Linear connections on Lie algebroids

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Abstract

We refer to the anchored vector bundles and Lie algebroids. The main purpose of this paper is to establish several remarkable properties of linear connections on Lie algebroids.

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1 Introduction

The Lie algebroids can be regarded as natural generalizations of Lie algebras and tangent bundles to manifolds.

In the paper [2] (J.Cortés and E.Martinez, 2004, MR 2099990 (2005 h:93038)) is introduced the notion of linear connection on a Lie algebroid.

The paper contains three sections. In the first section the category of anchored vector bundles is constructed. The second section is dedicated to Lie algebroids and its main properties. In the third section we investigate the linear connections on Lie algebroids and some properties of its torsion and curvature are presented. Finally we give some ways for construction of new linear connections starting from linear connections given on Lie algebroids.

The study of linear connections on Lie algebroids is important in the geometrical description of Lagrangian and Hamiltonian mechanical systems on Lie algebroids, see for instance [2, 5, 9].

2 Anchored vector bundles

Definition 2.1. Let (E, p, M) be a vector bundle and (TM, π_M, M) the tangent bundle to M. A morphism of vector bundles $\rho : E \to TM$ is called *anchor* of vector bundle E, i.e. ρ is a differentiable map such that $\pi_M \circ \rho = p$. An *anchored vector bundle* is a pair (E, ρ), where (E, p, M) is a vector bundle and $\rho : E \to TM$ is anchor.

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An anchored vector bundle (E,ρ) is said *transitive*, if its anchor ρ is a surjective submersion.

If (E,ρ) is an anchored vector bundle over M, then the anchor $\rho : E \to TM$ is a morphism of vector bundles. Then the map ρ defines a morphism of $\mathcal{F}(M)$- modules between the $\mathcal{F}(M)$-modules $\Gamma(E)$ and $\Gamma(TM)$ of sections of E and TM respectively, denoted with $\overline{\varphi} : \Gamma(E) \to \Gamma(TM)$ given by $s \in \Gamma(E) \longrightarrow \overline{\rho}(s) \in \Gamma(TM)$, where $\overline{\rho}(s)(x) = \rho(s(x))$, $(\forall)\, x \in M$. The morphism $\overline{\rho}$ is called *induced morphism* between $\Gamma(E)$ and $\Gamma(TM) = \mathcal{X}(M)$ by ρ. We will sometimes denote $\overline{\rho} : \Gamma(E) \to \mathcal{X}(M)$ also with the symbol ρ.

Example 2.1. (i) Let V be a real vector space of finite dimension. Then V is an anchored vector bundle over a manifold formed by one point. In this case, the anchor is zero map.

(ii) Let TM be tangent bundle to manifold M. Then the pair (TM, id_{TM}) is an anchored vector bundle with the identity map on TM as anchor.

Definition 2.2. Let (E,p,M) and $(E^{'},p^{'},M)$ be two anchored vector bundles over the same base M with the anchors $\rho : E \to TM$ and $\rho^{'} : E^{'} \to TM$. A *morphism of anchored vector bundles over* M or a M *- morphism of anchored vector bundles* between (E,ρ) and $(E^{'},\rho^{'})$ is a morphism of vector bundles $\varphi : (E,p,M) \longrightarrow (E^{'},p^{'},M)$ such that $\rho^{'} \circ \varphi = \rho$

Proposition 2.1. *If* $(E_i,p_i,M)\,, i = 1,2,3$ *are anchored vector bundles over* M *with anchors* $\rho_i : E_i \to TM, i = 1,2,3,$ *and* $\varphi : (E_1,\rho_1) \to (E_2,\rho_2)$ *and* $\psi : (E_2,\rho_2) \to (E_3,\rho_3)$ *are* M *- morphisms of anchored vector bundles, then* $\psi \circ \varphi : (E_1,\rho_1) \to (E_3,\rho_3)$ *is a* M *- morphism of anchored vector bundles.* □

The anchored vector bundles over the same base M and M - morphisms of anchored vector bundles form a category, denoted with $\mathcal{AVB}\,(M)$, and called the *category of anchored vector bundles over* M .

Direct product of two anchored vector bundles over the same base. Let (E_i,p_i,M), $i = 1,2$ be two anchored vector bundles over M, with anchors $\rho_i : E_i \to TM$. Consider the direct product $(E_1 \times E_2, p_1 \times p_2, M \times M)$ of vector bundles (E_1,p_1,M) and (E_2,p_2,M). We construct the map $\rho_1 \times \rho_2 : E_1 \times E_2 \longrightarrow T(M \times M) \simeq TM \times TM$ given by $(\rho_1 \times \rho_2)(z_1,z_2) = (\rho_1(z_1), \rho_2(z_2))$ for all $z_1 \in E_1$, $z_2 \in E_2$. It is easy to prove that $\rho_1 \times \rho_2 : E_1 \times E_2 \longrightarrow TM \times TM$ is a morphism of vector bundles. Using the relations $\pi_M \circ \rho_1 = p_1$ and $\pi_M \circ \rho_2 = p_2$ we have $(\pi_M \times \pi_M) \circ (\rho_1 \times \rho_2) = (p_1 \times p_2)$. Hence $(E_1 \times E_2, \rho_1 \times \rho_2)$ is an anchored vector bundle with $\rho_1 \times \rho_2 : E_1 \times E_2 \longrightarrow T(M \times M) \simeq TM \times TM$ as anchor. □

Direct sum of two anchored vector bundles with same base over the tangent bundle. Let (E_1,p_1,M) and (E_2,p_2,M) be two anchored vector bundles over M with the anchors $\rho_1 : E_1 \to TM$ and $\rho_2 : E_2 \to TM$ such that (E_2,ρ_2) is transitive. Consider the Whitney sum $(E_1 \oplus E_2, p_1 \oplus p_2, M)$ of vector bundles E_1 and E_2 over M. We have $E_1 \oplus E_2 = \{(z_1,z_2) \in E_1 \times E_2 \mid\ p_1(z_1) = p_2(z_2)\}$ and $(p_1 \oplus p_2)(z_1,z_2) = p_1(z_1)\,, (\forall)\,(z_1,z_2) \in E_1 \oplus E_2$. Also $\rho_1 : E_1 \to TM$ and $\rho_2 : E_2 \to TM$ are M- morphisms of vector bundles with property that :

$$\mathbf{Im}\ \rho_{1,x} + \mathbf{Im}\ \rho_{2,x} = T_xM, (\forall)\, x \in M,$$

since $\rho_{2,x} : E_{2,x} \to T_xM$ is surjective.

Let $E_1 \oplus_{TM} E_2 = \{(z_1, z_2) \in E_1 \oplus E_2 \mid \rho_1(z_1) = \rho_2(z_2)\}$. It is known (see, Mackenzie, 1987,[6]) that $(E_1 \oplus_{TM} E_2, p_1 \oplus p_2, M)$ is a un vector bundle over M, called *direct sum of vector bundles E_1 and E_2 over the tangent bundle TM* .

Consider the map $\rho_{\oplus_{TM}} : E_1 \oplus_{TM} E_2 \longrightarrow TM$ given by $\rho_{\oplus_{TM}}(z_1, z_2) = \rho_1(z_1)$, $(\forall)(z_1, z_2) \in E_1 \oplus_{TM} E_2$. We have that $(E_1 \oplus_{TM} E_2, p_1 \oplus p_2, M)$ is an anchored vector bundle with anchor $\rho_{\oplus_{TM}} : E_1 \oplus_{TM} E_2 \longrightarrow TM$. □

The prolongation of an anchored vector bundle over a surjective submersion. Let $\pi : P \to M$ be a surjective submersion. It follows that π is a fibration, that is P is a fibred manifold over M. Let (E, p_E, M) an anchored vector bundle with anchor $\rho : E \to TM$. Consider the subset

$$\mathcal{P}^\pi E = \{(z, v) \in E \times TP \mid \rho(z) = T\pi(v)\}$$

where $T\pi : TP \to TM$ is the tangent map to $\pi : P \to M$.

Denote by $\tau^\pi : \mathcal{P}^\pi E \to P$ the canonical projection, i.e. $\tau^\pi(z, v) = \tau_P(v)$, $(\forall)(z, v) \in \mathcal{P}^\pi E$, where $\tau_P : TP \to P$. It is easy to prove that $(\mathcal{P}^\pi E, \tau^\pi, P)$ is a vector bundle over P with projection τ^π. For every point $p \in P$ with property that $\pi(p) = x$, the local fibre $(\mathcal{P}^\pi E)_p$ of bundle $(\mathcal{P}^\pi E, \tau^\pi, P)$ is

$$(\mathcal{P}^\pi E)_p = \{(z, v) \in E_x \times T_pP \mid \rho(z) = T_p\pi(v)\}.$$

We will use sometimes the notation (p, z, v) for (z, v). Thus the map $\tau^\pi : \mathcal{P}^\pi E \to P$ is given by $\tau^\pi(p, z, v) = p$, $(\forall)(p, z, v) \in (\mathcal{P}^\pi E)_p$, i.e. τ^π is the projection on first factor.

Define the map $\rho^\pi : \mathcal{P}^\pi E \to TP, \rho^\pi(p, z, v) = v$, $(\forall)(p, z, v) \in (\mathcal{P}^\pi E)_p$, i.e. ρ^π is the projection pe on third factor. We have that ρ^π is a morphism of vector bundles between $(\mathcal{P}^\pi E, \tau^\pi, P)$ and (TP, τ_P, P).

We verify that $(\mathcal{P}^\pi E, \rho^\pi)$ is an anchored vector bundle with the anchor ρ^π.

Let the map $\mathcal{T}\pi : \mathcal{P}^\pi E \to E$ given by $\mathcal{T}\pi(p, z, v) = z$, $(\forall)(p, z, v) \in (\mathcal{P}^\pi E)_p$, i.e. $\mathcal{T}\pi$ is the projection on second factor.
Then $(\mathcal{T}\pi, \pi) : (\mathcal{P}^\pi E, \tau^\pi, P) \to (E, p_E, M)$ is a morphism of anchored vector bundles. □

3 Lie algebroids

We start this section with the concept of Lie algebroid.

Definition 3.1. *([6])*Let (E, p, M) be an anchored vector bundle with the anchor $\rho : E \to TM$. The anchored vector bundle (E, ρ) endowed with a Lie bracket $[\cdot, \cdot]_E$ on the space $\Gamma(E)$ of sections of E such that the following conditions are verified:

(1) $\Gamma(E)$ has Lie algebra structure to respect the bracket $[\cdot, \cdot]_E$;

(2) the morphism $\overline{\rho} : \Gamma(E) \to \Gamma(TM) = \mathcal{X}(M)$ induced from anchor ρ, is a homomorphism of Lie algebras, that is

$$\overline{\rho}([\sigma, \eta]_E) = [\overline{\rho}(\sigma), \overline{\rho}(\eta)], \ (\forall)\, \sigma, \eta \in \Gamma(E), \tag{3.1}$$

(3) the anchor ρ verify the *Leibnitz identity*:

$$[\sigma, f\eta]_E = f[\sigma, \eta]_E + \overline{\rho}(\sigma)(f)\eta, \quad (\forall)\, f \in \mathcal{F}(M), \sigma, \eta \in \Gamma(E) \tag{3.2}$$

is called a *Lie algebroid over M*. □

A Lie algebroid (E, p, M) over M with the anchor ρ and the bracket $[\cdot,\cdot]_E$ will be denoted with $(E, [\cdot,\cdot]_E, \rho)$.

A Lie algebroid Lie $(E, [\cdot,\cdot]_E, \rho)$ is said to be *transitive*, if ρ is surjective.

Definition 3.2. Let $(E, [\cdot,\cdot]_E, \rho)$ and $(E^{'}, [\cdot,\cdot]_{E^{'}}, \rho^{'})$ be two Lie algebroids over M. A *morphism of Lie algebroids* over M, is a morphism $\varphi : (E, \rho) \to (E^{'}, \rho^{'})$ of anchored vector bundles with property that:

$$\varphi\left([\sigma, \eta]_E\right) = [\varphi(\sigma), \varphi(\eta)]_{E^{'}},\ (\forall)\, \sigma, \eta \in \Gamma(E). \tag{3.3}$$

φ it also called a M - *morphism of Lie algebroids* .

Using Proposition 2.1, it is easy to prove the following proposition.

Proposition 3.1. *If* $(E_i, [\cdot,\cdot]_{E_i}, \rho_i)$, $i = 1, 2, 3$ *are Lie algebroids over* M *and* $\varphi : (E_1, \rho_1) \to (E_2, \rho_2)$ *and* $\psi : (E_2, \rho_2) \to (E_3, \rho_3)$ *are* M *- morphisms of Lie algebroids, then* $\psi \circ \varphi : (E_1, \rho_1) \to (E_3, \rho_3)$ *is a* M*- morphism of Lie algebroids .* □

The Lie algebroids over the same manifold M and all M - morphisms of Lie algebroids form a category, denoted by $\mathcal{LA}oid(M)$, and called the *category of Lie algebroids over* M. Since every Lie algebroid over M is an anchored vector bundle over M, follows that $\mathcal{LA}oid(M)$ is a subcategory of the category $\mathcal{AVB}(M)$.

Example 3.1. (i) Every real Lie algebra of finite dimension $(A, [\cdot,\cdot]_A)$ over a manifold M formed from one point is a Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ with $\rho = 0$.

(ii) The anchored vector bundle (TM, id_{TM}) (see, Example 2.1(ii)) with usually Lie bracket $[\cdot,\cdot]$ is a Lie algebroid $(TM, [\cdot,\cdot], id_{TM})$ over M.

(iii) Let M be a manifold and $\mathcal{A}$ a Lie algebra of finite dimension. The trivial fibration of Lie algebras $(E = M \times \mathcal{A}, pr_1, M)$ has a Lie algebroid structure over M having zero map as anchor ρ. If $\varphi : \mathcal{A} \to \mathcal{A}^{'}$ is a morphism of Lie algebras, then $id_M \times \varphi : M \times \mathcal{A} \to M \times \mathcal{A}^{'}$ is a morphism of Lie algebroids over M between $(E = M \times \mathcal{A}, pr_1, M)$ and $\left(E^{'} = M \times \mathcal{A}^{'}, pr_1, M\right)$.

Let us we construct some new Lie algebroids starting from given Lie algebroids. **The direct sum of two Lie algebroids with same base over tangent bundle.** Let $(E_1, [\cdot,\cdot]_{E_1}, \rho_1)$ and $(E_2, [\cdot,\cdot]_{E_2}, \rho_2)$ be two Lie algebroids over M with property that $(E_2, [\cdot,\cdot]_{E_2}, \rho_2)$ is transitive. Consider $(E_1 \oplus_{TM} E_2, p_1 \oplus p_2, M)$ the direct sum of anchored vector bundles (E_1, ρ_1) and (E_2, ρ_2) over tangent bundle $TM \xrightarrow{\pi_M} M$ with anchor $\rho_{\oplus_{TM}} : E_1 \oplus_{TM} E_2 \longrightarrow TM$ (see, Section 2). Hence $(E_1 \oplus_{TM} E_2, \rho_{\oplus_{TM}})$ is an anchored vector bundle.

A section of vector bundle $E_1 \oplus_{TM} E_2$ will denoted by $X_1 \oplus X_2$, where $X_1 \in \Gamma(E_1)$ and $X_2 \in \Gamma(E_2)$. Hence $X_1 \oplus X_2 \in \Gamma(E_1 \oplus_{TM} E_2)$. On the space $\Gamma(E_1 \oplus_{TM} E_2)$ define the Lie bracket $[\cdot,\cdot]_{\oplus_{TM}}$ by

$$[X_1 \oplus X_2, Y_1 \oplus Y_2]_{\oplus_{TM}} = [X_1, Y_1]_{E_1} \oplus [X_2, Y_2]_{E_2}. \tag{3.4}$$

We prove that $(E_1 \oplus_{TM} E_2, [\cdot,\cdot]_{\oplus_{TM}}, \rho_{\oplus_{TM}})$ is a Lie algebroid over M, called the *direct sum of Lie algebroids* $(E_1, [\cdot,\cdot]_{E_1}, \rho_1)$ and $(E_2, [\cdot,\cdot]_{E_2}, \rho_2)$. □

The prolongation of a Lie algebroid over a surjective submersion. Let $\pi : P \to M$ be a surjective submersion and $(E, [\cdot,\cdot]_E, \rho)$ a Lie algebroid over M

with anchor $\rho : E \to TM$. Consider $(\mathcal{P}^\pi E, \rho^\pi)$ the anchored vector bundle with anchor ρ^π (see, Section 2). Let $\Gamma(\mathcal{P}^\pi E)$ the space of sections of bundle $\mathcal{P}^\pi E$. An element $Z \in \Gamma(\mathcal{P}^\pi E)$ can be written in the form $Z(p) = (p, \sigma(\pi(p)), X(p))$ where $\sigma \in \Gamma(E), X \in \mathcal{X}(E), (\forall)\, p \in P$.

On the space $\Gamma(\mathcal{P}^\pi E)$ we define the bracket $[\cdot,\cdot]^\pi$ by

$$[Z_1, Z_2]^\pi(p) = (p, [\sigma_1, \sigma_2]_E(\pi(p)), [X_1, X_2](p)), (\forall)\, p \in P. \tag{3.5}$$

We have that $[Z_1, Z_2]^\pi(p) \in \Gamma(\mathcal{P}^\pi E)$ for $(\forall)\, p \in P$. It is easy to prove that $(\mathcal{P}^\pi E, [\cdot,\cdot]^\pi, \rho^\pi)$ is a Lie algebroid over M, called the *prolongation of Lie algebroid* $(E, [\cdot,\cdot]_E, \rho)$ *over* $\pi : P \to M$. □

Let $(E, [\cdot,\cdot]_E, \rho)$ be a Lie algebroid over M. If $(x^i),\ i = \overline{1,m}$ is a local coordinates system on M and $\{e_a | a = \overline{1,n}\}$ is a local basis of sections on the bundle E ($dim\ M = m,\ dim\ E = n$), then $(x^i, z^a),\ i = \overline{1,m}, a = \overline{1,m}$ are local coordinates on E. For an element $z \in E$ such that $x = p(z) \in U \subset M$, we have $z = z^a e_a(p(z))$.

In the chosen local coordinates system, the anchor ρ and the Lie bracket $[\cdot,\cdot]_E$ are determined by the differentiable functions ρ_a^i and $C_{ab}^c \in \mathcal{F}(M)$ given by:

$$\overline{\rho}(e_a) = \rho_a^i \tfrac{\partial}{\partial x^i} \quad \text{and} \quad [e_a, e_b]_E = C_{ab}^c e_c, \quad i = \overline{1,m}, a, b, c = \overline{1,n} \tag{3.6}$$

The functions ρ_a^i and $C_{ab}^c \in \mathcal{F}(M)$ given by the relations (3.6) are called *structure functions* of Lie algebroid $(E, [\cdot,\cdot]_E, \rho)$ in the chosen local coordinates system.

Proposition 3.2. *Let $(E, [\cdot,\cdot]_E, \rho)$ be a Lie algebroid over M, $(x^i),\ i = \overline{1,m}$ a local coordinates system on M and $\{e_a | a = \overline{1,n}\}$ a local basis of sections on E.*

The structure functions $\rho_a^i,\ C_{ab}^c \in \mathcal{F}(M)$ of the Lie algebroid E verify the following relations:

$$\rho_a^j \frac{\partial \rho_b^i}{\partial x^j} - \rho_b^j \frac{\partial \rho_a^i}{\partial x^j} = \rho_c^i C_{ab}^c \tag{3.7}$$

$$C_{ab}^c = -C_{ba}^c \qquad \textit{and} \qquad \sum_{ciclic(a,b,c)} \left(\rho_a^i \frac{\partial C_{bc}^d}{\partial x^i} + C_{ab}^e C_{ce}^d \right) = 0 \ . \tag{3.8}$$

Proof. By condition (3.1) from Definition 3.1, taking $\sigma = e_a$ and $\eta = e_b$ we have $\overline{\rho}([e_a, e_b]_E) = [\overline{\rho}(e_a), \overline{\rho}(e_b)]$. Taking account into the relations (3.6) and the fact that $\overline{\rho}$ is a morphism of $\mathcal{F}(M)$- modules, we obtain $\overline{\rho}([e_a, e_b]) = \overline{\rho}(C_{ab}^c e_c) = C_{ab}^c \overline{\rho}(e_c) = C_{ab}^c \rho_c^i e_i$.

On the other hand, applying the properties of the usual Lie bracket in the Lie algebra $\mathcal{X}(M)$, we have $[\overline{\rho}(e_a), \overline{\rho}(e_b)] = [\rho_a^j e_j, \rho_b^k e_k] = (\rho_a^j \frac{\partial \rho_b^k}{\partial x^j} - \rho_b^j \frac{\partial \rho_a^k}{\partial x^j}) e_k$.

Equaling the local expressions of two sides, we obtain the relation (3.7).

Using the fact that the bracket $[\cdot,\cdot]_E$ is antisymmetric, that is $[e_a, e_b]_E = -[e_b, e_a]_E$, and applying the second relation from (3.6) it follows immediately the first equality from (3.8).

Using the Jacobi identity for the bracket $[\cdot,\cdot]_E$ and the antisymmetry property of structure functions C_{ab}^c, we can obtain by direct calculation the second equality from (3.8). □

The equations (3.7) and (3.8) are called the *structure equations* of Lie algebroid $(E, [\cdot,\cdot]_E, \rho)$.

For more information about vector bundles and Lie algebroids, see [3],[7],[8].

4 Linear connections on Lie algebroids

Definition 4.1. *([2])* Let $(E, [\cdot,\cdot]_E, \rho)$ be a Lie algebroid over M with the anchor $\rho : E \to TM$ and the projection $p : E \to M$. A *linear connection* on the Lie algebroid E, is a map $\nabla : \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E)$, $(\sigma, \eta) \longmapsto \nabla(\sigma, \eta) = \nabla_\sigma \eta \in \Gamma(E)$ such that the following conditions hold :

(1) ∇ is $\mathbf{R}$-bilinear;

(2) $\nabla_{f\sigma}\eta = f\nabla_\sigma \eta$, for all $f \in \mathcal{F}(M)$ and $\sigma, \eta \in \Gamma(E)$,

i.e. ∇ is $\mathcal{F}(M)$-homogenous to respect the first argument;

(3) $\nabla_\sigma(f\eta) = (\overline{\rho}(\sigma)f)\eta + f\nabla_\sigma\eta$, for all $f \in \mathcal{F}(M)$ and $\sigma, \eta \in \Gamma(E)$,

i.e. ∇ satisfy a rule of Leibniz type with respect to the external operation which define the structure of $\mathcal{F}(M)$ - module on $\Gamma(E)$.

For $\sigma, \eta \in \Gamma(E)$, the section $\nabla_\sigma \eta \in \Gamma(E)$ is called the *covariant derivative of the section η with respect to section σ.*

Proposition 4.1. *Let ∇ be a linear connection on the Lie algebroid $(E, [\cdot,\cdot]_E, \rho)$. Then for all $a, b \in \mathbf{R}$ and $\sigma, \eta, \omega \in \Gamma(E)$ we have:*

$$\begin{array}{rcl} \nabla_{a\sigma+b\eta}\omega & = & a\nabla_\sigma\omega + b\nabla_\eta\omega \quad \textit{and} \\ \nabla_\sigma(a\eta + b\omega) & = & a\nabla_\sigma\eta + b\nabla_\sigma\omega. \end{array} \tag{4.1}$$

Proof. Taking account into ∇ is a map $\mathbf{R}$ -linear to respect the first argument, can write $\nabla_{a\sigma+b\eta}\omega = \nabla_\sigma(a\sigma + b\eta, \omega) = a\nabla(\sigma, \omega) + b\nabla(\eta, \omega) = a\nabla_\sigma\omega + b\nabla_\eta\omega$. Similarly can prove the second relation. □

Proposition 4.2. *Let $(E, [\cdot,\cdot]_E, \rho)$ be a Lie algebroid over M with the structure functions ρ_a^i and $C_{ab}^c \in \mathcal{F}(M)$ in a local coordinates system (x^i), $i = \overline{1, m}$ on M and a local basis of sections $\{e_a | a = \overline{1, n}\}$ on E. Let $\sigma, \eta \in \Gamma(E)$ such that $\sigma = \sigma^a e_a, \eta = \eta^b e_b$.*

(i) *The Lie bracket of sections σ and η is expressed locally in the following manner:*

$$[\sigma, \eta]_E = \left(\sigma^b \rho_b^i \tfrac{\partial \eta^a}{\partial x^i} - \eta^c \rho_c^i \tfrac{\partial \sigma^a}{\partial x^i} + C_{bc}^a \sigma^b \eta^c\right) e_a \quad . \tag{4.2}$$

(ii) *If ∇ is a linear connection on E, then the local expression of the covariant derivative $\nabla_\sigma \eta$ of the section η with respect to section σ is given by:*

$$\nabla_\sigma \eta = (\sigma^a \rho_a^i \frac{\partial \eta^c}{\partial x^i} + \Gamma_{ab}^c \sigma^a \eta^b) e_c, \quad \textit{where} \quad \Gamma_{ab}^c \in \mathcal{F}(M). \tag{4.3}$$

Proof. (i) Using the fact that $\overline{\rho}$ is a morphism of $\mathcal{F}(M)$- modules and applying the relations (3.2), (3.6), (3.8) we have successively :

$[\sigma, \eta]_E = [\sigma, \eta^c e_c]_E = \eta^c [\sigma, e_c]_E + \overline{\rho}(\sigma)(\eta^c) e_c =$

$= -\eta^c [e_c, \sigma]_E + \overline{\rho}(\sigma)(\eta^c) e_c = -\eta^c [e_c, \sigma^b e_b]_E + \overline{\rho}(\sigma^b e_b)(\eta^c) e_c =$

$= -\eta^c (\sigma^b [e_c, e_b]_E + \overline{\rho}(e_c)(\sigma^b) e_b) + \sigma^b \overline{\rho}(e_b)(\eta^c) e_c =$

$= \sigma^b \eta^c [e_b, e_c]_E - \eta^c \overline{\rho}(e_c)(\sigma^b) e_b + \sigma^b \overline{\rho}(e_b)(\eta^c) e_c =$

$= \sigma^b \eta^c C^a_{bc} e_a - \eta^c \rho^i_c \frac{\partial \sigma^b}{\partial x^i} e_b + \sigma^b \rho^i_b \frac{\partial \eta^c}{\partial x^i} e_c =$

$= C^a_{bc} \sigma^b \eta^c e_a - \eta^c \rho^i_c \frac{\partial \sigma^a}{\partial x^i} e_a + \sigma^b \rho^i_b \frac{\partial \eta^a}{\partial x^i} e_a = (\sigma^b \rho^i_b \frac{\partial \eta^a}{\partial x^i} - \eta^c \rho^i_c \frac{\partial \sigma^a}{\partial x^i} + C^a_{bc} \sigma^b \eta^c) e_a.$

(ii) Because $e_a, e_b \in \Gamma(E)$ and $\nabla(e_a, e_b) \in \Gamma(E)$ follows that $\nabla_{e_a} e_b = \nabla(e_a, e_b) = \Gamma^c_{ab} e_c$ with $\Gamma^c_{ab} \in \mathcal{F}(M)$.

Applying (3.2) and (3.3) from Definition 4.1 and the first relation from (3.6) we have:

$\nabla_\sigma \eta = \nabla_{\sigma^a e_a} \eta = \sigma^a \nabla_{e_a}(\eta^b e_b) = \sigma^a (\overline{\rho}(e_a)(\eta^b) e_b + \eta^b \nabla_{e_a} e_b) =$

$= \sigma^a (\rho^i_a \frac{\partial \eta^b}{\partial x^i} e_b + \eta^b \Gamma^c_{ab} e_c) = \sigma^a \rho^i_a \frac{\partial \eta^b}{\partial x^i} e_b + \sigma^a \eta^b \Gamma^c_{ab} e_c = (\sigma^a \rho^i_a \frac{\partial \eta^c}{\partial x^i} + \Gamma^c_{ab} \eta^b) e_c.$ □

The functions $\Gamma^c_{ab} \in \mathcal{F}(M)$ from the relations (4.3) are called *coefficients of connection* of the linear connection ∇ in the chosen local coordinates system.

If ∇ is a linear connection on the Lie algebroid E, define the map $T : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ by:

$$T(\sigma, \eta) = \nabla_\sigma \eta - \nabla_\eta \sigma - [\sigma, \eta]_E, \quad (\forall)\, \sigma, \eta \in \Gamma(E). \tag{4.4}$$

Proposition 4.3. *Let* $(E, [\cdot,\cdot]_E, \rho)$ *be a Lie algebroid with the structure functions* $C^c_{ab} \in \mathcal{F}(M)$*. If* ∇ *is a linear connection on* $(E, [\cdot,\cdot]_E, \rho)$*, then :*

(i) *the map* T *given by (4.4) is* $\mathbf{R}$ *-bilinear and antisymmetric ;*

(ii) *For all* $\sigma, \eta \in \Gamma(E)$ *such that* $\sigma = \sigma^a e_a$ *and* $\eta = \eta^b e_b$ *the following relations hold :*

$$T(\sigma, \eta) = (\Gamma^c_{ab} - \Gamma^c_{ba} - C^c_{ab}) \sigma^a \eta^b e_c; \tag{4.5}$$

$$T^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba} - C^c_{ab} \quad \textit{and} \quad T^c_{ab} = -T^c_{ba} \tag{4.6}$$

Proof. (i) For all $a, b \in \mathbf{R}$, $\sigma, \eta, \omega \in \Gamma(E)$ we have $T(a\sigma + b\eta, \omega) = aT(\sigma, \omega) + bT(\eta, \omega)$, that is T is linear with respect to the first argument. Indeed, applying the properties of Lie bracket $[\cdot,\cdot]_E$ and the relations (4.1) we have successively:

$T(a\sigma + b\eta, \omega) = \nabla_{a\sigma + b\eta} \omega - \nabla_\omega (a\sigma + b\eta) - [a\sigma + b\eta, \omega]_E =$

$= a\nabla_\sigma \omega + b\nabla_\eta \omega - (a\nabla_\omega \sigma + b\nabla_\omega \eta) - (a[\sigma, \omega]_E + b[\eta, \omega]_E) =$

$= a(\nabla_\sigma \omega - \nabla_\omega \sigma - [\sigma, \omega]_E) + b(\nabla_\eta \omega - \nabla_\omega \eta - [\eta, \omega]_E) = aT(\sigma, \omega) + bT(\eta, \sigma).$

Similarly prove that T is linear with respect to the second and the third argument.

Applying (4.4) follows immediately that $T(\sigma, \eta) = -T(\eta, \sigma)$, i.e. T is antisymmetric.

(ii) Using (4.4) and (4.3) we have suasively:
$T(\sigma, \eta) = \nabla_\sigma \eta - \nabla_\eta \sigma - [\sigma, \eta]_E = (\sigma^a \rho^i_a \frac{\partial \eta^c}{\partial x^i} + \Gamma^c_{ab} \sigma^a \eta^b) e_c - (\eta^a \rho^i_a \frac{\partial \sigma^c}{\partial x^i} + \Gamma^c_{ab} \eta^a \sigma^b) e_c - (\sigma^a \rho^i_a \frac{\partial \eta^c}{\partial x^i} - \eta^a \rho^i_a \frac{\partial \sigma^c}{\partial x^i} + C^c_{ab} \sigma^a \eta^b) e_c = (\Gamma^c_{ab} - \Gamma^c_{ba} - C^c_{ab}) \sigma^a \eta^b e_c.$
Therefore, the relation (4.5) holds.
From the fact that $T(e_a, e_b) \in \Gamma(E)$ follows that $(\exists)\, T^c_{ab} \in \mathcal{F}(M)$ such that $T(e_a, e_b) = T^c_{ab} e_c$.

On the other hand, in the relation (4.5) replace $\sigma = e_a = \delta^u_a e_u, \eta = e_b = \delta^v_b e_v$ and we obtain $T(e_a, e_b) = \delta^u_a \delta^v_b (\Gamma^c_{uv} - \Gamma^c_{vu} - C^c_{uv}) e_c = (\Gamma^c_{ab} - \Gamma^c_{ba} - C^c_{ab}) e_c.$

Therefore $T_{ab}^c = \Gamma_{ab}^c - \Gamma_{ba}^c - C_{ab}^c$. Hence the first relation from (4.6) holds.

From $C_{ab}^c = -C_{ba}^c$ and the first equality of (4.6) follows

$$T_{ab}^c = \Gamma_{ab}^c - \Gamma_{ba}^c - C_{ab}^c = \Gamma_{ab}^c - \Gamma_{ba}^c + C_{ba}^c = -T_{ba}^c. \qquad \square$$

From Proposition 4.3 implies that T is a tensor of type $(2,1)$. The tensor T is called the *torsion of the linear connection* ∇. The differentiable functions $T_{ab}^c \in \mathcal{F}(M)$ are called *coefficients of torsion* of the linear connection ∇.

Proposition 4.4. *Let ∇ be a linear o connection on a Lie algebroid Lie $(E, [\cdot,\cdot]_E, \rho)$. Then for every $\sigma \in \Gamma(E)$ such that $\sigma = \sigma^a e_a$ the following relations hold:*

$$(4.7) \qquad C_{ab}^c \sigma^a \sigma^b = 0 \quad \textit{and} \quad (\Gamma_{ab}^c - \Gamma_{ba}^c - C_{ab}^c)\sigma^a\sigma^b = 0, \ \textit{for all } a,b,c = \overline{1,n}.$$

Proof. Applying the relations (4.2) and taking account into $[\sigma,\sigma]_E = 0$, we obtain the first equality from (4.7).

For $\sigma = \sigma^a e_a$, apply (4.5) and obtain $T(\sigma,\sigma) = (\Gamma_{ab}^c - \Gamma_{ba}^c + C_{ba}^c)\sigma^a\sigma^b e_c$. Since $T(\sigma,\sigma) = 0$, follows immediately the second equality from (4.7). $\square$

If ∇ is a linear connection on Lie algebroid E, define the map

$$R : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E), \quad (\sigma,\eta,\omega) \longmapsto R(\sigma,\eta,\omega) = R(\sigma,\eta)\omega,$$

where the section $R(\sigma,\eta)\omega$ is given by:

$$(4.8) \qquad R(\sigma,\eta)\omega = \nabla_\sigma\nabla_\eta\omega - \nabla_\eta\nabla_\sigma\omega - \nabla_{[\sigma,\eta]_E}\omega, \quad \text{for all } \sigma,\eta,\omega \in \Gamma(E).$$

Proposition 4.5. *If ∇ is a linear connection on the Lie algebroid $(E, [\cdot,\cdot]_E, \rho)$, then the following assertions hold :*

(i) *the map R is $\mathbf{R}$ -linear in every argument ;*

(ii) *the map R is antisymmetric with respect to the first two arguments, that is :*

$$(4.9) \qquad R(\sigma,\eta,\omega) = -R(\eta,\sigma,\omega), \quad \textit{for all } \sigma,\eta,\omega \in \Gamma(E);$$

(iii) *R has the property :*

$$(4.10) \qquad R(\sigma,\sigma,\omega) = 0, \quad \textit{for all } \sigma,\omega \in \Gamma(E).$$

Proof. (i) The equality $R(a\sigma_1 + b\sigma_2,\eta,\omega) = aR(\sigma_1,\eta,\omega) + bR(\sigma_2,\eta,\omega)$ for all $a,b \in \mathbf{R}$ and $\sigma_1,\sigma_2,\eta,\omega \in \Gamma(E)$ can verified by direct calculation, using the properties of covariant derivative and the properties of the Lie bracket. Hence, R is linear with respect to the first argument. Similarly can prove that R is linear with respect to the other arguments.

(ii) Because $\nabla_{[\sigma,\eta]_E} = -\nabla_{[\eta,\sigma]_E}$ we have $R(\sigma,\eta,\omega) = R(\sigma,\eta)\omega =$

$= \nabla_\sigma\nabla_\eta\omega - \nabla_\eta\nabla_\sigma\omega - \nabla_{[\sigma,\eta]_E}\omega = \nabla_\sigma\nabla_\eta\omega - \nabla_\eta\nabla_\sigma\omega + \nabla_{[\eta,\sigma]_E}\omega =$

$= -(\nabla_\sigma\nabla_\eta\omega - \nabla_\sigma\nabla_\eta\omega - \nabla_{[\eta,\sigma]_E}\omega) = -R(\eta,\sigma)\omega = -R(\eta,\sigma,\omega).$

Hence (4.9) holds.

(iii) Equality (4.10) follows immediately from (4.9). $\square$

The map R defined by (4.8) is called *curvature* of the linear connection ∇.

Proposition 4.6. *Let $(E_i, [\cdot,\cdot]_{E_i}, \rho_i), i = 1,2$, be two Lie algebroids over M with property that $(E_2, [\cdot,\cdot]_{E_2}, \rho_2)$ is transitive.*

If $\nabla^i : \Gamma(E_i) \times \Gamma(E_i) \to \Gamma(E_i), i = 1,2$ is a linear connection on E_1 resp. E_2, then the map $\nabla^{\oplus} : \Gamma(E_1 \oplus_{TM} E_2) \times \Gamma(E_1 \oplus_{TM} E_2) \to \Gamma(E_1 \oplus_{TM} E_2)$ given by:

$$\nabla^{\oplus}(X_1 \oplus X_2, Y_1 \oplus Y_2) = \nabla^{\oplus}_{X_1 \oplus X_2}(Y_1 \oplus Y_2) = \nabla^1_{X_1} Y_1 \oplus \nabla^2_{X_2} Y_2 \tag{4.11}$$

for all $X_1 \oplus X_2, Y_1 \oplus Y_2 \in \Gamma(E_1 \oplus_{TM} E_2)$ is a linear connection on the direct sum $E_1 \oplus_{TM} E_2$ of Lie algebroids E_1 and E_2 over TM.

Proof. It is not hard to verify the conditions from definition of a linear connection. For instance, verify the condition (2) of Definition 4.1.

Denoting $E = E_1 \oplus_{TM} E_2$, for $f \in \mathcal{F}(M),\ X = X_1 \oplus X_2 \in \Gamma(E)$ and $Y = Y_1 \oplus Y_2 \in \Gamma(E)$ we have:

$\nabla^{\oplus}_{fX} Y = \nabla^{\oplus}(fX, Y) = \nabla^{\oplus}(fX_1 \oplus fX_2, Y_1 \oplus Y_2) =$

$= \nabla^1_{fX_1} Y_1 \oplus \nabla^2_{fX_2} Y_2 = (f\nabla^1_{X_1} Y_1) \oplus (f\nabla^2_{X_2} Y_2) =$

$= f(\nabla^1_{X_1} Y_1 \oplus \nabla^2_{X_2} Y_2) = f\nabla^{\oplus}_{X_1 \oplus X_2}(Y_1 \oplus Y_2) = f\nabla_X Y.$ □

Proposition 4.7. *Let $(E, [\cdot,\cdot]_E, \rho)$ be a Lie algebroid over M, $\pi : P \to M$ a surjective submersion and $(\mathcal{P}^\pi E, [\cdot,\cdot]^\pi, \rho^\pi)$ the prolongation of E by π.*

If $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is a linear connection on E, then the map $\nabla^\pi : \Gamma(\mathcal{P}^\pi E) \times \Gamma(\mathcal{P}^\pi E) \to \Gamma(\mathcal{P}^\pi E)$, $(Z, W) \longmapsto \nabla^\pi(Z, W) = \nabla^\pi_Z W$ defined by :

$$\nabla^\pi(Z, W)(p) = (p, \nabla_\sigma \eta(\pi(p)), [X, Y](p)), \quad (\forall)\, p \in P \tag{4.12}$$

where $Z(p) = (p, \sigma(\sigma(p)), X(p))$, $W(p) = (p, \eta(\pi(p)), Y(p))$ with $\sigma, \eta \in \Gamma(E)$ and $X, Y \in \mathcal{X}(E)$.

Proof. It is easy to verify the conditions from the definition of a linear connection for ∇^π, taking account of the properties of the linear connection ∇ and the properties of the Lie brackets $[\cdot,\cdot]_E$ on E and $[\cdot,\cdot]$ on $\mathcal{X}(E)$. For example, we verify the condition (2) of Definition 4.1. For $f \in \mathcal{F}(M), (Z, W) \in F \times F$, where $F = \Gamma(\mathcal{P}^\pi E)$ we have:

$\nabla^\pi_{fZ} W(p) = (p, \nabla_{f\sigma} \eta(\pi(p)), [X, Y](p)) = (p, f(\pi(p))\nabla_\sigma \eta(\pi(p)), [X, Y](p))$ and

$f\nabla^\pi_Z w(p) = f(\pi(p))(p, \nabla_\sigma \eta(\pi(p)), [X, Y](p)) = (p, f(\pi(p))\nabla_\sigma \eta(\pi(p), [X, Y](p)).$

Then $\nabla^\pi_{fZ} W(p) = f\nabla^\pi_Z W(p)$ for all $p \in P$. Hence $\nabla^\pi_{fZ} W = f\nabla^\pi_Z W$. □

For more details concerning the applications of Lie algebroids in differential geometry and quantum mechanics, the reader can consult the papers [1],[2],[5],[9].

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