $x^3 = y^3$ system (11) becomes:

$$\dot{y}(t) = A_1y(t) + B_1y(t-\tau) + F(y(t),y(t-\tau)) + G(y(t),y(t-\tau)) \tag{19}$$

where A_1, B_1 are the matrices given in (19), $y(t) = (y^1(t),y^2(t),y^3(t))^T$, $y(t-\tau) = (y^1(t-\tau),y^2(t-\tau),y^3(t-\tau))^T$, $F(y(t),y(t-\tau)) = (F^1(y(t),y(t-\tau)),F^2(y(t),y(t-\tau)),F^3(y(t),y(t-\tau))$, with

$$\begin{array}{l} F^1(y(t),y(t-\tau)) = \\ d_{23}y^2(t)y^3(t) + e_{23}y^2(t-\tau)y^3(t) + f_{23}y^2(t)y^3(t-\tau) + h_{23}y^2(t-\tau)y^3(t-\tau), \\ F^2(y(t),y(t-\tau)) = \\ d_{13}y^1(t)y^3(t) + e_{13}y^1(t-\tau)y^3(t) + f_{13}y^1(t)y^3(t-\tau) + h_{13}y^1(t-\tau)y^3(t-\tau), \\ F^3(y(t),y(t-\tau)) = \\ d_{13}y^1(t)y^2(t) + e_{12}y^1(t-\tau)y^2(t) + f_{12}y^1(t)y^2(t-\tau) + h_{12}y^1(t-\tau)y^2(t-\tau), \end{array} \tag{20}$$

and

$$\begin{array}{ll} d_{23} = a_2\mu_2\alpha_3 - a_3\mu_3\alpha_2, & e_{23} = a_2\mu_2\alpha_3 - a_3\mu_3\beta_2, \\ f_{23} = a_2\mu_2\beta_3 - a_3\eta_3\alpha_2, & h_{23} = a_2\eta_2\beta_3 - a_3\eta_3\beta_2, \end{array}$$

$$\begin{array}{ll} d_{13} = a_3\mu_3 - a_1\mu_1\alpha_3, & e_{13} = a_3\mu_3 - a_1\mu_1\beta_3, \\ f_{13} = -a_1\mu_1\alpha_3, & h_{13} = -a_1\eta_1\beta_3, \end{array}$$

$$\begin{array}{ll} d_{12} = a_1\mu_1\alpha_2 - a_2\mu_2, & e_{12} = a_1\mu_1\alpha_2, \\ f_{12} = a_1\mu_1\beta_2 - a_2\eta_2, & h_{12} = a_1\eta_1\beta_2. \end{array}$$

The function $G(y(t),y(t-\tau))$ has as components polynomials of third degree in the coordinates of $y(t)$ and $y(t-\tau)$.

Let $\tau = \tau_0 + \varepsilon$ be with $\varepsilon \geq 0$ sufficiently small and the center manifold in τ given by:

$$y(\theta) = z\phi(\theta) + \bar{z}\phi(\theta) + \frac{1}{2}w_{20}(\theta)z^2 + w_{11}(\theta)z\bar{z} + \frac{1}{2}w_{02}(\theta)\bar{z}^2 + \dots \tag{21}$$

where $\theta \in [-\tau, 0]$, $z \in C^2$, $\phi(\theta) = \phi(0)e^{\lambda_1\theta}$, and $\phi(0)$ is the eigenvector of the matrix $A_1 + e^{-\lambda_1\tau_0}B_1$.

By direct computation it results $\phi(0) = (0, v_2, 1)^T$ where

$$v_2 = -\frac{m(a_1 - a_3)e^{-\lambda_1\tau_0}}{\beta_2 a_1^2 m^2 + \lambda_1}.$$

The eigenvector of $\psi(0)$ of the adjunct matrix $A_1 + e^{-\lambda_1\tau_0}B_1$ is $\psi(0) = (0, w_2, w_3)$, where

$$w_2 = \frac{m(a_1 - a_2)e^{-\lambda_2\tau_0}w_3}{\beta_2 a_1^2 m^2 + \lambda_2}w_3.$$

Using the following scalar product :

$$< \psi(s), \phi(s) >= \psi(0)\phi(0) + \int_{-\tau_o}^{0} \psi(s+\tau_0)B_1\phi(s)ds$$

we can determinate w_3 so that $< \psi(s), \phi(s) >=< \bar{\psi}(s), \bar{\phi}(s) >= 1$, $< \psi(s), \bar{\phi}(s) >=< \bar{\psi}(s), \phi(s) >= 0$.

By direct computation it results:

$$w_3 = \frac{1}{\eta}, \eta = 1 - \frac{m^2(a_1 - a_3)(a_1 - a_2)}{\beta_2^2 a_1^4 m^4 + \omega_0^2}(1 + \beta_2 a_1 m^2\tau_0 + \lambda_2\tau_0 + \tau_0(\beta_2 a_1 m^2 + \lambda_2)e^{2\lambda_1\tau_0}).$$

Replacing y with $y(0)$ and $\tilde{y}$ with $y(t-\tau)$ given by (21) in (20) we have:

$$\begin{aligned} F^1(y,\tilde{y}) &= \frac{1}{2}F_{20}^1 z^2 + F_{11}^1 z\bar{z} + \frac{1}{2}F_{02}^1\bar{z}^2 + \frac{1}{2}F_{21}^1 z^2\bar{z} + \dots \\ F^2(y,\tilde{y}) &= \frac{1}{2}F_{21}^2 z^2\bar{z} + \dots \\ F^3(y,\tilde{y}) &= \frac{1}{2}F_{21}^3 z^2\bar{z} + \dots \end{aligned}$$

where

$$F_{21}^1 = 2(d_{23} + (e_{23} + f_{23})e^{-\lambda_1\tau_0} + h_{23}e^{-2\lambda_1\tau_0})v_2$$

$$\begin{aligned} F_{21}^2 &= d_{13}(w_{20}^1(0) + 2w_{11}^1(0)) + e_{13}(w_{20}^1(-\tau_0) + 2w_{11}^1(-\tau_0)) + \\ &+ f_{13}(w_{20}^1(0)e^{\lambda_1\tau_0} + 2w_{11}^1(0)e^{-\lambda_1\tau_0}) + \\ &+ h_{13}(w_{20}^1(-\tau_0)e^{\lambda_1\tau_0} + 2w_{11}^1(-\tau_0)e^{-\lambda_1\tau_0}) \\ F_{21}^3 &= d_{12}(w_{20}^1(0)\bar{v}_2 + 2w_{11}^1(0)v_2) + e_{12}(w_{20}^1(-\tau_0)\bar{v}_2 + w_{11}^1(-\tau_0)v_2) + \\ &+ f_{12}(w_{20}^1(0)e^{\lambda_1\tau_0}\bar{v}_2 + 2w_{11}^1(0)e^{-\lambda_1\tau_0}v_2) + \\ &+ h_{12}(w_{20}^1(-\tau_0)e^{\lambda_1\tau_0}\bar{v}_2 + 2w_{11}^1(-\tau_0)e^{-\lambda_1\tau_0}v_2). \end{aligned}$$

The invariance property of central manifold [4] leads to:

$$w_{20}^1(\theta) = \frac{1}{2\lambda_2}F_{20}^1 e^{2\lambda_2\theta}, w_{11}^1(\theta) = 0, \quad \theta \in [-\tau_0, 0].$$

Proposition 4. *(i). The normal form of the system (19) on the central manifold is:*

$$\dot{z}(t) = \lambda_1 z(t) + \frac{1}{2}g_{21}z(t)^2\bar{z}(t) \tag{22}$$

where

$$g_{21} = w_2 F_{21}^2 + w_3 F_{21}^3$$
$$F_{21}^2 = \frac{1}{2\lambda_2} F_{20}^1 (d_{13} + e_{13} e^{-2\lambda_2 \tau_0} + f_{13} e^{\lambda_1 \tau_0} + h_{13} e^{(\lambda_1 - 2\lambda_2)\tau_0})$$
$$F_{21}^3 = \frac{1}{2\lambda_2} F_{20}^1 \bar{v}_2 (d_{12} + e_{12} e^{-2\lambda_2 \tau_0} + f_{12} e^{\lambda_1 \tau_0} + h_{12} e^{(\lambda_1 - 2\lambda_2)\tau_0}).$$

(ii) The solution of differential system with time delay (11) in the neighborhood of $M_1(m, 0, 0)$ *is:*

$$x^1(t) = m + Re(w_{20}^1(0) z^2(t))$$
$$x^2(t) = 2Re(z(t) v_2)$$
$$x^3(t) = 2Re(z(t)),$$

where $z(t)$ *is a solution of equation (22).*

(iii) The Lyapunov coefficient is $C_1 = \dfrac{g_{21}}{2}$*. The elements which characterize the limit cycle are:*

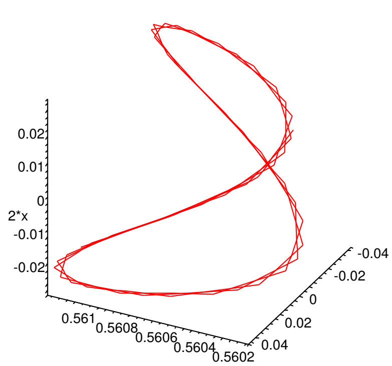
$$\tilde{\mu}_2 = -\frac{Re(C_1)}{Re\left(\dfrac{d\lambda}{d\tau}\right)_{\tau=\tau_0, \lambda=\lambda_1}}, \beta_2 = 2Re(C_1)$$
$$T_2 = -\frac{Im(C_1) + \tilde{\mu}_2 Im\left(\dfrac{d\lambda}{d\tau}\right)_{\tau=\tau_0, \lambda=\lambda_1}}{\omega_0}.$$

(iv) If $\tilde{\mu}_2 > 0 (< 0)$ *then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for* $\tau > \tau_0 (< \tau_0)$*; the solutions are orbitally stable (unstable) if* $\beta_2 < 0 (> 0)$*; the period increases (decreases) if* $T_2 > 0 (< 0)$*.*

For $\varepsilon_0 = 0.2$, $\varepsilon_1 = 0$, $\varepsilon_2 = 0.2$, $\varepsilon_3 = 0.3$, $\varepsilon_4 = 0$, $\varepsilon_5 = 0$, $\varepsilon_6 = 0.3$, $\varepsilon_7 = 0$, $a_1 = 0.6$, $a_2 = 0.4$, $a_0 = 0.2$, $\delta_0 = \dfrac{a_2}{a_1}(\varepsilon_0 + \varepsilon_3)$, $\delta_1 = \dfrac{a_2}{a_3}(\varepsilon_0 + \varepsilon_1) - \dfrac{a_2}{a_1}(\varepsilon_0 + \varepsilon_3)$, $\delta_2 = \delta_3 = \delta_4 = \delta_6 = \delta_7 = 0$, $\delta_5 = 1 - \delta_0 - \delta_1$ it results: $p = 0.860662958$ and $m = 0.66$, $\omega_0 = 0.16557$, $\tau_0 = 2.9090$, $\beta_2 = 1.145$, $\mu_2 = -4.524$, $T_2 = -10.046$. There is a subcritical bifurcation, orbitally unstable solution with decrease period.

The phase plot $(x^1(t), x^2(t), x^3(t))$ is given in fig 1:

Fig1. Phase plot (x(t)^1,x(t)^2,x(t)^3)



The analysis of the system (11) in the neighborhood of the equilibrium points $M_2(0, m, 0)$ and $M_3(0, 0, m)$ will be given in the next papers.

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