Invariant conformal geometry on Finsler manifolds

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Abstract

The electric capacity of a conductor in the 3-dimensional Euclidean space $I\!R^3$ is defined as a ratio of a given positive charge on the conductor to the value of potential on the surface. This definition of the capacity is independent of the given charge. The capacity of a set as a mathematical notion was defined first by N. Wiener (1924) and was developed by O. Forstman [8]], C. J. de La Vallee Poussin, and several other French mathematicians in connection with potential theory. This paper develops the theory of conformal invariants initiated in [6] for Finsler manifolds. More precisely we prove: The capacity of a compact set and the capacity of the condenser of two closed sets are conformally invariant. By mean of the notion of capacity, we construct and study four conformal invariant functions ρ_M, ν_M, μ_M and λ_M which have similarities with the classical invariants on S^n, $\mathbb{R}$ or H^n. Their properties and especially their continuity are efficient tools for solving some problems of conformal geometry in the large.

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Introduction

The notion of conformal capacity was introduced by Loewner [13] and has been extensively developed for $\mathbb{R}$ (for instance [9], [10], [15], [18]). Particularly it was used by G.D Mostow to prove his famous theorem on the rigidity of hyperbolic spaces [15]. J.Ferrand proved that, the capacity of compact sets in Riemannian manifolds is invariant under conformal mappings and then she used this notion to prove her famous theorem in Riemannian conformal geometry [4]. Here, inspiring her method, we define an equivalent notion of capacity in Finsler geometry and prove its invariance property under conformal mappings.

1 Preliminaries

1.1 Finsler metric

Let M be a n-dimensional C^∞ manifold. For a point $x \in M$, denoted by T_xM the tangent space of M at x. The tangent bundle of M is the union of tangent spaces.

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$$TM := \cup_{x\in M} T_xM.$$

We will denote the elements of TM by (x, y) where $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) := x$. Throughout this paper, we use *Einstein summation convention* for the expressions with indices.
A *Finsler structure* on a manifold M is a function $F : TM_0 \to [0, \infty)$ with the following properties:
(i) F is C^∞ on TM_0.
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM:

$$\forall\lambda > 0 \quad F(x, \lambda y) = \lambda F(x, y).$$

(iii) The Hessian of F^2 with elements $g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^iy^j}$ is positively defined on TM_0.

Then the pair (M, F) is called a *Finsler manifold.* F is Riemannian if $g_{ij}(x, y)$ are independent of $y \neq 0$.

1.2 Notations and definitions on conformal geometry of Finsler manifolds

A diffeomorphism $f : (M, g) \to (N, h)$ between n-dimensional Finsler manifolds (M, g) and (N, h) is called *conformal* if each $(f_*)_p$ for $p \in M$ is angle-preserving, and in this case two Finsler manifolds are called *conformally equivalent* or simply *conformal.* If $M = N$ then$f$ is called a *conformal transformation or conformal automorphism.* It can be easily checked that a diffeomorphism is conformal if and only if [1], $f^*h = e^{2\sigma}g$ for some function $\sigma : M \to \mathbb{R}$. The diffeomorphism f is called an *isometry* if $f^*h = g$.
Now let's consider two Finsler manifolds (M, g) and $(\overline{M}, \overline{g})$ with Finsler structures F and $\bar{F}$ and with line elements (x, y) and $(\bar{x}, \bar{y})$ respectively. Throughout this paper we shall always assume that coordinate systems on (M, g) and $(\overline{M}, \overline{g})$ have been chosen so that $\overline{x^i} = x^i$ and $\overline{y^i} = y^i$ holds, unless a contrary assumption is explicitly made. Using this assumption we can show them by (M, g) and $(M, \overline{g})$ or simply by M and $\overline{M}$. Then this two manifolds are conformal if $\overline{F}(x, y) = e^\sigma F(x, y)$ or equivalently

$$\overline{g} = e^{2\sigma(x)}\ g\ .$$

Locally we have $\overline{g}_{ij}(x, y) = e^{2\sigma(x)}\ g_{ij}(x, y)$, and $\overline{g}^{ij}(x, y) = e^{-2\sigma(x)}\ g^{ij}(x, y)$.

1.3 Some vector spaces and their properties

1.3.1 Pull-back space π^*TM

Let $\pi : TM \longrightarrow M$ be the natural projection from TM to M.
The *pull-back tangent space* π^*TM defined by

$$\pi^*TM := \{(x, y, v) |\, y \in T_xM_0,\ v \in T_xM\}.$$

[1] This result is due to Knebelman [11]. In fact the sufficient condition implies that the function $\sigma(x, y)$ be independent of direction y, or equivalently $\partial\sigma/\partial y^i = 0$.

The *pull-back cotangent space* π^*T^*M defined by

$$\pi^*T^*M := \{\pi^*\theta |\, \theta \in T^*M\}.$$

Both π^*TM and π^*T^*M are n-dimensional vector spaces over TM_0.

1.3.2 Sphere bundle SM

Let us denote by S_xM the set consisting of all rays $[y] := \{\lambda y |\, \lambda > 0\}$, where $y \in T_xM_0$. Let

$$SM = \bigcup_{x \in M} S_xM.$$

SM has a natural $(2n-1)$ dimensional manifold structure, called *Sphere bundle* over M. We denote the elements of SM by $(x, [y])$ where $y \in T_xM_0$ [2].

Lemma 1. *[3] The Sphere bundle of a differentiable manifold is orientable.*

1.3.3 Pull-back space p^*TM

Let $p : SM \longrightarrow M$ denotes the natural projection from SM to M. The *pull-back tangent space* p^*TM is defined by

$$p^*TM := \{(x, [y], v) |\, y \in T_xM_0,\, v \in T_xM\}.$$

The *pull-back cotangent space* p^*T^*M is defined by

$$p^*T^*M := \{p^*\theta |\, \theta \in T^*M\}.$$

Both p^*TM and p^*T^*M are n-dimensional vector spaces over SM.
Let we define the function η as follows

$$\eta : TM_0 \longrightarrow SM,$$

$$\eta(x, y) = (x, [y]).$$

We use the following lemma for replacing the C^∞ functions on TM_0 by those on SM.

Lemma 2. *[16] Let $f \in C^\infty(TM_0)$. Then there exist a function $g \in C^\infty(SM)$ satisfying $\eta^*g = f$ if and only if*

$$f(x, y) = f(x, \lambda y), \qquad y \in T_xM_0, \lambda > 0,$$

where η^* is the pull-back of η.

Let $f \in C^\infty(M)$, the vertical lift of f is denoted by $f^V \in C^\infty(TM_0)$ and defined by

$$f^V : TM \longrightarrow I\!R$$

$$f^V(x, y) := f \circ \pi(x, y) = f(x).$$

f^V is independent of y and from lemma 2 there is a function g on $C^\infty(SM)$ related to f^V by means of $\eta^*g = f^V$. We denote g in the sequel by f^V for simplicity.

1.4 Nonlinear connection

1.4.1 On the tangent bundle TM

Consider $\pi_* : TTM \longrightarrow TM$ and let we put $ker\pi^v_* = \{z \in TTM | \, \pi^v_*(z) = 0\}, \ \forall v \in TM$, then the vertical vector bundle on M is defined by

$$VTM = \bigcup_{v \in TM} ker\pi^v_*.$$

A *non-linear connection* or a *horizontal distribution* on TM is a complementary distribution HTM for VTM on TTM. The non-linear nomination arise from the fact that HTM is spanned by the functions which are completely determined by the differentiable non-linear functions. These functions are called coefficients of the non-linear connection and will be noted in the sequel by N^j_i. It is clear that HTM is a horizontal vector bundle. By definition we have the decomposition $TTM = VTM \oplus HTM$.

Using the induced coordinates (x^i, y^i) on TM, where x^i and y^i are called respectively *position* and *direction* of a point on TM, we have the local field of frames $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}$ on TTM. Let $\{dx^i, dy^i\}$ be the dual of $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$. It is well known that we can choose a local field of frames $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y_i}\}$ adapted to the above decomposition i.e. $\frac{\delta}{\delta x^i} \in \mathcal{X}(HTM)$ and $\frac{\partial}{\partial y_i} \in \mathcal{X}(VTM)$. They are sections of horizontal and vertical sub-bundle on HTM and VTM, defined by $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x_i} - N^j_i \frac{\partial}{\partial y_j}$, where $N^j_i(x, y)$ are the coefficients of non linear connection. Clearly

$$N^i{}_j = \gamma^i{}_{jk} y^k - C^i{}_{jk} \gamma^k{}_{rs} y^r y^s,$$

where $\gamma^i{}_{jk} := \frac{1}{2} g^{is} (\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j})$ and $C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$.

1.4.2 On the sphere bundle SM

Using the coefficients of non linear connection on TM one can define a non linear connection on SM by using the objects which are invariant under positive re-scaling $y \mapsto \lambda y$. Our preference for being on SM dictates us to work with

$$\frac{N^i{}_j}{F} := \gamma^i{}_{jk} l^k - C^{\,i}{}_{jk} \gamma^k{}_{rs} l^r l^s,$$

where $l^i = \frac{y^i}{F}$.

We prefer also to work with the local field of frames $\{\frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^j}\}$ and $\{dx^i, \frac{\delta y^j}{F}\}$ which are invariant under the positive re-scaling of y and can be used as a local field of frame for tangent bundle p^*TM and cotangent bundle p^*T^*M over SM respectively.

1.5 Riemannian metrics on SM

It turns out that the manifold TM_0 has a natural Riemannian metric (known in the literature as *Sasaki metric* [2], [14])

$$\widetilde{g} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F},$$

where $g_{ij}(x,y)$ are the Hessian of Finsler structure F^2. They are functions on TM_0 and invariant under positive re-scaling of y, therefore they can be considered as functions on SM. With respect to this metric, the *horizontal subspace* spanned by $\frac{\delta}{\delta x^j}$ is orthogonal to the *vertical subspace* spanned by $F\frac{\partial}{\partial y^i}$.
The metric $\widetilde{g}$ is invariant under the positive re-scaling of y and can be considered as a Riemannian metric on $S(M)$.

1.6 The Hilbert form

Consider the pull-back vector bundle p^*TM over SM. The pull-back tangent bundle p^*TM has a canonical section l defined by

$$l_{(x,[y])} = (x, [y], \frac{y}{F(x,y)}).$$

We use the local coordinate system (x^i, y^i) for SM, where y^i being homogeneous coordinates up to a positive factor. Let $\partial_i := (x, [y], \frac{\partial}{\partial x^i})$. $\{\partial_i\}$ is a natural local field of frames for p^*TM. The natural dual co-frame for p^*T^*M is $\{dx^i\}$. The Finsler structure $F(x,y)$ induces a canonical 1-form on SM defined by

$$\omega := l_i dx^i,$$

where $l_i = g_{ij}l^j$.
ω is called *Hilbert form* of F. Using $g_{ij} = FF_{y^iy^j} + F_{y^i}F_{y^j}$ and $\frac{\delta F}{\delta x^i} = 0$, with straight forward calculation we get

$$d\omega = -(g_{ij} - l_i l_j)dx^i \wedge \frac{\delta y^j}{F}. \tag{1.1}$$

1.7 The gradient vector field

For a Riemannian manifold $(S(M), \widetilde{g})$, the gradient vector field of a function $f \in C^\infty(S(M))$ is given by

$$\widetilde{g}(\nabla f, \widetilde{X}) = df(\widetilde{X}), \qquad \forall\ \widetilde{X} \in \mathcal{X}(SM).$$

Using the local coordinate system $(x^i, [y^i])$ for SM, the vector field $\widetilde{X} \in \mathcal{X}(SM)$ is given by $\widetilde{X} = X^i(x,y)\frac{\delta}{\delta x^i} + Y^i(x,y)F\frac{\partial}{\partial y^j}$ where $X^i(x,y)$ and $Y^i(x,y)$ are C^∞ functions on SM. Using straight forward calculation we get locally

$$\nabla f = g^{ij}\frac{\delta f}{\delta x^i}\frac{\delta}{\delta x^j} + F^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^j}.$$

The norm of ∇f with respect to the Riemannian metric $\widetilde{g}$ is given by

$$|\nabla f|^2 = \widetilde{g}(\nabla f, \nabla f) = g^{ij}\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + F^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}. \tag{1.2}$$

2 Conformal invariants

In what follows (M, g) denotes a connected Finsler manifold of class C^1 and dimension $n \geq 2$. Let $(S(M), \widetilde{g})$ be its Riemannian Sphere bundle, we set at first some definitions and notations.
Let's consider the Volume element $\eta(g)$ on $S(M)$ defined as follows [1]

$$\eta(g) := \frac{(-1)^N}{(n-1)!}\, \omega \wedge (d\omega)^{n-1},$$

where $N = \frac{n(n-1)}{2}$ and ω is a Hilbert form of F.

Let $H(M) = \mathcal{C}(M) \cap W_n^1(M)$ be the linear space of continuous real valued functions u on M admitting a generalized L^n-integrable differential, satisfying

$$I(u, M) = \int_{S(M)} \mid \nabla u^V \mid^n \eta(g) < \infty,$$

where u^V is the vertical lift of u.

If M is non-compact then $H_0(M)$ is the subspace of functions $u \in H(M)$ such that its vertical lift u^V has a compact support in $S(M)$.

Definition 1. *A function $u \in \mathcal{C}(M)$ will be called monotone if for any relatively compact domain D of M*

$$\sup_{x \in \partial D} u(x) = \sup_{x \in D} u(x), \qquad \inf_{x \in \partial D} u(x) = \inf_{x \in D} u(x).$$

We denote by $H^*(M)$ the set of monotone functions $u \in H(M)$.

Definition 2. *The capacity of a compact subset C of a non-compact Finslerian manifold M is defined by*

$$Cap_{M}(C) := \inf_{u} I(u, M),$$

where the infimum is taken over the functions $u \in H_0(M)$ with $u = 1$ on C and $0 \leq u(x) \leq 1$ for all x, these functions being said to be admissible for C.

Definition 3. *Let (C_0, C_1) be a pair of closed sets in Finslerian manifold M. The capacity of the condenser $\Gamma(C_0, C_1, M)$ is defined by*

$$Cap_{M}(C_0, C_1) = \inf_{u \in A(C_0, C_1)} I(u, M),$$

where the infimum is taken over the set $A(C_0, C_1)$ of all functions $u \in H(M)$ satisfying $u = 0$ on C_0 and $u = 1$ on C_1 and $0 \leq u(x) \leq 1$ for all x, these functions are called admissible for condenser $\Gamma(C_0, C_1, M)$. If $A(C_0, C_1) = \emptyset$ and particulary if $C_0 \bigcap C_1 \neq \emptyset$, we set $Cap_{M}(C_0, C_1) = +\infty$.

Definition 4. *A relative continuum is a closed subset C of M such that $C \cup \{\infty\}$ is connected in Alexandrov's compactification $\overline{M} = M \cup \{\infty\}$. For avoiding ambiguities the connected closed sets of M which are not reduced to one point will be called continua.*

In what follows we want to associate the conformal invariant functions determined entirely by conformal structure of manifold M, at every double, triple and quaternary points of M.

Definition 5. *For all (x_1, x_2) in M^2 we set*

$$\mu_{_M}(x_1, x_2) = \inf_{C \in \alpha(x_1, x_2)} Cap_{_M}(C),$$

where $\alpha(x_1, x_2)$ is the set of all compact continua subsets of M, containing x_1 and x_2. And we set

$$\lambda_{_M}(x_1, x_2) = \inf_{C_0, C_1} Cap_{_M}(C_0, C_1),$$

where C_0 and C_1 are relative continua resp. containing x_1 and x_2.

Definition 6. *Let $\triangle = \{(x, x, x) |\ x \in M\}$ be the diagonal of M^3. For any $(x_1, x_2, x_3) \in M^3 \backslash \triangle$ we set*

$$\nu_{_M}(x_1, x_2, x_3) = \inf_{C_0, C_1} Cap_{_M}(C_0, C_1),$$

where C_0 is a relative continuum containing x_3 and C_1 a compact continuum containing x_1 and x_2.

Definition 7. *Let $\triangle$ be the set of all points (x_1, x_2, x_3, x_4) of M^4 such that at least three coordinates of which are equal , and $\overline{I\!R}_+ = I\!R_+ \cup \{+\infty\}$. We define a function $\rho_{_M} : M^4 \backslash \triangle \longrightarrow \overline{I\!R}_+$ by setting $\rho_{_M}(x_1, x_2, x_3, x_4) = +\infty$ if $\{x_1, x_2\} \cap \{x_3, x_4\} \neq \emptyset$ and in all other cases*

$$\rho_{_M}(x_1, x_2, x_3, x_4) = \inf_{C_0,\, C_1} Cap_{_M}(C_0, C_1),$$

where C_0 is a compact continuum containing x_1, x_2 and C_1 a compact continuum containing x_3, x_4.

Definition 8. *For any subset S of M and any $u \in \mathcal{C}(M)$, we denote by $\omega(u, S)$ the oscillation of u on S.*

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3 Conformal properties of capacity

Let $f : M \longrightarrow M'$ be a diffeomorphism between two manifolds and h the mapping

$$h : S(M) \longrightarrow S(M'),$$

$$h(x, [y]) = (f(x), [f_*(y)]),$$

where f_* is the differential of f (the tangent map, [16]). Since f_* is linear, h is well defined.

Let f be a conformal map between Finsler manifolds (M, g) and (M', g'), with the Finsler structures F and F' respectively. With respect to the function λ on M and ω' be a Hilbert form related to the Finsler structure F'. In other word $\omega' = g'_{ij}\frac{y'^j}{\sqrt{g'_{mn}y'^m y'^n}}dx'^i$, we have

$$h^*\omega' = \sqrt{\lambda}\omega.$$

from (1.1) we get

$$h^*d\omega' = \sqrt{\lambda}d\omega.$$

So if $\eta(g)$ and $\eta(g')$ denotes the volume elements of $S(M)$ and $S(M')$ respectively, then we find that

$$h^*(\eta(g')) = (\sqrt{\lambda})^n\eta(g). \tag{3.1}$$

Therefore the mapping h is orientation preserving diffeomorphism from $S(M)$ to $S(M')$. With above notions we have the following lemma.

Lemma 3. *If* $u \in H_0(M')$ *then we have*

I) $\mid \nabla u^V \mid^n = (g'^{ij}\frac{\delta u^V}{\delta x'^i}\frac{\delta u^V}{\delta x'^j})^{\frac{n}{2}}$,

II) $(uof)^V = u^V oh$,

III) $h^*\frac{\delta u^V}{\delta x'^i} = \frac{\delta(u\circ f)^V}{\delta x^i}$.

Proof. The first assertion follows from (1.2), II) and III) can be easily verified by direct calculations. □

From the above lemma we have

$$h^* \mid \nabla u^V \mid^n = (\sqrt{\lambda})^{-n} \mid \nabla(u \circ f)^V \mid^n . \tag{3.2}$$

Now we can prove the following theorem. It shows that, the capacity of a compact set and the capacity of the condenser of two closed sets are conformally invariant, i.e. they only depend on the conformal structure.

Theorem 1. *Let* f *be a conformal map between two Finsler manifolds* (M, g) *and* (M', g')*. Then we have*

$$Cap_{_M}(C) = Cap_{_{M'}}(f(C)), \quad Cap_{_M}(C_0, C_1) = Cap_{_{M'}}(f(C_0), f(C_1)),$$

for every compact subset C *and closed subsets* C_0 *and* C_1 *of* M*.*

Proof. Let $f : (M, g) \longrightarrow (M', g')$ be a conformal map. First we prove

$$I(u, M') = I(u \circ f, M), \tag{3.3}$$

for every $u \in H_0(M')$. By definition

$$I(u, M') = \int_{S(M')} |\nabla u^V|^n \eta(g'),$$

Since $S(M)$ and $S(M')$ are two orientable n-dimensional smooth manifolds with boundary and h is a smooth and orientation preserving diffeomorphism between them, we have (see for example p. 245, [12])

$$\int_{S(M')} |\nabla u^V|^n \eta(g') = \int_{S(M)} h^*(|\nabla u^V|^n \eta(g')).$$

Using equation (3.1) and (3.2) gives

$$\int_{S(M)} h^*(|\nabla u^V|^n \eta(g')) = \int_{S(M)} |\nabla(u \circ f)^V|^n \eta(g) = I(u \circ f, M).$$

Let C be a compact set in M by definition

$$Cap_M(C) = \inf_{v \in H_0 M,\ v|_C = 1} I(v, M), \qquad Cap_{M'}(f(C)) = \inf_{u \in H_0 M',\ u|_{f(C)} = 1} I(u, M').$$

Putting

$$A = \{I(v, M) | v \in H_0 M, v|_C = 1\},$$
$$B = \{I(u, M') | u \in H_0 M', u|_{f(C)} = 1\},$$

since $f^{-1}($ support $u) =$ support $(u \circ f)$ for all $I(u, M') \in B$, we have $(u \circ f) \in H_0(M)$. On the other hand $(u \circ f)|_C = 1$ and from relation (3.3), $I(u, M') = I(u \circ f, M)$. Hence $B \subseteq A$.

By the same argument we can prove $A \subseteq B$. Therefore $Cap_M(C) = Cap_{M'}(f(C))$.
Let C_0 and C_1 be closed subsets of M. By putting

$$A = \{I(v, M) | v \in H_0 M, v|_{C_0} = 0, v|_{C_1} = 1\},$$

$$B = \{I(u, M') | u \in H_0 M', u|_{f(C_0)} = 0, u|_{f(C_1)} = 1\},$$

with the same argument we can prove $Cap_M(C_0, C_1) = Cap_{M'}(f(C_0), f(C1))$. □

By mean of the notion of capacity, we can study the properties of four conformal invariant functions ρ_M, ν_M, μ_M and λ_M which have similarities with the classical invariants on S^n, $\mathbb{R}$ or H^n [4], [15]. Their properties and especially their continuity are efficient tools for solving some problems of conformal geometry.

In the following theorem we prove that the functions ρ_M, ν_M, μ_M and λ_M depend only on the conformal structure of M and therefore invariant under any conformal mapping.

Theorem 2. *Let f be a conformal mapping from the Finsler manifold M to the Finsler manifold M' , we have for all x_1, x_2, x_3, x_4 in M*
$\rho_M(x_1, x_2, x_3, x_4) = \rho_{M'}(f(x_1), f(x_2), f(x_3), f(x_4))$,
$\nu_M(x_1, x_2, x_3) = \nu_{M'}(f(x_1), f(x_2), f(x_3))$,
$\mu_M(x_1, x_2) = \mu_{M'}(f(x_1), f(x_2))$,
$\lambda_M(x_1, x_2) = \lambda_{M'}(f(x_1), f(x_2))$.

Proof. The proof is a straight forward conclusion of theorem 1 and definitions 5, 6 and 7. □

References

[1] H. Akbar-Zadeh, *Generalized Einstein manifolds*, Journal of Geometry and Physics 17 (1995), 342-380.

[2] D. Bao, S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, New York, Inc, 2000.

[3] D. Bao, B. Lackey, *Special eigenforms on the sphere bundle of a Finsler manifold*, Contemporary Mathematics, Vol. 196, 1996.

[4] J. Ferrand, *Conformal capacities and conformally invariant functions on Riemannian manifolds*, Geometriae Dedicata 61 (1996), 103-120.

[5] J. Ferrand, *Conformal capacities and conformally invariant metrics*, Pacific J. Math. 180, 1 (1997), 41-50.

[6] J. Ferrand, *Invariants conformes globaux sur les varietes Riemanniennes*, J. Differential Geometry 8 (1973), 487-510.

[7] J. Ferrand, G. J. Martin and M. Vuorinen, *Lipschitz conditions in conformally invariant metrics*, *J. Analyse Math.* 56 (1991), 187-210.

[8] O. Forstman, *Potentiel d'equilibre et capacite des ensembles avec quelques applications a la theorie des fonctions*, Medd. Lunds Univ. Mat. Sem. 3 (1935), 1-11.

[9] F. W. Gehring, *Extremal length definitions for the conformal capacity of rings in space*, *Michigan Math. J.* 9 (1962), 137-150.

[10] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer Math. Soc. 103 (1962), 353-393.

[11] M. S. Knebelman, *Conformal geometry of generalized metric space*, Proc. Nat. Acad. Sci. USA 15 (1929), 376-379.

[12] J.M. Lee, *Introduction to Smooth Manifolds*, *University of Washington*, 2000.

[13] C. Loewner, *On the conformal capacity in space*, J. Math. Mech. 8 (1959), 411-414.

[14] R. Miron, *The geometry of high order Hamilton mechanic*, Kluwer, FTPH, 2003.

[15] G. D. Mostow, *Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Etudes Sci. Publ. Math. 34 (1968), 53-104.

[16] M. Nakahara, *Geometry Topology and Physics*, Graduate student series in Physics, University of Sussex, 1990.

[17] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math. 1319, Springer-Verlag, 1988.

[18] M. Vuorinen, *On Teichmuller's modulus problem in* $\mathbb{R}^n$, Math. Scand. 63 (1988), 315-333.

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