Geometry of shape spaces

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Abstract

This is a brief survey on the geometry of shape spaces. For three types of transformations between images of the same scene, we regard the shape of a configuration of landmarks, often extracted from a digital image, as a point on a shape space. Thsese shape spaces are described as familiar symmetric spaces.

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1 Introduction

High level image analysis, an area of rapid growth of applied sciences, involves a blend of techniques of geometry, statistics and computer science, each having its important contribution to this field.
In this paper we introduce the reader to some of the basic concepts in shape analysis, arising in the particular context of pattern recognition from digital images.
We will assume that the features have been already extracted, using a *landmark* detection method, and at the end of the feature extraction algorithm, we analyze a configuration of points in plane or in three dimensions.
Depending on the type of images considered (frontal view, perspective view, or range image), we analyze the shape of the configuration extracted (direct similarity shape, projective shape or 3D similarity shape). This shape is regarded as a point on a shape space, and the first main objective of this paper is the description of the corresponding shape spaces.
Averaging on shape spaces is useful is scene recognition [8], medical imaging [2] and scene enhancement [13], a motivation of this study.

2 The planar direct similarity shape space

In this section we will follow the approach in [1]. A planar k-ad is an ordered set $(z_1, z_2, \ldots, z_k)$ of k points in $\mathbb{R}^2 = \mathbb{C}$, at least two of which are distinct. We denote by $\mathcal{C}_{k,2}$ the set of planar k-ads. The *direct similarity planar shape space*, Σ_2^k, is the

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shape space $\Sigma_2^k = \Sigma_G(\mathcal{C}_{k,2})$, where G is the group of direct similarities of $\mathbb{R}^2$. A planar k-ad $(z_1, z_2, \ldots, z_k)$ has the same direct similarity shape with a planar k-ad $(\zeta_1, \zeta_2, \ldots, \zeta_k) \in L_k$, centered at 0, where

$$L_k = \{w = (w_1, w_2, \ldots, w_k) \in \mathbb{C}^k \mid \sum_{j=1}^k w_j = 0\}.$$

We set $\zeta_j = z_j - \overline{z}, forall j = 1, \ldots, k$ (translation). We may then assume that two k-ads $z = (z_1, z_2, \ldots, z_k)$ and $z' = (z'_1, z'_2, \ldots, z'_k)$ having the same direct similarity shape, are both in L_k. Two k-ads $z, z' \in L_k$ have the same direct similarity shape, iff for $j = 1, \ldots, k$, z'_j can be obtained from z_j by a multiplication by a complex number of modulus 1 (rotation) and then by a multiplication by a positive real number (scaling). That is $z, z' \in L$ have the same direct similarity shape iff there is a $w \in \mathbb{C}\backslash\{0\}$, s.t. $z' = wz$. In the language of projective geometry, that is same as saying $[z'_1 : z'_2 : \ldots : z'_k] = [z_1 : z_2 : \ldots : z_k]$ in $P(L)$, where $L = \{z \in \mathbb{C}^k \mid \sum_{j=1}^k z_j = 0\}$. Then we have the following result

Theorem 2.1 *The following equality holds:*

$$\Sigma_2^k = P(L_k), \quad L_k = \{(z_1, z_2, \ldots, z_k) \in \mathbb{C}^k \mid z_1 + z_2 + \ldots + z_k = 0\}.$$

Advantages of the new description of Σ_2^k*.* The above identification of Σ_2^k with a complex projective space differs from both [2] and [5]. Kendall pointed out in [5] that there is no unique way to identify Σ_2^k with $\mathbb{C}P^{k-2}$; he used a Helmert submatrix H to *remove location* (see [3, p. 34]). The Helmert submatrix is only an artifact used to drop one complex dimension, and to represent the shape space as $\mathbb{C}P^{k-2}$. The new description is edge registration free, unlike the one in [2]. It is also useful, at the level of shape spaces, in understanding the effect of adding a new point to a given k-ad.

3 The linear shape space and the affine shape space

In this section we will follow the approach in [6], [7]. A k-ad is an ordered set $(x_1, x_2, \ldots, x_k)$ of k points. We will consider the set $\mathcal{C}_{k,m,0}$ of k-ads in general position in $\mathbb{R}^m$. The *linear shape space*, $L\Sigma_m^k$, is the shape space $L\Sigma_m^k = \Sigma_G(\mathcal{C}_{k,m,0})$, where $G = GL(m, \mathbb{R})$ is the general linear group of $\mathbb{R}^m$. Therefore two k-ads $(x_1, x_2, \ldots, x_k)$, $(x'_1, x'_2, \ldots, x'_k)$ have the same linear shape if there is a matrix $A \in GL(m, \mathbb{R})$ such that $(x'_1, x'_2, \ldots, x'_k) = (Ax_1, Ax_2, \ldots, Ax_k)$, which implies that $\overline{x'} = A\overline{x}$. If we regard the configurations $x = (x_1, x_2, \ldots, x_k)$, $x' = (x'_1, x'_2, \ldots, x'_k)$ as $m \times k$ matrices, it follows that $x'^T = x^T A$, that is to say the vector subspaces of $\mathbb{R}^k$ spanned respectively by the columns of the matrices x^T respectively x'^T are the same, which leads to the following result

Proposition 3.1 *The linear shape space $L\Sigma_m^k$ can be identified with the Grassman manifold $G_m(\mathbb{R}^k)$ of m-dimensional vector subspaces of $\mathbb{R}^k$.*

The *affine shape space*, $A\Sigma_m^k$, is the shape space $A\Sigma_m^k = \Sigma_G(\mathcal{C}_{k,m,0})$, where $G = Aff(m, \mathbb{R})$ is the group of affine transformations of $\mathbb{R}^m$.
Therefore two k-ads $(x_1, x_2, \ldots, x_k), (x'_1, x'_2, \ldots, x'_k)$ have the same linear shape if there is a matrix $A \in GL(m, \mathbb{R})$ such that $(x'_1, x'_2, \ldots, x'_k) = (Ax_1, Ax_2, \ldots, Ax_k)$, which implies that $\overline{x'} = A\overline{x}$.

Let $(y_1, y_2, \ldots, y_k), (y'_1, y'_2, \ldots, y'_k)$ be given by

$$y_j = x_j - \overline{x}, y'_j = x'_j - \overline{x}', \forall j = 1, \ldots, k.$$

It follows that if $(x_1, x_2, \ldots, x_k), (x'_1, x'_2, \ldots, x'_k)$ have the same affine shape, then the k-ads $(y_1, y_2, \ldots, y_k), (y'_1, y'_2, \ldots, y'_k)$, centered at 0, have the same linear shape, that is $(y_1, y_2, \ldots, y_k), (y'_1, y'_2, \ldots, y'_k) \in L_{m,k}$, where

$$L_{m,k} = \{w = (w_1, w_2, \ldots, w_k) \in (\mathbb{R}^m)^k \mid \sum_{j=1}^{m} w_j = 0\}.$$

Since $y_k = -(y_1 + y_2 + \ldots + y_{k-1})$, we obtain the following

Proposition 3.2 *The affine shape space $L\Sigma_m^k$ can be identified with the Grassman manifold $G_m(\mathbb{R}^{k-1})$ of m-dimensional vector subspaces of $\mathbb{R}^{k-1}$.*

4 The projective shape space

For our introductory approach to projective shape, including notations, definitions and results given without proof, we send the reader to [8] and [12]. The coordinates of interest in projective shape analysis, are projective coordinates. These are invariant with respect to the group of projective (general linear) transformations PGL(m). A *projective transformation* α of $\mathbb{R}P^m$ is defined in terms of a $(m+1) \times (m+1)$ nonsingular matrix $A \in GL(m+1, \mathbb{R})$ by

$$\alpha([X^1 : \ldots : X^{m+1}]) = [A(X^1, \ldots, X^{m+1})^T].$$

Note that k points in $\mathbb{R}P^m$ with $k \geq m+2$ are in *general position* if any subset of $m+1$ of these points is not included in a linear variety of dimension $m-1$.

Definition 4.1 *A projective frame in $\mathbb{R}P^m$ is an ordered system of $m+2$ points in general position.*

Example 2.1. Let $(e_1, \ldots, e_{m+1})$ be the standard basis of $\mathbb{R}^{m+1}$. The *standard* projective frame is $([e_1], \ldots, [e_{m+1}], [e_1 + \ldots + e_{m+1}])$. The last point of the frame is referred to as the *unit* point.

Proposition 4.1 *PGL(m) acts simply transitively on the set of projective frames in $\mathbb{R}P^m$.*

Definition 4.2 *The projective coordinate(s) of a point $p \in \mathbb{R}P^m$ w.r.t. a projective frame $\pi = (p_1, \ldots, p_{m+2})$ is (are) defined as*

$$p^\pi = \alpha^{-1}(p),$$

where $\alpha \in PGL(m)$ is given above (see [8]).

Assume $x_1, \ldots, x_{m+2}$ are points in general position and $x = (x^1, \ldots, x^m)$ is an arbitrary point in $\mathbb{R}^m$. Note that in our notation, the superscripts are reserved for the components of a point whereas the subscripts are for the labels of points. In order to determine the projective coordinates of $[x : 1]$ w.r.t. the projective frame associated with $(x_1, \ldots, x_{m+2})$ we set $\tilde{x} = (x^1, \ldots, x^m, 1)^T$ and consider the $(m+1)\times(m+1)$ matrix $U_m = [\tilde{x}_1, \ldots, \tilde{x}_{m+1}]$, the j-th column of which is $\tilde{x}_j = (x_j, 1)^T, j = 1, \ldots, m+1$. We define an intermediate system of homogeneous coordinates

$$v(x) = U_m^{-1}\tilde{x} \tag{4.1}$$

and write $v(x) = (v^1(x), \ldots, v^{m+1}(x))^T$.

Next we set

$$z^j(x) = \frac{v^j(x)}{v^j(x_{m+2})} / \left|\left|\frac{v^j(x)}{v^j(x_{m+2})}\right|\right|, \quad j = 1, \ldots, m+1, \tag{4.2}$$

so that the last point x_{m+2} is now used.
The projective coordinate(s) of x are given by the point $[z^1(x) : \ldots. : z^{m+1}(x)]$, where $(z^1(x))^2 + \ldots + (z^{m+1}(x))^2 = 1$. In this way, we identify the position of a point p with respect to π with an unoriented direction (axis). Let $G(k,m)$ denote the set of all ordered systems of k points $(p_1, \ldots, p_k)$ for which $(p_1, \ldots, p_{m+2})$ is a projective frame, $k > m+2$. $PGL(m)$ acts simply transitively on $G(k,m)$ by $\alpha(p_1, \ldots, p_k) = (\alpha p_1, \ldots, \alpha p_k)$.

Definition 4.3 *The projective shape space $P\Sigma_m^k$, or space of projective k-ads in $\mathbb{R}P^m$ is the quotient $G(k,m)/PGL(m)$.*

Proposition 4.2 *$P\Sigma_m^k$ is a manifold diffeomorphic with $(\mathbb{R}P^m)^q$ where $q = k-m-2$.*

Proof. We define $F : P\Sigma_m^k \to (\mathbb{R}P^m)^q$ by $F((p_1, \ldots, p_k)\text{mod } \alpha) = (p_{m+3}^\pi, \ldots, p_k^\pi)$, where $\pi = (p_1, \ldots, p_{m+2})$ and

$$p_i^\pi = [z^1(x_i) : \ldots : z^{m+1}(x_i)] \tag{4.3}$$

and where

$$z(x_i) = (z^1(x_i), \ldots, z^{m+1}(x_i))^T, ||z(x_i)|| = 1, i = m+3, \ldots, k,$$

with $z^j(\,\cdot\,)$ given at (4.3). The mapping F is a well defined diffeomorphism between the two homogeneous spaces.

This representation of projective shape given in [8] is an alternative to the invariant representation used in [4], [9], [11], [10] and the connection between the two representations is described in [8].

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