An adapted connection
on a strict complex contact manifold

Dorel Fetcu

Abstract
We find a class of connections on a strict complex almost contact manifold, which we call adapted connections, and, by using them, we prove some results about the normality of such a manifold.

Mathematics Subject Classification: 53B05, 53D99.
Key words: complex contact manifolds, adapted connection.

1 Introduction
Concerning the complex contact manifold we shall recall some notions as they are presented in [1].
A complex contact manifold is a complex manifold of odd complex dimension 2n+1 together with an open covering \( \{ \mathcal{O}_\alpha \} \) by coordinate neighborhoods such that:

1. On each \( \{ \mathcal{O}_\alpha \} \) there is a holomorphic 1-form \( \theta_\alpha \) such that
   \[ \theta_\alpha \wedge (d\theta_\alpha)^n \neq 0; \]
2. On \( \mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset \) there is a non-vanishing holomorphic function \( f_{\alpha\beta} \) such that
   \[ \theta_\alpha = f_{\alpha\beta} \theta_\beta. \]

A complex contact manifold with a global complex form is called strict complex contact manifold.

On the other hand if \( M \) is a complex manifold with almost complex structure \( J \), Hermitian metric \( g \) and open covering \( \{ \mathcal{O}_\alpha \} \) by coordinate neighborhoods, \( M \) is called a complex almost contact metric manifold if it satisfies the following two conditions:

1. On each \( \{ \mathcal{O}_\alpha \} \) there exist 1-forms \( u_\alpha \) and \( v_\alpha = u_\alpha \circ J \) with orthogonal dual vector fields \( U_\alpha \) and \( V_\alpha = -JU_\alpha \) and (1,1) tensor fields \( G_\alpha \) and \( H_\alpha = G_\alpha J \) such that
   \[ G_\alpha^2 = H_\alpha^2 = -I + u_\alpha \otimes U_\alpha + v_\alpha \otimes V_\alpha, \]
   \[ G_\alpha J = -JG_\alpha, \quad G_\alpha U_\alpha = 0, \quad g(X, G_\alpha Y) = -g(G_\alpha X, Y), \]

\( \theta \) Fifth Conference of Balkan Society of Geometers, August 29 - September 2, 2005, Mangalia-Romania, pp. 54-61.
An adapted connection

2. On $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$,

\[
u_\beta = au_\alpha - bv_\alpha, \quad v_\beta = bu_\alpha + av_\alpha,
\]

\[
G_\beta = aG_\alpha - bH_\alpha, \quad v_\beta = bG_\alpha + aH_\alpha,
\]

where $a$ and $b$ are functions with $a^2 + b^2 = 1$. We call a complex almost contact manifold with global structure tensors a strict complex almost contact manifold.

It is proved that a complex contact manifold admits a complex almost contact metric structure for which the local contact form $\theta$ is of the form $u - iv$ and the local tensor fields $G$ and $H$ are related to $du$ and $dv$ by

\[
du(X, Y) = g(X, GY) + (\sigma \wedge v)(X, Y),
\]

\[
dv(X, Y) = g(X, HY) - (\sigma \wedge u)(X, Y),
\]

where $\sigma(X) = g(\nabla_X U, V)$. We call a complex contact manifold with a complex almost contact metric structure satisfying these conditions, a complex contact metric manifold. Note that if the complex contact structure is strict then $\sigma = 0$.

Let $S_1$ and $S_2$ be two tensor fields defined by

\[
S_1(X, Y) = N_G(X, Y) + 2g(X, GY)U - 2g(X, HY)V + 2(v(Y)HX - v(X)HY)
\]

\[+ \sigma(GY)HX - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX,
\]

\[
S_2(X, Y) = N_H(X, Y) - 2g(X, GY)U + 2g(X, HY)V + 2(u(Y)GX - u(X)GY)
\]

\[+ \sigma(HX)GY - \sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX,
\]

where $N_G$ and $N_H$ are the Nijenhuis tensor fields of $G$ and $H$, respectively.

A complex contact metric structure is normal if

\[
S_1(X, Y) = S_2(X, Y) = 0, \quad X, Y \in \mathcal{H},
\]

\[
S_1(U, X) = S_2(V, X) = 0, \quad X \in \chi(M),
\]

where $\mathcal{H}$ is the subbundle named the horizontal subbundle and defined by the subspaces $\{ X \in T_PM, P \in M; \theta_\alpha(X) = 0 \}$.

A normal complex contact metric manifold whose complex contact structure is given by a global complex contact form is called a complex Sasakian manifold.

For a complex Sasakian manifold we have the following formulas, (see [3]),

\[
g((\nabla_X G)Y, Z) = -2\nu(X)g(HGY, Z) - \nu(Y)g(X, Z) -
\]

\[-v(Y)g(JX, Z) + u(Z)g(X, Y) + v(Z)g(JX, Y),
\]

\[
g((\nabla_X H)Y, Z) = -2\nu(X)g(HGY, Z) + \nu(Y)g(JX, Z) -
\]

\[-v(Y)g(X, Z) + u(Z)g(X, JY) + v(Z)g(X, Y),
\]

and

\[
g((\nabla_X J)Y, Z) = -2\nu(X)g(HY, Z) + 2\nu(X)g(GY, Z)
\]

where

\[
\sigma(GY) = \nu(Y)g(X, Z) - \nu(Z)g(X, Y);
\]

\[
\sigma(HY) = v(Y)g(X, Z) - v(Z)g(X, Y);
\]

\[
\sigma(GHY) = \nu(Y)g(JX, Z) - \nu(Z)g(JX, Y);
\]

\[
\sigma(GHX) = v(Y)g(JX, Z) - v(Z)g(JX, Y);
\]

\[
\sigma(HGY) = \nu(Y)g(X, JY) - \nu(Z)g(X, JY);
\]

\[
\sigma(HGX) = v(Y)g(X, JY) - v(Z)g(X, JY);
\]
2 Adapted connections on a strict complex almost contact manifold

In [4], [6] and [5] the authors introduced a class of adapted connections on an almost contact manifold. In this paper we define, in a similar way, a class of adapted connections on a strict complex almost contact manifold.

Let \((M, J, G, H, U, V, u, v)\) be a strict complex almost contact manifold and let us denote by 
\[ p = I - u \otimes U - v \otimes V \]
and 
\[ q = u \otimes U + v \otimes V \]
the projectors on the distributions \(H\) and \(V = \text{span}\{U, V\}\), respectively. It is easy to see that

\[
\begin{align*}
    p^2 &= p, & q^2 &= q, & pq &= qp = 0, \\
    G^2 &= H^2 = -p, & pG &= Gp = G, & pH &= Hp = H, \\
    qG &= Gq = 0, & qH &= Hq = 0, \\
    GH &= -HG = -pJ,
\end{align*}
\]

(2.1)

Definition 2.1. We call an affine connection on the strict complex manifold \(M\) an adapted connection if

\[
\begin{align*}
    (\nabla_X G)Y &= -2v(X)pJY - u(Y)pX - v(Y)pJX + \\
    &+ \frac{1}{2}[du(GX, pY) - du(X, GY)]U - \\
    &- \frac{1}{2}[dv(GX, pY) + dv(X, GY)]V, \\
    (\nabla_X H)Y &= 2u(X)pJY + u(Y)pJX - v(Y)pX - \\
    &- \frac{1}{2}[du(HX, pY) - du(X, HY)]U + \\
    &+ \frac{1}{2}[dv(HX, pY) - dv(X, HY)]V, \\
    (\nabla_X u)Y &= \frac{1}{2}[du(X, Y) + du(GX, GY)], \\
    (\nabla_X v)Y &= \frac{1}{2}[dv(X, Y) + dv(HX, HY)], \\
    \nabla_X U &= -GX - \frac{1}{2}du(X, U)U - \frac{1}{2}dv(X, U)V, \\
    \nabla_X V &= -HX - \frac{1}{2}du(X, V)U - \frac{1}{2}dv(X, V)V,
\end{align*}
\]

(2.2)

for any \(X, Y \in \chi(M)\).

Note that on a complex Sasakian manifold the Levi-Civita connection is an adapted connection.

Remark 2.1. If \(\nabla\) is an adapted connection on \(M\) we have

\[
\begin{align*}
    (\nabla_X J)Y &= 2v(X)GY - 2u(X)HY + \\
    &+ \frac{1}{2}[dv(X, Y) + dv(HX, HY) - du(GX, HY) - du(X, JY)]U - \\
\end{align*}
\]
An adapted connection

\[-\frac{1}{2}[du(X,Y) + du(GX,GY) - dv(GX,HY) + dv(X,JX)]V,\]

for any \(X,Y \in \chi(M)\).

In order to prove the existence of the adapted connections on a strict complex almost contact manifold \((M,J,G,H,U,V,u,v)\) let us define first the following tensor fields of type (2, 2) on \(M\)

\[
\begin{align*}
\phi^G &= \frac{1}{2}(I \otimes I - G \otimes G), \quad \Psi^G = \frac{1}{2}(I \otimes I + G \otimes G), \\
\phi^H &= \frac{1}{2}(I \otimes I - H \otimes H), \quad \Psi^H = \frac{1}{2}(I \otimes I + H \otimes H), \quad \Theta = \frac{1}{2}(I \otimes I - p \otimes p).
\end{align*}
\]

(2.3)

Just like in the real case (see [6]), it is easy to prove that

\[
\begin{align*}
\phi^G + \Psi^G &= I \otimes I, \quad (\phi^G)^2 = \phi^G - \frac{1}{2}\Theta, \quad (\Psi^G)^2 = \Psi^G - \frac{1}{2}\Theta, \\
\phi^G \Psi^G &= \Psi^G \phi^G = \phi^G \Theta = \Theta \phi^G = \Psi^G \Theta = \Theta \Psi^G = \Theta^2 = \frac{1}{2}\Theta, \\
(\Psi^G + \Theta)(\phi^G - \Theta) &= I \otimes I, \\
(\Psi^G + \Theta)(\phi^G - \Theta) &= (\phi^G - \Theta)(\Psi^G + \Theta) = 0, \\
(\Psi^G + \Theta)(\Psi^G + \Theta) &= I \otimes I, \quad (\phi^G - \Theta)(\phi^G - \Theta) = \phi^G - \Theta,
\end{align*}
\]

(2.4)

and that the similar equations holds for \(\phi^H\) and \(\Psi^H\). Note that the previous results are obtained by using the expressions of the tensor fields in local coordinates. For example \([(\phi^G)^2]_{ik}^j = (\phi^G)[k]^i_r (\phi^G)[r]^j_i\), where \((\phi^G)[k]^i_r = \frac{1}{2}(\delta^k_i \delta^r_j - G^j_k G^i_r)\).

**Theorem 2.2.** If \(\tilde{\nabla}\) is a connection on the strict complex almost contact manifold \(M\) then the family of the adapted connections on \(M\) is given by

\[
\nabla = \tilde{\nabla} + P,
\]

(2.5)

where \(P\) is a tensor field of type (1, 2) on \(M\), given by \(P(X,Y) = P_X(Y)\), where \(P_X\) is a tensor field of type (1, 1) defined as follows

\[
P_X = B^G_X + B^H_X - (\Psi^H + \Theta)B^G_X + (\phi^H - \Theta)R_X,
\]

with \(R\) an arbitrary tensor field of type (1, 2), and

\[
B^G_X = \frac{1}{2}[(\tilde{\nabla}_X G)G - v(X)H - \frac{1}{2}[i_{GX} du \circ G + i_X du - u \circ (\tilde{\nabla}_X G^2)] \otimes U + \frac{1}{2}[i_{GX} dv \circ G - i_X dv + v \circ (\tilde{\nabla}_X G^2)] \otimes V - (GX + \tilde{\nabla}_X U) \otimes u - (HX + \tilde{\nabla}_X V) \otimes v,\]

\[
B^H_X = \frac{1}{2}[(\tilde{\nabla}_X H)H - u(X)G + \frac{1}{2}[i_{HX} du \circ H + i_X du - u \circ (\tilde{\nabla}_X H^2)] \otimes U + \frac{1}{2}[i_{HX} dv \circ H + i_X dv + v \circ (\tilde{\nabla}_X H^2)] \otimes V - (GX + \tilde{\nabla}_X U) \otimes u - (HX + \tilde{\nabla}_X V) \otimes v.
\]
Proof. From $\nabla_X = \tilde{\nabla}_X + P_X$ and since $\nabla$ is an adapted connection it follows that
\[
P_X(U) = -GX - \frac{1}{2} i_X du(U) U - \frac{1}{2} dv(U) V - \tilde{\nabla}_X U \tag{2.6}
\]
and
\[
P_X(V) = -HX - \frac{1}{2} i_X du(V) U - \frac{1}{2} i_X dv(V) V - \tilde{\nabla}_X V. \tag{2.7}
\]
We also obtain that
\[
P_X \circ G - G \circ P_X = -\tilde{\nabla}_X G - 2 v(X) pJ - pX \otimes u - pJX \otimes v + \nonumber
\]
\[+ \frac{1}{2} [i_{G} du - (i_{G} du(U)) u - (i_{G} du(V)) v - i_{X} du \circ G] \otimes U - \nonumber
\]
\[- \frac{1}{2} [i_{G} dv - (i_{G} dv(U)) u - (i_{G} dv(V)) v + i_{X} dv \circ G] \otimes V \tag{2.8}
\]
and
\[
P_X \circ H - H \circ P_X = -\nabla_X H + 2 u(X) pJ + pX \otimes u - pX \otimes v - \nonumber
\]
\[- \frac{1}{2} [i_{H} du - (i_{H} du(U)) u - (i_{H} du(V)) v + i_{X} du \circ H] \otimes U + \nonumber
\]
\[+ \frac{1}{2} [i_{H} dv - (i_{H} dv(U)) u - (i_{H} dv(V)) v - i_{X} dv \circ H] \otimes V. \tag{2.9}
\]
The equations 2.6 and 2.8 are equivalent with
\[
P_X + G \circ P_X \circ G = (\tilde{\nabla}_X G)G - 2v(X)pH - \nonumber
\]
\[- \frac{1}{2} [i_{G} du \circ G - i_{X} du \circ G^2 + i_{X} du(U) u + i_{X} du(U) v] \otimes U + \nonumber
\]
\[+ \frac{1}{2} [i_{G} dv \circ G + i_{X} dv \circ G^2 - i_{X} dv(U) u - i_{X} dv(U) v] \otimes V - \nonumber
\]
\[- (GX + \tilde{\nabla}_X U) \otimes u - (HX + \tilde{\nabla}_X V) \otimes v, \tag{2.10}
\]
which can be written
\[
\Psi^G P_X = \frac{1}{2} A_X,
\]
where we have denoted with $A_X$ the right side of equation 2.10. It follows from 2.4, that $2 \Theta \Psi^G P_X = \Theta P_X = \Theta A_X$ and then $(\Psi^G + \Theta) P_X = \frac{1}{2} A_X + \Theta A_X = B^G_X$. Hence
\[
P_X = B^G_X + (\phi^G - \Theta) Q_X,
\]
where $Q_X$ is a tensor field of type $(1,1)$. Replacing $P_X$ in the equation for $H$ which is an analogous of 2.10 and trough a similar computation one obtains that
\[
(\phi^G - \Theta) Q_X = B^H_X - (\Psi^H + \Theta) B^G_X + (\phi^H - \Theta) R_X,
\]
where $R$ is an arbitrary tensor field of type $(1,2)$. Thus one obtains the desired result.

Remark 2.3. If in the proof of the previous theorem we use first the equation for $H$ one obtains for $P_X$ the following formula, which is equivalent with that in the theorem
\[
P_X = B^H_X + B^G_X - (\Psi^G + \Theta) B^H_X + (\phi^H - \Theta) M_X.
\]

Remark 2.4. If $\tilde{\nabla}$ is an adapted connection on $M$ then the family of all adapted connections on $M$ is given by $\nabla = \tilde{\nabla} + (\phi^H - \Theta) R_X$ or $\nabla = \tilde{\nabla} + (\phi^G - \Theta) M_X$. 

3 The torsion of an adapted connection

Let \((M, J, G, H, U, V, u, v)\) be a strict complex almost contact manifold and let \(\nabla\) be an adapted connection on \(M\). Let \(T\) be the torsion of \(\nabla\), given by

\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad X, Y \in \chi(M).
\]

After a straightforward computation one obtains

\[
T(X, Y) + GT(X, GY) + GT(GX, Y) - T(GX, GY) = N_G(X, Y) + 2du(X, Y)U - 2dv(HX, HY)V
\]

and

\[
T(X, Y) + HT(X, HY) + HT(HX, Y) - T(HX, HY) = N_H(X, Y) - 2du(GX, GY)U + 2dv(X, Y)V;
\]

for any \(X, Y \in \mathcal{H}\). Similarly we have

\[
T(X, U) + GT(GX, U) = N_G(X, U) + 2du(X, U)U,
\]

\[
T(X, V) + HT(HX, V) = N_H(X, V) + 2dv(X, V)V,
\]

for any \(X \in \chi(M)\).

Using this formulas and the fact that the Levi-Civita connection on a complex Sasakian manifold is adapted we can state the following

**Theorem 3.1.** A strict complex contact manifold \(M\) is a complex Sasakian manifold if and only if there exist a torsion free adapted connection on \(M\).

In order to improve this result let us define for the strict complex almost contact manifold \((M, J, G, H, U, V, u, v)\) the tensor fields

\[
S_1(X, Y) = N_G(X, Y) + 2du(X, Y)U - 2dv(HX, HY)V,
\]

\[
S_2(X, Y) = N_H(X, Y) - 2du(GX, GY)U + 2dv(X, Y)V,
\]

for any \(X, Y \in \chi(M)\).

From 3.1, 3.2, 3.3 and 3.4 one obtains

**Proposition 3.2.** If on a strict complex almost contact manifold there exist a torsion free adapted connection then

\[
S_1(X, Y) = S_2(X, Y), \quad X, Y \in \mathcal{H}
\]

and

\[
S_1(U, X) = S_2(V, X) = 0, \quad X \in \chi(M).
\]

**Proposition 3.3.** On a strict complex almost contact manifold we have

\[
u(S_1(X, Y)) = 2u(T(X, Y)), \quad v(S_2(X, Y)) = 2v(T(X, Y)), \quad X, Y \in \chi(M),
\]

\[
q(S_1(U, X)) = q(T(U, X)), \quad q(S_2(V, X)) = q(T(V, X)), \quad X \in \chi(M).
\]
Proof. From the definition of the torsion $T$ one obtains $u(T(X,Y)) = (2du(X,Y) + (\nabla_X u)Y - (\nabla_Y u)X) = du(X,Y) - du(GX,GY)$, since the connection is adapted. On the other hand we have $u(S_1(X,Y)) = u([GX,GY]) + 2du(X,Y) = 2du(X,Y) - 2du(GX,GY)$. The last two statements follows directly from 3.3 and 3.4.

In the following let us consider $X,Y \in \mathcal{H}$ and $S_{1X}(Y) = S_1(X,Y)$, $T_X(Y) = T(X,Y)$. From 3.1 one obtains that

\begin{equation}
\tag{3.5}
p(S_{1X}) = pT_X + GT_X \circ G + GT_{GX} - pT_{GX} \circ G
\end{equation}

Since $G \otimes G = p \otimes p - 2(\phi^G - \Theta)$ we have

\begin{equation}
\tag{3.6}
GT_{GX}(GY) = pT_X(Y) - 2(\phi^G - \Theta)YJ X_X,
\end{equation}

where $(\phi^G - \Theta)Y(X) = (\phi^G - \Theta)(Y,X)$. After a straightforward computation it follows

\begin{equation}
\tag{3.7}
pT_{GX}(GY) = -pT_XY + 2(\phi^G - \Theta)YJ GT_{GX} - 2(\phi^G - \Theta)XJ TX_Y \text{ and since } 2pT_{GX}(GY) = pT_{GX}(GY) - pT_{GY}(GX) \text{ we have}
\end{equation}

\begin{equation}
\tag{3.8}
pS_{1X}(Y) = 4pT_X(Y) - (\phi^G - \Theta)Y(3T_X + GT_{GX}) + (\phi^G - \Theta)X(3T_X + GT_{GX}).
\end{equation}

In the same way

\begin{equation}
\tag{3.9}
pS_{2X}(Y) = 4pT_X(Y) - (\phi^H - \Theta)Y(3T_X + HT_{HX}) + (\phi^H - \Theta)X(3T_X + HT_{HY}),
\end{equation}

for any $X,Y \in \mathcal{H}$.

By a similar computation one obtains

\begin{equation}
\tag{3.10}
pS_{1X}(U) = pT_X(U) + pT_{p,X} + 2(\phi^G - \Theta)XJ TX_U,
\end{equation}

\begin{equation}
\tag{3.11}
pS_{2X}(V) = pT_X(V) + pT_{p,X} + 2(\phi^H - \Theta)XJ TX_V.
\end{equation}

Assume that $S_{1X}(Y) = S_{2X}(Y) = 0$, $X,Y \in \mathcal{H}$, and $S_{1U}(X) = S_{2V}(X) = 0$, $X \in \chi(M)$. From the second assumption it follows, by using Proposition 3.3 that $qT(X,Y) = 0$, $X,Y \in \chi(M)$. Using the previous results, the fact that $2T_X(Y) = T_X(Y) - T_Y(X)$ and the first assumption we have

\begin{equation}
\tag{3.12}
pT_X(Y) - \frac{1}{2}[(\phi^G - \Theta)Y(3T_{pX} + \frac{1}{4}GT_{GX} + 2u(X)T_U) + (\phi^H - \Theta)Y(\frac{3}{4}T_{pX} + \frac{1}{4}HT_{HX} + 2v(X)T_V)].
\end{equation}
An adapted connection

\[
\frac{1}{2}((\phi^G - \Theta)_{X}(\frac{3}{4}T_{pY} + \frac{1}{4}GT_{GY} + 2u(Y)T_{U}) + \\
(\phi^H - \Theta)_{X}(\frac{3}{4}T_{pY} + \frac{1}{4}HT_{HY} + 2v(Y)T_{V})) = \\
pT_{X}(Y) - \frac{1}{2}C(Y, X) + \frac{1}{2}C(X, Y) = 0,
\]

where

\[
C(Y, X) = (\phi^G - \Theta)_{Y}(\frac{3}{4}T_{pX} + \frac{1}{4}GT_{GX} + 2u(X)T_{U}) + \\
(\phi^H - \Theta)_{Y}(\frac{3}{4}T_{pX} + \frac{1}{4}HT_{HX} + 2v(X)T_{V})
\]

Let us consider the affine connection \( \bar{\nabla} \) on \( M \) defined by \( \bar{\nabla} \)X Y = \( \nabla \)X Y - \( \frac{1}{2} \)C(Y, X). It is easy to see that \( \bar{\nabla} \) is torsion free and, from Theorem 2.2, \( \bar{\nabla} \) is an adapted connection. We just have obtained

**Theorem 3.4.** On a strict complex almost contact manifold \( M \) there exist a torsion free adapted connection if and only if

\[ S_1(X, Y) = S_2(X, Y) = 0, \quad X, Y \in \mathcal{H} \]

and

\[ S_1(U, X) = S_2(V, X) = 0, \quad X \in \chi(M). \]

**References**


**Author’s address:**

Dorel Fetcu
Department of Mathematics, ”Gh.Asachi” Technical University
11 Carol I Blvd., 700506 Iași, România.
email: dfetcu@math.tuiasi.ro