

Fluid flow versus Geometric Dynamics

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Abstract

This paper investigates the way in which geometric dynamics on Riemannian manifolds can be applied to fluid flow. The equations of motion from fluid mechanics (momentum equations), expressed on general curvilinear coordinates, are compared with the equations of motion in a gyroscopic field of forces. Attempts are made to seek conditions under which both describe the same motion, especially in the case when trajectories are not field lines. It is proved that all pathlines and streamlines are geodesics in the least square sense and, in some circumstances, other trajectories included in geometric dynamics have physical support.

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1 Deterministic description of fluid flow

The basic mathematical point of view of any fluid flow is that it can be described by a *point transformation*, namely, a family of C^∞ transformations of coordinates indexed after the time parameter t [1, Aris 1989]. Accepting the idea that we can point out one individual particle of the flow at a certain moment of time, $t = 0$, we are able to define its initial position as $\xi (\xi^1, \xi^2, \xi^3)$. Later, at the moment t , we will find the same particle at position

$$(1.1) \quad x = \varphi(\xi, t).$$

The initial coordinates (ξ^1, ξ^2, ξ^3) of the particle will be referred as *material coordinates* and, when convenient, the particle itself may be called particle ξ . Convected coordinates or Lagrangian coordinates are also used to denote the particle. In the same coordinate system, the *spatial coordinates* (x^1, x^2, x^3) can be referred to as particle's *position* or *place*. In the case of a deterministic description of a continuous movement of the particle, it is accepted that the function (1.1) is a bijective application (change of coordinates). Physically this means that a continuous arc of particles does not break up during the motion and that, for short time, particles in the neighborhood

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of a given particle continue to stay in this neighborhood during the motion. Inverting the function (1.1) we obtain

$$\xi = \varphi^{-1}(x, t).$$

The deterministic character of the motion can be associated to the single valuedness property of the function $\xi \mapsto x$, which means that a particle cannot split up and cannot occupy two places at the same time nor can two distinct particles occupy the same place at same instant. Though practically we use derivatives of order one, two and three, we accept the class C^∞ . Exceptions may be accepted on a finite number of singular surfaces, lines or points. The necessary and sufficient mathematical condition in order to locally invert the function (1.1) is that the Jacobian satisfies the condition

$$J = \frac{Dx}{D\xi} \neq 0.$$

The partial function $t \mapsto \varphi(\xi, t)$ may be looked at as the parametric equation of a curve (particle path or trajectory) in R^3 , having t as parameter. The curve goes through point ξ at the initial moment $t = 0$. Any *material property* of the fluid may be followed along particle's path. For example: the *density* $\rho = \rho(\xi, t)$ in the neighborhood of a particle (the density, as seen by an observer riding on the particle, is a function of time), the position $x = \varphi(\xi, t)$ etc. Generally, the material description of any property can be represented by a function $F(\xi, t)$. It can be changed into a *spatial description* $f(x, t)$ via the composition

$$(1.2) \quad F(\xi, t) = F(\varphi^{-1}(\xi, t), t) = f(x, t).$$

The physical interpretation of relation (1.2) points out that the value of the property at position x and time t is the value characteristic to the particle ξ that occupies the position x at time t . The spatial description,

$$(1.3) \quad f(x, t) = f(\varphi(\xi, t), t) = F(\xi, t),$$

means that the value as seen by the particle at time t is the value at the position it occupies at that time. Associated with these two descriptions there are two derivatives with respect to time: partial derivative $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t}\right)_{x=ct}$ and material derivative $\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_{\xi=ct}$. Thus, $\frac{\partial f}{\partial t}$ is the rate of change of property $f(x, t)$ at a fixed point x , whereas $\frac{dF}{dt}$ is the rate of change of $F(\xi, t)$ as observed when moving with the particle ξ (that is at constant ξ). The latter is also known as *convected derivative*. Hence, the material velocity of the particle along its path is given by

$$(1.4) \quad V(\xi, t) = \left(\frac{\partial x}{\partial t}\right)_{\xi=ct} = \frac{dx}{dt} = \frac{\partial \varphi}{\partial t} \Big|_{\xi=ct} = \dot{\varphi}(\xi, t) = v(x, t).$$

Combining the previous relation with definition (1.3) allows us to establish a connection between the material derivative and the spatial derivative

$$(1.5) \quad \frac{dF}{dt}(\xi, t) = \frac{\partial F}{\partial t}(\xi, t) \Big|_{\xi=ct} = \frac{\partial f}{\partial t}(x, t) \Big|_{\xi=ct} = \frac{\partial f}{\partial x^i} \left(\frac{\partial x^i}{\partial t}\right)_{\xi=ct} + \frac{\partial f}{\partial t} \Big|_{x=ct}$$

$$= \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial t} \Big|_{x=ct}$$

or

$$(1.6) \quad \frac{dF}{dt} = \frac{\partial f}{\partial x^i} \dot{\varphi}^i + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} + (\dot{\varphi} \cdot \nabla) f = \frac{df}{dt}.$$

The spatial velocity field is defined by $v(x, t) = V(\xi, t)$, where ξ and x are related by the relation (1.1). The acceleration in the particle coordinate system is

$$(1.7) \quad A(\xi, t) = \frac{dV}{dt} = \frac{dv}{dt} = \frac{\partial v}{\partial x^i} \dot{\varphi}^i + \frac{\partial v}{\partial t} = \frac{\partial v}{\partial t} + (\dot{\varphi} \cdot \nabla) v.$$

The relation (1.6) points out the connection between the Lagrangian description and the Eulerian description of the flow. On the other hand, it can be used to write

$$(1.8) \quad \frac{dv}{dt} = \frac{\partial v}{\partial t} + (\dot{\varphi} \cdot \nabla) v,$$

which, along the trajectory, becomes $\frac{dv}{dt} = \frac{\partial v}{\partial t} + (v \cdot \nabla) v$ since $\dot{\varphi}(\xi, t) = v(x, t)$.

2 Arnold's point of view regarding incompressible fluids

Arnold [3, 1966], [2, 1969] showed that the Euler equations for an incompressible fluid could be given a Lagrangian and Hamiltonian description similar to that of rigid-body. For ideal fluids, the configuration space G (infinite dimensional) is the group $Diff_{vol}(\Omega)$ of volume preserving transformations of the fluid control volume (a region in R^2 or R^3 or a Riemannian manifold in general, possible with boundary). Group multiplication in G is the composition.

The reason we select $G = Diff_{vol}(\Omega)$ as the configuration space is similar to that for a rigid body; namely each element of G is a mapping of Ω to Ω that takes a reference point ξ to a current point $x = x(\xi) \in \Omega$; thus, knowing x tells us where each particle of fluid goes and hence gives us the *fluid configuration*. We ask that x be a diffeomorphism to exclude discontinuities, cavitation, and fluid interpenetration, and we ask that x be volume preserving to correspond to the assumption of incompressibility.

A *motion* of a fluid is a family of time-dependent elements of G , which we write as $x = \varphi(\xi, t)$. The material velocity field is defined by (1.4). If we suppress t in (1.1) and write $\dot{\varphi}$ for V , we observe that

$$(2.1) \quad v = \dot{\varphi} \circ \varphi^{-1}, \text{ i.e., } v_t = V_t \circ \varphi_t^{-1},$$

where $\varphi_t(\xi) = \varphi(\xi, t)$.

We can regard (2.1) as a map from the space of $(\varphi, \dot{\varphi})$ (*material or Lagrangian description*) to the space of $v \equiv (x, v)$ (*spatial or Eulerian description*). Like the rigid-body, the material to spatial map (2.2) takes the canonical bracket to a Lie Poisson bracket; one of our goals is to understand this reduction. Notice that if we replace φ by $\varphi \circ \eta$ for a fixed (time-dependent) $\eta \in Diff_{vol}(\Omega)$, then $\dot{\varphi} \circ \varphi^{-1}$ is independent of η ; this reflects the *right invariance* of the Eulerian description (v is invariant under

composition of φ by η on the right). This is also called the *particle relabeling symmetry* of fluid dynamics). The spaces TG and T^*G represent the Lagrangian (material) description, and we pass to the Eulerian (spatial) description by right translations and use the (+)Lie-Poisson bracket. One of the things we want to do later is to better understand the reason for the switch between right and left in going from the rigid-body to fluids.

Dynamics of a Fluid. The *Euler equations* for an ideal, incompressible, homogeneous fluid moving in the region Ω are

$$(2.2) \quad \frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\frac{\nabla p}{\rho},$$

with the constraint $\operatorname{div} v = 0$ and the boundary condition that the velocity vector field v is tangent to the boundary $\partial\Omega$.

The *pressure* p is determined implicitly by the divergence-free (volume - preserving) constraint (see [7, Chorin and Marsden 2000] for basic information on the derivation of Euler's equations). The associated Lie algebra \mathfrak{g} is the space of all solenoidal (divergence-free) vector fields tangent to the boundary. This Lie algebra is endowed with the negative Jacobi-Lie bracket of vector fields given by

$$[v, w]_L^i = \sum_{j=1}^n \left(w^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial w^i}{\partial x^j} \right).$$

The subscript L on the bracket refers to the fact that it is the *left Lie algebra* bracket on \mathfrak{g} . The most common convention for the Jacobi-Lie bracket of vector fields, also the one we adopt, has the opposite sign. Also we identify \mathfrak{g} and \mathfrak{g}^* using the pairing

$$\langle v, w \rangle = \int_{\Omega} v \cdot w \, d^3x.$$

Hamiltonian Structure. We introduce the (+)Lie-Poisson bracket, called the *ideal fluid bracket*, on functions of velocity v by

$$\{F, G\}(v) = \int_{\Omega} v \cdot \left[\frac{\delta F}{\delta v}, \frac{\delta G}{\delta v} \right]_L \, d^3x,$$

where $\frac{\delta F}{\delta v}$ is defined by

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [F(v + \varepsilon \delta v) - F(v)] = \int_{\Omega} \delta v \cdot \frac{\delta F}{\delta v} \, d^3x.$$

With the energy function chosen to be the kinetic energy

$$H(v) = \frac{1}{2} \int_{\Omega} \|v\|^2 \, d^3x,$$

one can verify that the Euler equations (2.2) are equivalent to the Poisson bracket equations

$$\dot{F} = \{F, H\},$$

for all functions F on g . To see this, it is convenient to use the orthogonal decomposition $w = Pw + \nabla p$ of a vector field w into a divergence-free part Pw in g and a gradient. Then the Euler equations can be written

$$\frac{\partial v}{\partial t} + P(v \cdot \nabla v) = 0.$$

One can express the Hamiltonian structure in terms of the velocity as a basic dynamic variable and show that the preservation of coadjoint orbits amounts to Kelvin's circulation theorem. Marsden and Weinstein [13, 1983] show that the Hamiltonian structure in terms of Clebsch potentials fits naturally into this Lie-Poisson scheme, and that Kirchhoff's Hamiltonian description of point vortex dynamics, vortex filaments, and vortex patches can be derived in a natural way from the Hamiltonian structure described above.

Lagrangian Structure. The general framework of the Euler-Poincaré and the Lie-Poisson equations gives other insights as well. For example, this general theory shows that the Euler equations are derivable from the "variational principle"

$$\delta \int_a^b \int_{\Omega} \frac{1}{2} \|v\|^2 d^3x = 0,$$

which is to hold for all variations δv of the form $\delta v = \dot{u} + [v, u]_L$ (sometimes called *Lin constraints*, [5]), where u is a vector field (representing the infinitesimal displacement of the particle) vanishing at the temporal endpoints (this Lagrange-d'Alembert principle is due to Newcomb).

There are important functional-analytic differences between working in material representation (that is, on T^*G) and in Eulerian representation (that is, on g^*) that are important for proving existence and uniqueness theorems, theorems on the limit of zero viscosity, and the convergence of numerical algorithms (see [9, Ebin and Marsden 1970], [6, Chorin, Hughes, Marsden, and McCracken 1978]). Finally, we note that for *two-dimensional flows*, a collection of Casimir functions is given by

$$(2.3) \quad C(\omega) = \int_{\Omega} \Phi(\omega(x)) d^2x,$$

for $\Phi : R \rightarrow R$, any smooth function and $\omega k = \nabla \times v$, the vorticity. For three dimensional flows the function (2.3) is no longer a Casimir function.

3 Pathlines and streamlines in fluid dynamics

Suppose $X(x, t)$ is a given velocity field for a specific fluid flow in a domain Ω of R^2 or R^3 . First we need to select the evolution parameter and to introduce the evolution ODE. Second, we need the initial position of the particle, leading us to a Cauchy problem. From the point of view of fluid mechanics, two situations are to consider regarding kinematics of a material particle: pathline and streamline.

Pathlines are solutions of the Cauchy problems

$$\frac{dx}{dt} = X(x(t), t), \quad x(0) = x_0$$

and streamlines are solutions of different Cauchy problems

$$(3.1) \quad \frac{dx}{ds} = X(x(s), t), \quad x(0) = x_0.$$

In the second case, s is the parameter along the streamline, different from time parameter t which is held constant while the equations (3.1) are integrated. The parameter t can produce bifurcation in the equilibrium set or Hopf bifurcation of the flow. When the field X is autonomous, the pathlines coincide with streamlines.

4 Geometric dynamics derived from a velocity field in Riemannian space

Suppose $X(x, t)$ is a given contravariant vector field on a Riemannian manifold (M, g) , depending on the time parameter t , and

$$(4.1) \quad \frac{dx}{dt} = X(x(t), t)$$

is the associated first order nonautonomous differential system (pathlines). In order to obtain the prolongation by derivation of the system (4.1), we only have to take the temporal derivative (along a solution) of both members. The derivative of the left-hand side yields the contravariant components of the acceleration of the system

$$(4.2) \quad A^i = \frac{d\dot{x}^i}{dt} + \Gamma_{jk}^i \dot{x}^k \dot{x}^j,$$

where Γ_{jk}^i are the second order Christoffel symbols produced by the Riemannian metric g . The covariant differentiation of the right-hand side of 4.1 reads

$$\frac{\delta X^i}{\delta t} = \frac{\partial X^i}{\partial t} + X_{,j}^i \frac{dx^j}{dt} = \frac{\partial X^i}{\partial t} + \left(\frac{\partial X^i}{\partial x^j} + \Gamma_{jk}^i X^k \right) \frac{dx^j}{dt}$$

If we add and subtract the Riemannian adjoint tensor $g^{ih} g_{kj} X_{,h}^k$ of $X_{,j}^i$ in the previous relations, we obtain

$$(4.3) \quad \frac{\delta X^i}{\delta t} = \frac{\partial X^i}{\partial t} + (X_{,j}^i - g^{ih} g_{kj} X_{,h}^k) \frac{dx^j}{dt} + g^{ih} g_{kj} X_{,h}^k \frac{dx^j}{dt}.$$

The skew-symmetric (with respect to the Riemannian metric) coefficient

$$F_j^i = X_{,j}^i - g^{ih} g_{kj} X_{,h}^k$$

is an external tensor that characterizes the helicity of the vector field X . If we replace in the last term of (4.3), the derivative of the position vector by the corresponding velocity field component and the left-hand side with the acceleration given by (4.2), we obtain the prolongation specific to geometric dynamics formalism ([16, Udriste 2000a]- [24, waf2005])

$$(4.4) \quad \ddot{x}^i + \Gamma_{jk}^i \dot{x}^k \dot{x}^j = \frac{\partial X^i}{\partial t} + F_j^i \dot{x}^j + g^{ih} g_{kj} X_{,h}^k X^j.$$

The second order ODE (4.4) represents an Euler-Lagrange prolongation of first order ODE (4.1), containing three force fields: one gyroscopic field of forces, represented by $F_j^i \dot{x}^j$, one conservative field of forces

$$(4.5) \quad g^{ih} \frac{\partial f}{\partial x^h} = g^{ih} g_{kj} X_{,h}^k X^j,$$

and finally, the field of temporal variation $\frac{\partial X^i}{\partial t}$. The function $f = \frac{1}{2}g(X, X)$ is the potential energy density associated with X and with Riemannian structure g . The Lagrangian L producing (4.4) as Euler-Lagrange equations, i.e.,

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0,$$

is given by

$$L = \frac{1}{2}g(\dot{x} - X, \dot{x} - X),$$

i.e., it is a least squares Lagrangian. The equations (4.4) show that the field lines (4.1) are geodesics in a suitable *structure*, and that the set of all such geodesics is larger than the set of pathlines.

The Cauchy problem in the case of geometric dynamics equations, brings out two situations. One case is illustrated by conditions of type

$$x(0) = x_0, \quad \left. \frac{dx}{dt} \right|_{t=0} = X(x(0), 0),$$

that will describe nothing else but a pathline. The second, is a totally different situation,

$$(4.6) \quad x(0) = x_0, \quad \left. \frac{dx}{dt} \right|_{t=0} = Y(x(0), 0),$$

where Y represents some arbitrary initial conditions different from the values of the velocity field X , at first instant. We call this situation an *injection* with respect to the flow generated by X .

Now, let us change the previous point of view starting with the system which describes the streamlines,

$$\frac{dx}{ds} = X(x(s), t).$$

In this sense the evolution parameter s is different from the time parameter t . We can repeat the previous ideas to produce a geometric dynamics with evolution parameter s , but with forces depending on the time-parameter t .

5 Dynamics derived from a force field in Riemannian space

The purpose of dynamics is to assess the motion of a system under the influence of forces acting on it. Ingredients:

x = position, t = time, v = velocity, a = acceleration,
 m = mass, F = force field, α = vector field parameter

The trajectories of the system are generated by solving the second order differential system of equations,

$$a = \frac{F}{m} = X,$$

once the force field $F(x, t)$ or $F(x, v, t)$ or $F(x, v, \alpha, t)$ is known. In the general case we have $F(x(t), v(x, t), \alpha(x, t), t)$.

Following this idea, we can imagine that the force field comes from a generalized potential $U = U(x(t), \dot{x}(t), t)$. In this case, the Lagrangian has the following general form

$$L(t, x(t), \dot{x}(t)) = \frac{1}{2}m \|\dot{x}(t)\|^2 - U(x(t), \dot{x}(t), t).$$

Apart from Euler-Lagrange equations, the Hamilton principle regarding the stationarity of the action functional, yields the definition of the functional derivative δ_f of any scalar field Λ ([11, Gottlieb et. al. 1997])

$$\frac{\delta_f \Lambda}{\delta_f x^i} = \frac{\partial \Lambda}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial \Lambda}{\partial \dot{x}^i} \right).$$

Consequently, the Euler-Lagrange equations can be written $\frac{\delta_f L}{\delta_f x^i} = 0$ and the force field is defined by $\delta_f U / \delta_f x = -F$. This definition produces the covariant components

$$F_i(t, x(t), \dot{x}(t)) = -\frac{\partial U}{\partial x^i} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{x}^i} \right).$$

Thus, the covariant components of the force field are

$$F_i(t, x(t), \dot{x}(t)) = -\frac{\partial U}{\partial x^i} + \frac{\partial^2 U}{\partial t \partial \dot{x}^i} + \dot{x}^k \frac{\partial^2 U}{\partial x^k \partial \dot{x}^i} + \ddot{x}^k \frac{\partial^2 U}{\partial \dot{x}^k \partial \dot{x}^i}.$$

The natural hypothesis that F does not depend on the acceleration $\ddot{x}(t)$ is equivalent to $\frac{\partial^2 U}{\partial \dot{x}^k \partial \dot{x}^i} = 0$, i.e., $U(t, x(t), \dot{x}(t)) = W(t, x(t)) + \dot{x}^i A_i(t, x(t))$ and hence

$$F_i = -\frac{\partial W}{\partial x^i} + \frac{\partial A_i}{\partial t} + \dot{x}^k \left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \right).$$

Here W is the so called *simple potential* and A is the *potential vector* associated with W . The components of the force field generated by the generalized potential U contains a conservative part, a temporal variation of the potential vector field and a gyroscopic force field, similar to force field generated by the velocity field (4.4)-(4.5).

The previous explanations shows that the Euler-Lagrange equations

$$\frac{\delta_f L}{\delta_f x^i} = 0$$

differ from those in geometric dynamics generated by the vector field (flow) A only by gradient terms.

6 Leap from fluid flow to geometric dynamics

Let us consider an alternative to (1.8), the so called Helmholtz representation of fluid flow ([10, Florea and Panaitescu 1979])

$$(6.1) \quad \frac{\partial v}{\partial t} + \nabla \left(\frac{v \cdot v}{2} \right) + (\nabla \times v) \times v = f - \frac{\nabla p}{\rho},$$

where $v(x, t)$ is the velocity field, p is the pressure field, f an external force field and ρ is the constant field of density. Usually, equation (6.1) represents the main tool in finding the velocity field in a given domain Ω in R^2 or R^3 . The left hand side of it comprises the acceleration field composed by a local field $\partial v / \partial t$ and the transport or convective field, the remaining two terms. Simple vector algebra shows that $(v \cdot \nabla) v$ from (1.8) transforms accordingly in the convective part of the acceleration field, such that

$$(6.2) \quad \frac{dv}{dt} = \frac{\partial v}{\partial t} + (v \cdot \nabla) v = \frac{\partial v}{\partial t} + \nabla \left(\frac{v \cdot v}{2} \right) + (\nabla \times v) \times v,$$

represents nothing but a series of identities. We would like to transform the identity (6.2) in a differential equation expressing a dynamic. Taking into account the way vector operators gradient and curl act, the only possibility resides in replacing the material property velocity, in Lagrange representation, $v = \dot{x}$, as follows

$$(6.3) \quad \frac{d\dot{x}}{dt} = \frac{\partial v}{\partial t} + \nabla \left(\frac{v \cdot v}{2} \right) + (\nabla \times v) \times \dot{x}.$$

Relation (6.3) is the equivalent of geometric dynamics (4.4) if we identify the contravariant components of the acceleration on a Riemannian manifold, $d\dot{x}/dt$, with the left hand side of (4.4),

$$\frac{d\dot{x}^i}{dt} = \ddot{x}^i + \Gamma_{jk}^i \dot{x}^k \dot{x}^j.$$

The contravariant components of the gradient are

$$\nabla \left(\frac{v \cdot v}{2} \right) : \frac{1}{2} g^{ij} (g_{lm} v^l v^m)_{,j} = g^{ij} g_{lm} v_{,j}^l v^m$$

similar to the last term of (4.4). The outer product of the curl of the velocity field and \dot{x} can be written in tensorial notation as

$$(\nabla \times v) \times \dot{x} : \varepsilon_{ijk} (\varepsilon^{jlm} g_{lp} v_{,m}^p) \dot{x}^k = (\delta_k^l \delta_i^m - \delta_i^l \delta_k^m) g_{lp} v_{,m}^p \dot{x}^k,$$

or more

$$(\nabla \times v) \times \dot{x} : (\delta_k^l \delta_i^m - \delta_i^l \delta_k^m) g_{lp} v_{,m}^p \dot{x}^k = (g_{kp} v_{,i}^p - g_{ip} v_{,k}^p) \dot{x}^k.$$

Its contravariant components are expressed as

$$(\nabla \times v) \times \dot{x} : g^{jk} (g_{kp} v_{,i}^p - g_{ip} v_{,k}^p) \dot{x}^k = (v_{,i}^j - g^{jk} g_{ip} v_{,k}^p) \dot{x}^k.$$

One can easily identify the last expression with the *gyroscopic force* $F_j^i \dot{x}^j$, with F_j^i given by (4.5) and $X^j = v^j$. If we reassemble equation (6.2) by contravariant components we get the equations of geometric dynamics

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^k \dot{x}^j = \frac{\partial v^i}{\partial t} + F_j^i \dot{x}^j + g^{ih} g_{kj} v_{,h}^k v^j.$$

We conclude that solutions of (6.3) are geodesics in the least-square sense as described in §3.

7 Comparison between discrete-phase model and geometric dynamics

The solution of (4.4) with initial conditions (4.6) is a trajectory that is not a pathline. We wonder if this situation may have any physical support. One possible answer lies in analyzing a discrete-phase flow, meaning the injection of a different material particle at some fixed point in the basic flow-field. The physics of discrete-phase flow represents one of the most challenging issues for scientists. The deterministic approach of this type of flows takes into account the interaction between the basic fluid flow and the discrete-phase through several forces like drag, virtual mass, Bassett or external forces ([12, Kuan 1986]). In this case, the particle follows deterministic trajectory found via the velocity v_p of the particle along its path which is a solution of the Lagrangian equations of motion

$$(7.1) \quad \frac{dv_p}{dt} = C_D \frac{\rho}{\rho_p} |v - v_p| (v - v_p) + \frac{1}{2} \frac{\rho}{\rho_p} \frac{d(v - v_p)}{dt} - \frac{\nabla p}{\rho_p} + f_B + f_c,$$

where v is the velocity of basic flow, C_D is the drag coefficient, ρ the density of the fluid, ρ_p the density of the particle injected, p pressure, f_B is the volumic Bassett force accounting for the deviation of the flow from the steady flow pattern around a sphere. Its particular expression may be found in [12, Kuan 1986] and f_c other volumic external forces e.g., gravity. In the case of a perfect fluid and for negligible diameter of the particle, the drag coefficient vanishes, likewise Bassett force and other external forces, such that (7.1) becomes

$$(7.2) \quad \frac{dv_p}{dt} = \frac{1}{2} \frac{\rho}{\rho_p} \frac{d(v - v_p)}{dt} - \frac{\nabla p}{\rho_p}.$$

Assuming the knowledge of basic flow and that its pattern in the neighborhood of the discrete-phase injected is not affected by this injection, it is much simpler to solve the motion of the injected particle following the geometric dynamics equations rather than (7.2).

8 Application of Udriste dynamics to the potential flow around a circle

In this case the Cartesian components of the velocity field are

$$\frac{dx}{dt} = u = -U_\infty \left[1 - a^2 \frac{x^2 - y^2}{(x^2 + y^2)^2} \right], \quad \frac{dy}{dt} = v = 2U_\infty a^2 \frac{xy}{(x^2 + y^2)^2},$$

where a is the circle radius. The equations of Udriste geometric dynamics ([15], [16]) are

$$(8.1) \quad \frac{d^2x}{dt^2} = -U_\infty^2 a^2 \frac{x(-x^2 + a^2 + 3y^2)}{(x^2 + y^2)^4}, \quad \frac{d^2y}{dt^2} = -U_\infty^2 a^2 \frac{y(-3x^2 + a^2 + y^2)}{(x^2 + y^2)^4}.$$

Considering $a = 0.2m$ and $U_\infty = 1m/s$ let us approximate the trajectories for the following initial conditions:

- a) $x_0 = 0.5m, y_0 = 0.1m, u_0 = -0.858m/s, v_0 = 0.0591m/s$, streamline
- b) $x_0 = 0.5m, y_0 = 0.1m, u_0 = -0.858m/s, v_0 = 0.0m/s$, injection
- c) $x_0 = 0.5m, y_0 = 0.1m, u_0 = -0.858m/s, v_0 = -0.0591m/s$, injection
- d) $x_0 = 0.5m, y_0 = 0.1m, u_0 = -0.5m/s, v_0 = 0.0591m/s$, injection
- e) $x_0 = 0.5m, y_0 = 0.1m, u_0 = -0.1m/s, v_0 = 0.0591m/s$, injection
- f) $x_0 = 0.5m, y_0 = 0.1m, u_0 = 0.0m/s, v_0 = 0.0m/s$, injection
- g) $x_0 = 0.5m, y_0 = 0.1m, u_0 = -0.84m/s, v_0 = 0.0m/s$, streamline at stagnation point
- h) $x_0 = 0.193m, y_0 = 0.051m, u_0 = -0.134m/s, v_0 = 0.5m/s$, streamline on the circle

The results of the numerical integration of (8.1) are given in figures 1, 2 and 3.

Fig. 1.

Fig. 2.

Fig. 3.

9 Conclusions

We have investigated the basic equations of fluid flow and proved that streamlines and, in some cases, pathlines are the solutions of the equations of geometric dynamics. Work has further to be done when considering the interaction between the basic flow field and particle's own flow pattern.

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