An adapted connection on a strict complex contact manifold

Dorel Fetcu

Abstract

We find a class of connections on a strict complex almost contact manifold, which we call adapted connections, and, by using them, we prove some results about the normality of such a manifold.

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1 Introduction

Concerning the complex contact manifold we shall recall some notions as they are presented in [1].

A complex contact manifold is a complex manifold of odd complex dimension 2n+1 together with an open covering $\{\mathcal{O}_\alpha\}$ by coordinate neighborhoods such that:

1. On each $\{\mathcal{O}_\alpha\}$ there is a holomorphic 1-form θ_α such that

$$\theta_\alpha \wedge (d\theta_\alpha)^n \neq 0;$$

2. On $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \varnothing$ there is a non-vanishing holomorphic function $f_{\alpha\beta}$ such that $\theta_\alpha = f_{\alpha\beta}\theta_\beta$.

A complex contact manifold with a global complex form is called strict complex contact manifold.

On the other hand if M is a complex manifold with almost complex structure J, Hermitian metric g and open covering $\{\mathcal{O}_\alpha\}$ by coordinate neighborhoods, M is called a complex almost contact metric manifold if it satisfies the following two conditions:

1. On each $\{\mathcal{O}_\alpha\}$ there exist 1-forms u_α and $v_\alpha = u_\alpha \circ J$ with orthogonal dual vector fields U_α and $V_\alpha = -JU_\alpha$ and (1,1) tensor fields G_α and $H_\alpha = G_\alpha J$ such that

$$G_\alpha^2 = H_\alpha^2 = -I + u_\alpha \otimes U_\alpha + v_\alpha \otimes V_\alpha,$$

$$G_\alpha J = -JG_\alpha, \quad G_\alpha U_\alpha = 0, \quad g(X, G_\alpha Y) = -g(G_\alpha X, Y),$$

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2. On $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \varnothing$,

$$u_\beta = au_\alpha - bv_\alpha, \quad v_\beta = bu_\alpha + av_\alpha,$$

$$G_\beta = aG_\alpha - bH_\alpha, \quad v_\beta = bG_\alpha + aH_\alpha,$$

where a and b are functions with $a^2 + b^2 = 1$. We call a complex almost contact manifold with global structure tensors a strict complex almost contact manifold.

It is proved that a complex contact manifold admits a complex almost contact metric structure for which the local contact form θ is of the form $u - iv$ and the local tensor fields G and H are related to du and dv by

$$du(X,Y) = g(X,GY) + (\sigma \wedge v)(X,Y),$$

$$dv(X,Y) = g(X,HY) - (\sigma \wedge u)(X,Y),$$

where $\sigma(X) = g(\nabla_X U, V)$. We call a complex contact manifold with a complex almost contact metric structure satisfying these conditions, a complex contact metric manifold. Note that if the complex contact structure is strict then $\sigma = 0$.

Let S_1 and S_2 be two tensor fields defined by

$$\begin{aligned} S_1(X,Y) &= N_G(X,Y) + 2g(X,GY)U - 2g(X,HY)V + 2(v(Y)HX - v(X)HY) \\ &\quad +\sigma(GY)HX - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX, \\ S_2(X,Y) &= N_H(X,Y) - 2g(X,GY)U + 2g(X,HY)V + 2(u(Y)GX - u(X)GY) \\ &\quad +\sigma(HX)GY - \sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX, \end{aligned}$$

where N_G and N_H are the Nijenhuis tensor fields of G and H, respectively.

A complex contact metric structure is normal if

$$S_1(X,Y) = S_2(X,Y) = 0, \quad X, Y \in \mathcal{H},$$

$$S_1(U,X) = S_2(V,X) = 0, \quad X \in \chi(M),$$

where $\mathcal{H}$ is the subbundle named the horizontal subbundle and defined by the subspaces $\{X \in T_P\mathcal{O}_\alpha, P \in M; \theta_\alpha(X) = 0\}$.

A normal complex contact metric manifold whose complex contact structure is given by a global complex contact form is called a complex Sasakian manifold.

For a complex Sasakian manifold we have the following formulas, (see [3]),

$$\begin{aligned} g((\nabla_X G)Y,Z) &= -2v(X)g(HGY,Z) - u(Y)g(X,Z) - \\ &\quad -v(Y)g(JX,Z) + u(Z)g(X,Y) + v(Z)g(JX,Y), \end{aligned} \tag{1.1}$$

$$\begin{aligned} g((\nabla_X H)Y,Z) &= -2u(X)g(HGY,Z) + u(Y)g(JX,Z) - \\ &\quad -v(Y)g(X,Z) + u(Z)g(X,JY) + v(Z)g(X,Y), \end{aligned} \tag{1.2}$$

and

$$g((\nabla_X J)Y,Z) = -2u(X)g(HY,Z) + 2v(X)g(GY,Z) \tag{1.3}$$

2 Adapted connections on a strict complex almost contact manifold

In [4], [6] and [5] the authors introduced a class of adapted connections on an almost contact manifold. In this paper we define, in a similar way, a class of adapted connections on a strict complex almost contact manifold.

Let (M, J, G, H, U, V, u, v) be a strict complex almost contact manifold and let us denote by $p = I - u \otimes U - v \otimes V$ and $q = u \otimes U + v \otimes V$ the projectors on the distributions $\mathcal{H}$ and $\mathcal{V} = span\{U, V\}$, respectively. It is easy to see that

$$(2.1) \qquad \begin{cases} p^2 = p, q^2 = q, pq = qp = 0, \\ G^2 = H^2 = -p, pG = Gp = G, pH = Hp = H, \\ qG = Gq = 0, qH = Hq = 0, \\ GH = -HG = -pJ \end{cases} .$$

Definition 2.1. We call an affine connection on the strict complex manifold M an adapted connection if

$$(2.2) \qquad \begin{cases} (\nabla_X G)Y = -2v(X)pJY - u(Y)pX - v(Y)pJX + \\ \qquad + \frac{1}{2}[du(GX, pY) - du(X, GY)]U - \\ \qquad - \frac{1}{2}[dv(GX, pY) + dv(X, GY)]V, \\ (\nabla_X H)Y = 2u(X)pJY + u(Y)pJX - v(Y)pX - \\ \qquad - \frac{1}{2}[du(HX, pY) - du(X, HY)]U + \\ \qquad + \frac{1}{2}[dv(HX, pY) - dv(X, HY)]V, \\ (\nabla_X u)Y = \frac{1}{2}[du(X, Y) + du(GX, GY)], \\ (\nabla_X v)Y = \frac{1}{2}[dv(X, Y) + dv(HX, HY)], \\ \nabla_X U = -GX - \frac{1}{2}du(X, U)U - \frac{1}{2}dv(X, U)V, \\ \nabla_X V = -HX - \frac{1}{2}du(X, V)U - \frac{1}{2}dv(X, V)V, \end{cases}$$

for any $X, Y \in \chi(M)$.

Note that on a complex Sasakian manifold the Levi-Civita connection is an adapted connection.

Remark 2.1. If ∇ is an adapted connection on M we have

$$(\nabla_X J)Y = 2v(X)GY - 2u(X)HY +$$

$$+ \frac{1}{2}[dv(X, Y) + dv(HX, HY) - du(GX, HY) - du(X, JY)]U -$$

$$-\frac{1}{2}[du(X,Y)+du(GX,GY)-dv(GX,HY)+dv(X,JX)]V,$$

for any $X, Y \in \chi(M)$.

In order to prove the existence of the adapted connections on a strict complex almost contact manifold (M, J, G, H, U, V, u, v) let us define first the following tensor fields of type $(2, 2)$ on M

$$\begin{cases} \phi^G = \frac{1}{2}(I\otimes I - G\otimes G),\ \Psi^G = \frac{1}{2}(I\otimes I + G\otimes G), \\ \phi^H = \frac{1}{2}(I\otimes I - H\otimes H),\ \Psi^H = \frac{1}{2}(I\otimes I + H\otimes H), \\ \Theta = \frac{1}{2}(I\otimes I - p\otimes p). \end{cases} \tag{2.3}$$

Just like in the real case (see [6]), it is easy to prove that

$$\begin{cases} \phi^G + \Psi^G = I\otimes I,\ (\phi^G)^2 = \phi^G - \frac{1}{2}\Theta, (\Psi^G)^2 = \Psi^G - \frac{1}{2}\Theta, \\ \phi^G\Psi^G = \Psi^G\phi^G = \phi^G\Theta = \Theta\phi^G = \Psi^G\Theta = \Theta\Psi^G = \Theta^2 = \frac{1}{2}\Theta, \\ (\Psi^G + \Theta) + (\phi^G - \Theta) = I\otimes I, \\ (\Psi^G + \Theta)(\phi^G - \Theta) = (\phi^G - \Theta)(\Psi^G + \Theta) = 0, \\ (\Psi^G + \Theta)(\Psi^G + \Theta) = \Psi^G + \Theta,\ (\phi^G - \Theta)(\phi^G - \Theta) = \phi^G - \Theta, \end{cases} \tag{2.4}$$

and that the similar equations holds for ϕ^H and Ψ^H. Note that the previous results are obtained by using the expressions of the tensor fields in local coordinates. For example $[(\phi^G)^2]^{kl}_{hj} = (\phi^G)^{kr}_{ij}(\phi^G)^{il}_{hr}$, where $(\phi^G)^{kr}_{ij} = \frac{1}{2}(\delta^k_i\delta^r_j - G^k_i G^r_j)$.

Theorem 2.2. *If $\dot{\nabla}$ is a connection on the strict complex almost contact manifold M then the family of the adapted connections on M is given by*

$$\nabla = \dot{\nabla} + P, \tag{2.5}$$

where P is a tensor field of type $(1, 2)$ on M, given by $P(X, Y) = P_X(Y)$, where P_X is a tensor field of type $(1, 1)$ defined as follows

$$P_X = B^G_X + B^H_X - (\Psi^H + \Theta)B^G_X + (\phi^H - \Theta)R_X,$$

with R an arbitrary tensor field of type $(1, 2)$, and

$$B^G_X = \frac{1}{2}(\dot{\nabla}_X G)G - v(X)H - \frac{1}{2}[i_{GX}du\circ G + i_X du - u\circ(\dot{\nabla}_X G^2)]\otimes U +$$
$$+\frac{1}{2}[i_{GX}dv\circ G - i_X dv + v\circ(\dot{\nabla}_X G^2)]\otimes V -$$
$$-(GX + \dot{\nabla}_X U)\otimes u - (HX + \dot{\nabla}_X V)\otimes v,$$
$$B^H_X = \frac{1}{2}(\dot{\nabla}_X H)H - u(X)G + \frac{1}{2}[i_{HX}du\circ H - i_X du + u\circ(\dot{\nabla}_X H^2)]\otimes U +$$
$$-\frac{1}{2}[i_{HX}dv\circ H + i_X dv - v\circ(\dot{\nabla}_X H^2)]\otimes V -$$
$$-(GX + \dot{\nabla}_X U)\otimes u - (HX + \dot{\nabla}_X V)\otimes v.$$

Proof. From $\nabla_X = \dot{\nabla}_X + P_X$ and since ∇ is an adapted connection it follows that

$$P_X(U) = -GX - \frac{1}{2}i_X du(U)U - \frac{1}{2}dv(U)V - \dot{\nabla}_X U \tag{2.6}$$

and

$$P_X(V) = -HX - \frac{1}{2}i_X du(V)U - \frac{1}{2}i_X dv(V)V - \dot{\nabla}_X V. \tag{2.7}$$

We also obtain that

$$\begin{aligned} P_X \circ G - G \circ P_X = -\dot{\nabla}_X G - 2v(X)pJ - pX \otimes u - pJX \otimes v+ \\ +\frac{1}{2}[i_{GX}du - (i_{GX}du(U))u - (i_{GX}du(V))v - i_X du \circ G] \otimes U- \\ -\frac{1}{2}[i_{GX}dv - (i_{GX}dv(U))u - (i_{GX}dv(V))v + i_X dv \circ G] \otimes V \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} P_X \circ H - H \circ P_X = -\dot{\nabla}_X H + 2u(X)pJ + pJX \otimes u - pX \otimes v- \\ -\frac{1}{2}[i_{HX}du - (i_{HX}du(U))u - (i_{HX}du(V))v + i_X du \circ H] \otimes U+ \\ +\frac{1}{2}[i_{HX}dv - (i_{HX}dv(U))u - (i_{HX}dv(V))v - i_X dv \circ H] \otimes V. \end{aligned} \tag{2.9}$$

The equations 2.6 and 2.8 are equivalent with

$$\begin{aligned} P_X + G \circ P_X \circ G = (\dot{\nabla}_X G)G - 2v(X)pH- \\ -\frac{1}{2}[i_{GX}du \circ G - i_X du \circ G^2 + i_X du(U)u + i_X du(V)v] \otimes U+ \\ +\frac{1}{2}[i_{GX}dv \circ G + i_X dv \circ G^2 - i_X dv(U)u - i_X dv(V)v] \otimes V- \\ -(GX + \dot{\nabla}_X U) \otimes u - (HX + \dot{\nabla}_X V) \otimes v, \end{aligned} \tag{2.10}$$

which can be written

$$\Psi^G P_X = \frac{1}{2}A_X,$$

where we have denoted with A_X the right side of equation 2.10. It follows from 2.4, that $2\Theta\Psi^G P_X = \Theta P_X = \Theta A_X$ and then $(\Psi^G + \Theta)P_X = \frac{1}{2}A_X + \Theta A_X = B^G_X$. Hence $P_X = B^G_X + (\phi^G - \Theta)Q_X$, where Q_X is a tensor field of type $(1,1)$. Replacing P_X in the equation for H which is an analogous of 2.10 and trough a similar computation one obtains that

$$(\phi^G - \Theta)Q_X = B^H_X - (\Psi^H + \Theta)B^G_X + (\phi^H - \Theta)R_X,$$

where R is an arbitrary tensor field of type $(1,2)$. Thus one obtains the desired result.

Remark 2.3. If in the proof of the previous theorem we use first the equation for H one obtains for P_X the following formula, which is equivalent with that in the theorem

$$P_X = B^H_X + B^G_X - (\Psi^G + \Theta)B^H_X + (\phi^G - \Theta)M_X.$$

Remark 2.4. If $\dot{\nabla}$ is an adapted connection on M then the family of all adapted connections on M is given by $\nabla = \dot{\nabla} + (\phi^H - \Theta)R_X$ or $\nabla = \dot{\nabla} + (\phi^G - \Theta)M_X$.

3 The torsion of an adapted connection

Let (M, J, G, H, U, V, u, v) be a strict complex almost contact manifold and let ∇ be an adapted connection on M. Let T be the torsion of ∇, given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad X,Y \in \chi(M).$$

After a straightforward computation one obtains

$$T(X,Y) + GT(X,GY) + GT(GX,Y) - T(GX,GY) = N_G(X,Y) + \\ +2du(X,Y)U - 2dv(HX,HY)V \tag{3.1}$$

and

$$T(X,Y) + HT(X,HY) + HT(HX,Y) - T(HX,HY) = N_H(X,Y) - \\ -2du(GX,GY)U + 2dv(X,Y)V, \tag{3.2}$$

for any $X, Y \in \mathcal{H}$. Similarly we have

$$T(X,U) + GT(GX,U) = N_G(X,U) + 2du(X,U)U, \tag{3.3}$$

$$T(X,V) + HT(HX,V) = N_H(X,V) + 2dv(X,V)V, \tag{3.4}$$

for any $X \in \chi(M)$.

Using this formulas and the fact that the Levi-Civita connection on a complex Sasakian manifold is adapted we can state the following

Theorem 3.1. *A strict complex contact manifold M is a complex Sasakian manifold if and only if there exist a torsion free adapted connection on M.*

In order to improve this result let us define for the strict complex almost contact manifold (M, J, G, H, U, V, u, v) the tensor fields

$$S_1(X,Y) = N_G(X,Y) + 2du(X,Y)U - 2dv(HX,HY)V,$$

$$S_2(X,Y) = N_H(X,Y) - 2du(GX,GY)U + 2dv(X,Y)V,$$

for any $X, Y \in \chi(M)$.

From 3.1, 3.2, 3.3 and 3.4 one obtains

Proposition 3.2. *If on a strict complex almost contact manifold there exist a torsion free adapted connection then*

$$S_1(X,Y) = S_2(X,Y), \quad X,Y \in \mathcal{H}$$

and

$$S_1(U,X) = S_2(V,X) = 0, \quad X \in \chi(M).$$

Proposition 3.3. *On a strict complex almost contact manifold we have*

$$u(S_1(X,Y)) = 2u(T(X,Y)), \quad v(S_2(X,Y)) = 2v(T(X,Y)), \quad X,Y \in \chi(M),$$

$$q(S_1(U,X)) = q(T(U,X)), \quad q(S_2(V,X)) = q(T(V,X)), \quad X \in \chi(M).$$

Proof. From the definition of the torsion T one obtains $u(T(X,Y)) = (2du(X,Y)+ (\nabla_X u)Y - (\nabla_Y u)X) = du(X,Y) - du(GX,GY)$, since the connection is adapted. On the other hand we have $u(S_1(X,Y)) = u([GX,GY]) + 2du(X,Y) = 2du(X,Y) - 2du(GX,GY)$. The last two statements follows directly from 3.3 and 3.4.

In the following let us consider $X, Y \in \mathcal{H}$ and $S_{1X}(Y) = S_1(X,Y)$, $T_X(Y) = T(X,Y)$. From 3.1 one obtains that

$$p(S_{1X}) = pT_X + GT_X \circ G + GT_{GX} - pT_{GX} \circ G \tag{3.5}$$

Since $G \otimes G = p \otimes p - 2(\phi^G - \Theta)$ we have

$$GT_{GX}(GY) = pT_X(Y) - 2(\phi^G - \Theta)_Y T_X, \tag{3.6}$$

where $(\phi^G - \Theta)_Y(X) = (\phi^G - \Theta)(Y,X)$. After a straightforward computation it follows $pT_{GX}GY = -pT_XY + 2(\phi^G - \Theta)_Y GT_{GX} - 2(\phi^G - \Theta)_X T_Y$ and since $2pT_{GX}(GY) = pT_{GX}(GY) - pT_{GY}(GX)$ we have

$$(pT_{GX} \circ G)Y = -pT_X(Y) - (\phi^G - \Theta)_X(T_Y + GT_{GY})+ \\ +(\phi^G - \Theta)_Y(T_X + GT_{GX}). \tag{3.7}$$

Thus, using 3.5, 3.6 and 3.7,

$$pS_{1X}(Y) = 4pT_X(Y) - (\phi^G - \Theta)_Y(3T_X + GT_{GX})+ \\ +(\phi^G - \Theta)_X(3T_Y + GT_{GY}). \tag{3.8}$$

In the same way

$$pS_{2X}(Y) = 4pT_X(Y) - (\phi^H - \Theta)_Y(3T_X + HT_{HX})+ \\ +(\phi^H - \Theta)_X(3T_Y + HT_{HY}), \tag{3.9}$$

for any $X, Y \in \mathcal{H}$.

By a similar computation one obtains

$$pS_{1X}(U) = pT_X(U) + pT_{pX}(U) + 2(\phi^G - \Theta)_X T_U, \tag{3.10}$$

$$pS_{2X}(V) = pT_X(V) + pT_{pX}(V) + 2(\phi^H - \Theta)_X T_V. \tag{3.11}$$

Assume that $S_{1X}(Y) = S_{2X}(Y) = 0$, $X, Y \in \mathcal{H}$, and $S_{1U}(X) = S_{2V}(X) = 0$, $X \in \chi(M)$. From the second assumption it follows, by using Proposition 3.3 that $qT(X,Y) = 0$, $X, Y \in \chi(M)$. Using the previous results, the fact that $2T_X(Y) = T_X(Y) - T_Y(X)$ and the first assumption we have

$$pT_X(Y) - \frac{1}{2}[(\phi^G - \Theta)_Y(\frac{3}{4}T_{pX} + \frac{1}{4}GT_{GX} + 2u(X)T_U)+ \\ +(\phi^H - \Theta)_Y(\frac{3}{4}T_{pX} + \frac{1}{4}HT_{HX} + 2v(X)T_V)]+ \\ +\frac{1}{2}[(\phi^G - \Theta)_X(\frac{3}{4}T_{pY} + \frac{1}{4}GT_{GY} + 2u(Y)T_U)+$$

$$+(\phi^H-\Theta)_X(\frac{3}{4}T_{pY}+\frac{1}{4}HT_{HY}+2v(Y)T_V)]=$$

$$=pT_X(Y)-\frac{1}{2}C(Y,X)+\frac{1}{2}C(X,Y)=0,$$

where

$$C(Y,X)=(\phi^G-\Theta)_Y(\frac{3}{4}T_{pX}+\frac{1}{4}GT_{GX}+2u(X)T_U)+$$

$$+(\phi^H-\Theta)_Y(\frac{3}{4}T_{pX}+\frac{1}{4}HT_{HX}+2v(X)T_V)$$

Let us consider the affine connection $\widetilde{\nabla}$ on M defined by $\widetilde{\nabla}_X Y=\nabla_X Y-\frac{1}{2}C(Y,X)$. It is easy to see that $\widetilde{\nabla}$ is torsion free and, from Theorem 2.2, $\widetilde{\nabla}$ is an adapted connection. We just have obtained

Theorem 3.4. *On a strict complex almost contact manifold M there exist a torsion free adapted connection if and only if*

$$S_1(X,Y)=S_2(X,Y)=0,\quad X,Y\in\mathcal{H}$$

and

$$S_1(U,X)=S_2(V,X)=0,\quad X\in\chi(M).$$

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Author's address:

Dorel Fetcu
Department of Mathematics, "Gh.Asachi" Technical University
11 Carol I Blvd., 700506 Iaşi, România.
email: dfetcu@math.tuiasi.ro