Structure equations of second order

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Abstract

We write the structure equations on the 2−jet (or 2−tangent) bundle of a differentiable manifold endowed with an arbitrary nonlinear connection N and also an arbitrary linear connection D, with the only restriction that D should preserve the distributions generated by N.

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1 Introduction

Let N be a nonlinear connection on the total space T^2M of the $2-$ tangent bundle (T^2M, π, M), $(Def.1.1.)$. Then, there exists an unique decomposition of tangent space of T^2M at the point $u = \left(x, y^{(1)}, y^{(2)}\right) \in T^2M$ in the following direct sum of the linear vector space:

$$T_uT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u), \quad \forall u \in T^2M.$$

An $\mathbf{N}-$ **linear connection** $D\Gamma(N)$ on T^2M is a linear connection on T^2M, which preserves by parallelism the horizontal and vertical distributions N_0, N_1 and V_2. It has *nine* sets of coefficients. Consequently, we obtain for its torsion *fourteen* sets of components and for its curvature *eighteen* sets of components.

Moreover, on T^2M there exists a natural $2-$tangent structure J given by

$$J\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial y^{(1)a}}, \quad J\left(\frac{\partial}{\partial y^{(1)a}}\right) = \frac{\partial}{\partial y^{(2)a}}, \quad J\left(\frac{\partial}{\partial y^{(2)a}}\right) = 0.$$

Hence, particularly, an $N-$linear connection on T^2M is called **JN** - *linear connection* if if is absolutely parallel with respect to $J : D_XJ = 0, \forall X \in \chi\left(T^2M\right)$.

A $JD\Gamma(N)-$linear connection has only *three* sets of coefficients, its torsion has *thirteen* sets of components and its curvature has *six* sets of components. All these correspond to Miron-Atanasiu's theory on $T^2M = Osc^2M$ ($2-$osculator bundle) (see the joint papers [8] − [14]).

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In this paper we get the structure equations of an $N-$linear connection on T^2M (Theorem 10.1) generalizing the same problem solved for a $JN-$ linear connection at the First Conference of Balkan Society of Geometers, [1].

Of course, for the physical applications to electrodynamics, elasticity, quantum field theories, etc., to work with an $N-$linear connection on T^2M is an advantage and it is not difficult (see [2],[3],[5] and, more generally, [4],[6],[7]).

2 Tangent bundle of the second order $\left(T^2M, \pi^2, M\right)$

Let M be a real differentiable $C^\infty-$manifold of dimension n. A point of M will be denoted by x and its local coordinate system by $(U, \varphi), \varphi(x) = (x^a)$. The indices $a, b, ...$ run over set $\{1, 2, ..., n\}$ and the Einstein convention of summation will be adopted all over this work.

Let $J_{0,x}(\mathbb{R}, M)$ be the set of germs of mappings $f : \mathbb{R} \to M$ with $f(0) = x$. We say that $f, g \in J_{0,x}(\mathbb{R}, M)$ are equivalent up to order 2 if there exists a chart (U, φ) around x such that

$$d_0^\beta(\varphi \circ f) = d_0^\beta(\varphi \circ g), (\beta = 1, 2), \tag{2.1}$$

where d denotes the Frechet differentiation. It can be seen if (2.1) holds for a chart (U, φ), it holds for any other chart (V, ψ) around x.

We denote by $j_{0,x}^2 f$ the equivalence class of f and set $J_{0,x}^2 = \left\{j_{0,x}^2 f, \forall f \in J_{0,x} (\mathbb{R}, M)\right\}$. Then we put

$$T^2M = U_{x\in M} J_{0,x}^2,$$

and define $\pi^2 : T^2M \to M$ by $\pi^2\left(J_{0,x}^2\right) = x$.

Definition 1.1. The set (T^2M, π^2, M) will be called the tangent bundle of order two of the manifold M.

For a local chart (U, φ) in $x \in M$ its lifted local chart in $u \in \left(\pi^2\right)^{-1}(x)$ will be denoted by $\left((\pi_2)^{-1}(U), \Phi\right)$, with $\Phi(u) = (u^\alpha), (u^\alpha) = \left(x^a, y^{(1)a}, y^{(2)a}\right) \subset \mathbb{R}^{3n}, (\alpha = 0, 1, 2)$. Thus a differentiable atlas $\mathcal{A}_M$ of the differentiable structure of the manifold M determines a differentiable atlas $\mathcal{A}_{T^2M}$ on T^2M and therefore the triple $\left(T^2M, \pi^2, M\right)$, is a differentiable manifold.

By (2.1), a transformation of local coordinates $u = (u^\alpha) = \left(x^a, y^{(1)a}, y^{(2)a}\right) \to \widetilde{u} = (\widetilde{u}^\alpha) = \left(\widetilde{x}^a, \widetilde{y}^{(1)a}, \widetilde{y}^{(2)a}\right), (\alpha = 0, 1, 2)$,on the manifold T^2M is given by

$$\begin{cases} \widetilde{x}^a &= \widetilde{x}^a\left(x^1, ..., x^n\right), \det(\dfrac{\partial \widetilde{x}^a}{\partial x^b}) \neq 0, \\ \widetilde{y}^{(1)a} &= \dfrac{\partial \widetilde{x}^a}{\partial x^b} y^{(1)b}, \\ 2\widetilde{y}^{(1)a} &= \dfrac{\partial \widetilde{y}^{(1)a}}{\partial x^b} y^{(1)b} + 2\dfrac{\partial \widetilde{y}^{(1)a}}{\partial y^{(1)b}} y^{(2)b}. \end{cases}$$

One can see that T^2M is of dimension $3n$.

Moreover, if M is a paracompact manifold, then T^2M is paracompact, too.

The null section $0 : M \to T^2M$ of the projection π^2 is defined by $0 : (x) \in M \to (x, 0, 0) \in T^2M$ we denote by $\widetilde{T^2M} = T^2M \setminus \{0\}$.

Let $\mathbb{J}$ be the natural 2−tangent structure on T^2M :

$$\mathbb{J}\left(\frac{\partial}{\partial x^a}\right)=\frac{\partial}{\partial y^{(1)a}},\quad \mathbb{J}\left(\frac{\partial}{\partial y^{(1)a}}\right)=\frac{\partial}{\partial y^{(2)a}},\quad \mathbb{J}\left(\frac{\partial}{\partial y^{(2)a}}\right)=0. \tag{2.2}$$

where $\left(\frac{\partial}{\partial x^a}|_u, \frac{\partial}{\partial y^{(1)a}}|_u, \frac{\partial}{\partial y^{(2)a}}|_u\right)$ is the natural basis of the tangent space TT^2M at the point $u \in T^2M$.

If N is a nonlinear connection on T^2M, then $N_0 = N, N_1 = \mathbb{J}(N_0)$ are two distributions geometrically defined on T^2M, everyone of local dimension n. Let us consider the distribution V_2 on T^2M locally generated by the vector fields $\left\{\frac{\partial}{\partial y^{(2)a}}\right\}$. Consequently, the tangent bundle to T^2M at a point $u \in T^2M$ is given by a direct sum of the vector space:

$$T_uT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u), \quad \forall u \in T^2M. \tag{2.3}$$

We consider $\{\delta_a, \delta_{1a}, \delta_{2a}\}$ an adapted basis to the decomposition (2.3) and its dual basis denoted by $\{dx^a, \delta y^{(1)a}, \delta y^{(2)a}\}$, where

$$\begin{aligned}
\delta_a &= \frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - \underset{1}{N}{}^b{}_a \frac{\partial}{\partial y^{(1)b}} - \underset{2}{N}{}^b{}_a \frac{\partial}{\partial y^{(2)b}},\\
\delta_{1a} &= \frac{\delta}{\delta y^{(1)a}} = \frac{\partial}{\partial y^{(1)b}} - \underset{1}{N}{}^b{}_a \frac{\partial}{\partial y^{(1)b}},\\
\delta_{2a} &= \frac{\partial}{\partial y^{(2)a}},
\end{aligned} \tag{2.4}$$

respectively

$$\begin{aligned}
\delta y^{(1)a} &= dy^{(1)a} + \underset{1}{N}{}^a{}_b dx^b,\\
\delta y^{(2)a} &= dy^{(2)a} + \underset{1}{N}{}^a{}_b dy^{(1)b} + \left(\underset{2}{N}{}^a{}_b + \underset{1}{N}{}^a{}_c \underset{1}{N}{}^c{}_b\right) dx^b.
\end{aligned} \tag{2.5}$$

Then, a vector field $X \in \chi\left(T^2M\right)$ is represented in the local adapted basis as

$$X = X^{(0)a}\delta_a + X^{(1)a}\delta_{1a} + X^{(2)a}\delta_{2a},$$

with the three right terms, called *d-vector fields*, belonging to the distributions N_0, N_1 and V_2, respectively.

A 1-form $\omega \in \mathcal{X}^*(T^2M)$ will be decomposed with three terms, called *d-1-forms*, as

$$\omega = \underset{(0)}{\omega_a}\, dx^a + \underset{(1)}{\omega_a}\, \delta y^{(1)a} + \underset{(2)}{\omega_a}\, \delta y^{(2)a}.$$

Similarly, a tensor field $T \in \mathcal{T}^r_s(T^2M)$ can be split with respect to (2.3) into components, which will be called *d-tensor fields*. Hence, the set

$$\left\{1, \delta_a, \delta_{1a}, \dot{\partial}_{2a}, dx^a, \delta y^{(1)a}, \delta y^{(2)a}\right\}$$

generates the algebra of the d-tensor fields over the ring of functions $\mathcal{F}(T^2M)$.

By a direct calculus we obtain

Proposition 1.1. *The Lie brackets of the vector fields of the adapted basis are given by*

$$
\begin{aligned}
[\delta_b, \delta_c] &= \overset{(0)}{\underset{(00)}{R}}{}^a{}_{bc}\delta_a + \overset{(1)}{\underset{(00)}{R}}{}^a{}_{bc}\delta_{1a} + \overset{(2)}{\underset{(00)}{R}}{}^a{}_{bc}\dot{\partial}_{2a},\\
[\delta_b, \delta_{1c}] &= \overset{(0)}{\underset{(10)}{B}}{}^a{}_{bc}\delta_a + \overset{(1)}{\underset{(10)}{B}}{}^a{}_{bc}\delta_{1a} + \overset{(2)}{\underset{(10)}{B}}{}^a{}_{bc}\dot{\partial}_{2a},\\
\left[\delta_b, \dot{\partial}_{2c}\right] &= \overset{(0)}{\underset{(20)}{B}}{}^a{}_{bc}\delta_a + \overset{(1)}{\underset{(20)}{B}}{}^a{}_{bc}\delta_{1a} + \overset{(2)}{\underset{(20)}{B}}{}^a{}_{bc}\dot{\partial}_{2a},\\
[\delta_{1b}, \delta_{1c}] &= \overset{(0)}{\underset{(11)}{R}}{}^a{}_{bc}\delta_a + \overset{(1)}{\underset{(11)}{R}}{}^a{}_{bc}\delta_{1a} + \overset{(2)}{\underset{(11)}{R}}{}^a{}_{bc}\dot{\partial}_{2a},\\
\left[\delta_{1b}, \dot{\partial}_{2c}\right] &= \overset{(0)}{\underset{(21)}{B}}{}^a{}_{bc}\delta_a + \overset{(1)}{\underset{(21)}{B}}{}^a{}_{bc}\delta_{1a} + \overset{(2)}{\underset{(21)}{B}}{}^a{}_{bc}\dot{\partial}_{2a},
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
\overset{(0)}{\underset{(00)}{R}}{}^a{}_{bc} &= 0,\ \overset{(1)}{\underset{(00)}{R}}{}^a{}_{bc} = \delta_{[c}\underset{1}{N}{}^a{}_{b]},\ \overset{(2)}{\underset{(00)}{R}}{}^a{}_{bc} = \delta_{[c}\underset{2}{N}{}^a{}_{b]} + \underset{1}{N}{}^a{}_f \overset{(1)}{\underset{(00)}{R}}{}^f{}_{bc},\\
\overset{(0)}{\underset{(10)}{B}}{}^a{}_{bc} &= 0,\ \overset{(1)}{\underset{(10)}{B}}{}^a{}_{bc} = \delta_{1c}\underset{1}{N}{}^a{}_b,\ \overset{(2)}{\underset{(10)}{B}}{}^a{}_{bc} = \delta_{1c}\underset{2}{N}{}^a{}_b - \delta_b\underset{1}{N}{}^a{}_c + \underset{1}{N}{}^a{}_f \overset{(1)}{\underset{(10)}{B}}{}^f{}_{bc},\\
\overset{(0)}{\underset{(10)}{B}}{}^a{}_{bc} &= 0,\ \overset{(1)}{\underset{(20)}{B}}{}^a{}_{bc} = \dot{\partial}_{2c}\underset{1}{N}{}^a{}_b,\ \overset{(2)}{\underset{(20)}{B}}{}^a{}_{bc} = \dot{\partial}_{2c}\underset{2}{N}{}^a{}_b + \underset{1}{N}{}^a{}_f \overset{(1)}{\underset{(20)}{B}}{}^f{}_{bc},\\
\overset{(0)}{\underset{(11)}{R}}{}^a{}_{bc} &= 0,\ \overset{(1)}{\underset{(11)}{R}}{}^a{}_{bc} = 0,\ \overset{(2)}{\underset{(11)}{R}}{}^a{}_{bc} = \delta_{1[c}\underset{1}{N}{}^a{}_{b]},\\
\overset{(0)}{\underset{(21)}{B}}{}^a{}_{bc} &= 0,\ \overset{(1)}{\underset{(21)}{B}}{}^a{}_{bc} = 0,\ \overset{(2)}{\underset{(21)}{B}}{}^a{}_{bc} = \overset{(1)}{\underset{(20)}{B}}{}^a{}_{bc}.
\end{aligned}
$$

Also, we can establish (see [2, pg.19, Prop.7.2]):

Proposition 1.2. *The exterior differentials of the 1-forms* dx^a, $\delta y^{(1)a}$, $\delta y^{(2)a}$, *which determine the adapted cobasis (2.5), are given by*

$$
\begin{aligned}
d(dx^a) &= 0,\\
d(\delta y^{(1)a}) &= \left\{\tfrac{1}{2}\overset{(1)}{\underset{(00)}{R}}{}^a{}_{bc}dx^c + \overset{(1)}{\underset{(10)}{B}}{}^a{}_{bc}\delta y^{(1)c} + \overset{(1)}{\underset{(20)}{B}}{}^a{}_{bc}\delta y^{(2)c}\right\} \wedge dx^b,\\
d(\delta y^{(2)a}) &= \left\{\tfrac{1}{2}\overset{(2)}{\underset{(00)}{R}}{}^a{}_{bc}dx^c + \overset{(2)}{\underset{(10)}{B}}{}^a{}_{bc}\delta y^{(1)c} + \overset{(2)}{\underset{(20)}{B}}{}^a{}_{bc}\delta y^{(2)c}\right\} \wedge dx^b +\\
&\quad + \left\{\tfrac{1}{2}\overset{(2)}{\underset{(11)}{R}}{}^a{}_{bc}\delta y^{(1)c} + \overset{(2)}{\underset{(21)}{B}}{}^a{}_{bc}\delta y^{(2)c}\right\} \wedge \delta y^{(1)b}.
\end{aligned}
$$

3 N-linear connections on the manifold T^2M

Definition 2.1. A linear connection D on T^2M is called an N-linear connection if it preserves by parallelism the horizontal and vertical distributions N_0, N_1 and V_2 on T^2M.

Let us notice that an N-linear connection, in the sense of the above definition, is not necessary compatible to the 2-tangent structure $\mathbb{J}$, (2.2). An N-linear connection which is also compatible to $\mathbb{J}$ is called, [3], a $\mathbb{J}N-$linear connection.

An N-linear connection is locally given by its nine sets of coefficients

$$D\Gamma(N) = \left(\underset{(00)}{L}{}^a{}_{bc}, \underset{(10)}{L}{}^a{}_{bc}, \underset{(20)}{L}{}^a{}_{bc}, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}^a{}_{bc}, \underset{(02)}{C}{}^a{}_{bc}, \underset{(12)}{C}{}^a{}_{bc}, \underset{(22)}{C}{}^a{}_{bc} \right),$$

where

$$\begin{cases} D_{\delta_c}\delta_b = \underset{(00)}{L}{}^a{}_{bc}\delta_a, D_{\delta_c}\delta_{1b} = \underset{(10)}{L}{}^a{}_{bc}, D_{\delta_c}\delta_{2b} = \underset{(20)}{L}{}^a{}_{bc}\delta_{2b}, \\ D_{\delta_{1c}}\delta_b = \underset{(01)}{C}{}^a{}_{bc}\delta_a, D_{\delta_{1c}}\delta_{1b} = \underset{(11)}{C}{}^a{}_{bc}\delta_{1a}, D_{\delta_{1c}}\delta_{2b} = \underset{(21)}{C}{}^a{}_{bc}\delta_{2a}, \\ D_{\delta_{2c}}\delta_b = \underset{(02)}{C}{}^a{}_{bc}\delta_a, D_{\delta_{2c}}\delta_{1b} = \underset{(12)}{C}{}^a{}_{bc}\delta_{1a}, D_{\delta_{2c}}\delta_{2b} = \underset{(22)}{C}{}^a{}_{bc}\delta_{2a}. \end{cases}$$

In the particular case when $D\Gamma(N)$ is $\mathbb{J}-$compatible, we have

$$\begin{aligned} \underset{(00)}{L}{}^a{}_{bc} = \underset{(10)}{L}{}^a{}_{bc} = \underset{(20)}{L}{}^a{}_{bc} =: \ & L^a{}_{bc}, \\ \underset{(01)}{C}{}^a{}_{bc} = \underset{(11)}{C}{}^a{}_{bc} = \underset{(21)}{C}{}^a{}_{bc} =: \ & \underset{(1)}{C}{}^a{}_{bc}, \\ \underset{(02)}{C}{}^a{}_{bc} = \underset{(12)}{C}{}^a{}_{bc} = \underset{(22)}{C}{}^a{}_{bc} =: \ & \underset{(2)}{C}{}^a{}_{bc}. \end{aligned} \tag{3.1}$$

Let h, v_1, v_2,be the projectors defined by the distributions N_0, N_1, V_2. If $X \in \chi\left(T^2M\right)$ we denote $X^H = hX,\ X^{V_1} = v_1X,\ X^{V_2} = v_2X$ and

$$\begin{cases} \underset{0}{D}{}^H_X Y = D_{X^H}Y^H, \underset{0}{D}{}^{V_1}_X Y = D_{X^{V_1}}Y^H, \underset{0}{D}{}^{V_2}_X Y = D_{X^{V_2}}Y^H, \\ \underset{\beta}{D}{}^H_X Y = D_{X^H}Y^{V_\beta}, \underset{\beta}{D}{}^{V_1}_X Y = D_{X^{V_1}}Y^{V_\beta}, \underset{\beta}{D}{}^{V_2}_X Y = D_{X^{V_2}}Y^{V_\beta}, \\ \qquad\qquad (\beta = 1, 2). \end{cases}$$

$\underset{\alpha}{D}{}^H, \underset{\alpha}{D}{}^{V_1}, \underset{\alpha}{D}{}^{V_2}$ are called, respectively, $\mathbf{h}_\alpha-, \mathbf{v}_{1\alpha}-$ and $\mathbf{v}_{2\alpha}-$*covariant derivatives*, $(\alpha = 0, 1, 2)$.

In local coordinates, for a d-tensor field

$$T = T^{a_1 \dots a_r}_{b_1 \dots b_s}(x, y^{(1)}, y^{(2)})\delta_{a_1} \otimes \dots \otimes \dot{\partial}_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s},$$

we have

$$\underset{\alpha}{D}{}^H_X T = X^{(0)d} T^{a_1 .. a_r}_{b_1 \dots b_s | \alpha d} \delta_{a_1} \otimes \dots \otimes \delta_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s},$$

where

$$\begin{aligned} & T^{a_1 \dots a_r}_{b_1 \dots b_s \,| \alpha d} = \delta_d T^{a_1 \dots a_r}_{b_1 \dots b_s} + \underset{(\alpha 0)}{L}{}^{a_1}{}_{cd} T^{c a_2 \dots a_r}_{b_1 \dots b_s} + \dots + + \underset{(\alpha 0)}{L}{}^{a_r}{}_{cd} T^{a_1 \dots a_{r-1} c}_{b_1 \dots b_s} - \\ & - \underset{(\alpha 0)}{L}{}^c{}_{b_1 d} T^{a_1 \dots a_r}_{c b_2 \dots b_s} - \dots - \underset{(\alpha 0)}{L}{}^c{}_{b_s d} T^{a_1 \dots a_r}_{b_1 \dots b_{s-1} c}, \end{aligned}$$

and

$$\underset{\alpha}{D}{}^{V_\beta}_X T = X^{(\beta)d} T^{a_1 \dots a_r}_{b_1 \dots b_s} \overset{(\beta)}{|}_{\alpha d} \delta_{a_1} \otimes \dots \otimes \dot{\partial}_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s},$$

where

$$T^{a_1...a_r}_{b_1...b_s} \overset{(\beta)}{|}_{\alpha d} = \delta_{\beta d} T^{a_1...a_r}_{b_1...b_s} + \underset{(\alpha\beta)}{C}{}^{a_1}{}_{cd} T^{ca_2...a_r}_{b_1...b_s} + ... +$$
$$+ \underset{(\alpha\beta)}{C}{}^{a_r}{}_{cd} T^{a_1...a_{r-1}c}_{b_1...b_s} - \underset{(\alpha\beta)}{C}{}^{c}{}_{b_1 d} T^{a_1...a_r}_{cb_2...b_s} - ... - \underset{(\alpha\beta)}{C}{}^{c}{}_{b_s d} T^{a_1...a_r}_{b_1...b_{s-1}c},$$
$$(\alpha = 0, 1, 2; \beta = 1, 2).$$

4 The d-tensors of torsion and curvature

In order to determine the local expressions of d-tensors of torsion and curvature of an $N-$linear connection $D\Gamma(N)$, we use the covariant derivatives in the adapted basis.

The torsion $\mathbb{T}$ of an $N-$linear connection D is expressed, as usually, by

$$\mathbb{T}(X, Y) = D_X Y - D_Y X - [X, Y], \forall X, Y \in \mathcal{X}(T^2 M).$$

It can be evaluated for the pairs of d-vector field $\delta_a, \delta_{1a}, \delta_{2a}$. We obtain:

Theorem 3.1. *The d-tensors of torsion of an N-linear connection D with coefficients* $D\Gamma(N) = (\underset{(\alpha 0)}{L}{}^{a}{}_{bc}, \underset{(\alpha 1)}{C}{}^{a}{}_{bc}, \underset{(\alpha 2)}{C}{}^{a}{}_{bc}), (\alpha = 0, 1, 2)$, *in the adapted basis (2.4), have the following expression:*

$$h\mathbb{T}(\delta_c, \delta_b) = \underset{(00)}{T}{}^{a}{}_{bc}\delta_a, \qquad v_\gamma\mathbb{T}(\delta_c, \delta_b) = \underset{(0\gamma)}{T}{}^{a}{}_{bc}\delta_{\gamma a},$$
$$h\mathbb{T}(\delta_{\beta c}, \delta_b) = \underset{(\beta 0)}{P}{}^{a}{}_{bc}\delta_a, \qquad v_\gamma\mathbb{T}(\delta_{\beta c}, \delta_b) = \underset{(\beta\gamma)}{P}{}^{a}{}_{bc}\delta_{\gamma a},$$
$$v_\gamma\mathbb{T}\left(\dot{\partial}_{2c}, \delta_{1b}\right) = \underset{(2\gamma)}{Q}{}^{a}{}_{bc}\delta_{\gamma a},$$
$$v_\gamma\mathbb{T}(\delta_{\beta c}, \delta_{\gamma b}) = \underset{(\beta\gamma)}{S}{}^{a}{}_{bc}\delta_{\gamma a},$$
$$(\beta, \gamma = 1, 2, \delta_{2a} = \dot{\partial}_{2a}),$$

where

$$\underset{(00)}{T}{}^{a}{}_{bc} = \underset{(00)}{L}{}^{a}{}_{bc} - \underset{(00)}{L}{}^{a}{}_{cb}, \ \underset{(01)}{T}{}^{a}{}_{bc} = \underset{(00)}{\overset{(1)}{R}}{}^{a}{}_{bc}, \ \underset{(02)}{T}{}^{a}{}_{bc} = \underset{(00)}{\overset{(2)}{R}}{}^{a}{}_{bc},$$
$$\underset{(10)}{P}{}^{a}{}_{bc} = \underset{(01)}{C}{}^{a}{}_{bc}, \ \underset{(11)}{P}{}^{a}{}_{bc} = \underset{(10)}{\overset{(1)}{B}}{}^{a}{}_{bc} - \underset{(10)}{L}{}^{a}{}_{cb}, \ \underset{(12)}{P}{}^{a}{}_{bc} = \underset{(10)}{\overset{(2)}{B}}{}^{a}{}_{bc},$$
$$\underset{(20)}{P}{}^{a}{}_{bc} = \underset{(02)}{C}{}^{a}{}_{bc}, \ \underset{(21)}{P}{}^{a}{}_{bc} = \underset{(20)}{\overset{(1)}{B}}{}^{a}{}_{bc}, \ \underset{(22)}{P}{}^{a}{}_{bc} = \underset{(20)}{\overset{(2)}{B}}{}^{a}{}_{bc} - \underset{(20)}{L}{}^{a}{}_{cb},$$
$$\underset{(11)}{S}{}^{a}{}_{bc} = \underset{(11)}{C}{}^{a}{}_{bc} - \underset{(11)}{C}{}^{a}{}_{cb}, \ \underset{(12)}{S}{}^{a}{}_{bc} = \underset{(11)}{\overset{(2)}{R}}{}^{a}{}_{bc},$$
$$\underset{(21)}{Q}{}^{a}{}_{bc} = \underset{(12)}{C}{}^{a}{}_{bc}, \ \underset{(22)}{Q}{}^{a}{}_{bc} = \underset{(20)}{\overset{(1)}{B}}{}^{a}{}_{bc} - \underset{(21)}{C}{}^{a}{}_{cb},$$
$$\underset{(21)}{S}{}^{a}{}_{bc} = 0 \quad, \underset{(22)}{S}{}^{a}{}_{bc} = \underset{(22)}{C}{}^{a}{}_{bc} - \underset{(22)}{C}{}^{a}{}_{cb}.$$

Analogously, the curvature $\mathbb{R}$ of D is given by

$$\mathbb{R}(X, Y)Z = (D_X D_Y - D_Y D_X)Z - D_{[X,Y]}Z, \forall X, Y, Z \in \mathcal{X}(T^2 M).$$

If $X, Y \in \{\delta_a, \delta_{1a}, \delta_{2a}\}$ we denote $\mathbb{R}(X, Y)$ by

$$\mathbb{R}(\delta_b,\delta_c)=\underset{(0)}{\mathbb{R}}_{bc},\qquad \mathbb{R}(\delta_{\beta b},\delta_c)=\underset{(\beta)}{\mathbb{P}}_{bc},$$
$$\mathbb{R}\left(\dot\partial_{2b},\delta_{1c}\right)=\underset{(2)}{\mathbb{Q}}_{bc},\quad \mathbb{R}(\delta_{\beta b},\delta_{\beta c})=\underset{(\beta)}{\mathbb{S}}_{bc},\quad \left(\beta=1,2,\delta_{2a}=\dot\partial_{2a}\right),$$

and the action of $\mathbb{R}(X,Y)$ on $Z\in\{\delta_a,\delta_{1a},\delta_{2a}\}$ we denote by

$$\begin{array}{ll}\underset{(0)}{\mathbb{R}}_{dc}\delta_{\alpha b}=\underset{(0\alpha)}{R}{}_b{}^a{}_{cd}\delta_{\alpha a}, & \underset{(\beta)}{\mathbb{P}}_{dc}\delta_{\alpha b}=\underset{(\beta\alpha)}{P}{}_b{}^a{}_{cd}\delta_{\alpha a},\\ \underset{(2)}{\mathbb{Q}}_{dc}\delta_{\alpha b}=\underset{(2\alpha)}{Q}{}_b{}^a{}_{cd}\delta_{\alpha a}, & \underset{(\beta)}{\mathbb{S}}_{dc}\delta_{\alpha b}=\underset{(\beta\alpha)}{S}{}_b{}^a{}_{cd}\delta_{\alpha a}.\end{array}\tag{4.1}$$

By direct computation, taking into account the Lie brackets (2.6), we get

Theorem 3.2. *An N-linear connection D with the coefficients*

$$D\Gamma(N)=(\underset{(\alpha0)}{L}{}^a{}_{bc},\underset{(\alpha1)}{C}{}^a{}_{bc},\underset{(\alpha2)}{C}{}^a{}_{bc}),(\alpha=0,1,2),$$

has the d-tensors of curvature (4.1) expressed by the following formulae:

$$\underset{(0\alpha)}{R}{}_b{}^a{}_{cd}=\delta_d\underset{(\alpha0)}{L}{}^a{}_{bc}-\delta_c\underset{(\alpha0)}{L}{}^a{}_{bd}+\underset{(\alpha0)}{L}{}^f{}_{bc}\underset{(\alpha0)}{L}{}^a{}_{fd}-\underset{(\alpha0)}{L}{}^f{}_{bd}\underset{(\alpha0)}{L}{}^a{}_{fc}+\underset{(\alpha1)}{C}{}^a{}_{bf}\underset{(00)}{\overset{(1)}{R}}{}^f{}_{cd}+$$
$$+\underset{(\alpha2)}{C}{}^a{}_{bf}\underset{(00)}{\overset{(2)}{R}}{}^f{}_{cd},$$
$$\underset{(\beta\alpha)}{P}{}_b{}^a{}_{cd}=\delta_{\beta d}\underset{(\alpha0)}{L}{}^a{}_{bc}-\underset{(\alpha\beta)}{C}{}^a{}_{bd|\alpha c}+\underset{(\alpha1)}{C}{}^a{}_{bf}\underset{(\beta1)}{\overset{\alpha}{P}}{}^f{}_{cd}+\underset{(\alpha2)}{C}{}^a{}_{bf}\underset{(\beta2)}{\overset{\alpha}{P}}{}^f{}_{cd},$$
$$\underset{(2\alpha)}{Q}{}_b{}^a{}_{cd}=\dot\partial_{2d}\underset{(\alpha1)}{C}{}^a{}_{bc}-\delta_{1c}\underset{(\alpha2)}{C}{}^a{}_{bd}+\underset{(\alpha1)}{C}{}^f{}_{bc}\underset{(\alpha2)}{C}{}^a{}_{fd}-\underset{(\alpha2)}{C}{}^f{}_{bd}\underset{(\alpha1)}{C}{}^a{}_{fc}+\underset{(\alpha2)}{C}{}^a{}_{bf}\underset{(21)}{P}{}^f{}_{cd},$$
$$\underset{(\beta\alpha)}{S}{}_b{}^a{}_{cd}=\delta_{\beta d}\underset{(\alpha\beta)}{C}{}^a{}_{bc}-\delta_{\beta c}\underset{(\alpha\beta)}{C}{}^a{}_{bd}+\underset{(\alpha\beta)}{C}{}^f{}_{bc}\underset{(\alpha\beta)}{C}{}^a{}_{fd}-\underset{(\alpha\beta)}{C}{}^f{}_{bd}\underset{(\alpha\beta)}{C}{}^a{}_{fc}+\underset{(\alpha2)}{C}{}^a{}_{bf}\underset{(\beta1)}{\overset{(2)}{R}}{}^f{}_{cd},$$
$$\Big(\alpha=0,1,2;\beta=1,2,\underset{(12)}{\overset{\alpha}{P}}{}^a{}_{bc}=\underset{(12)}{P}{}^a{}_{bc},\underset{(21)}{\overset{\alpha}{P}}{}^a{}_{bc}=\underset{(21)}{P}{}^a{}_{bc},\underset{(20)}{\overset{(2)}{R}}{}^a{}_{bc}=0,$$
$$y^{(0)}=x,\delta_{2a}=\dot\partial_{2a}\Big).$$

5 Structure equations of an N-linear connection on T^2M

For an $N-$linear connection D, with the coefficients $D\Gamma(N)=(\underset{(\alpha0)}{L}{}^a{}_{bc},\underset{(\alpha1)}{C}{}^a{}_{bc},\underset{(\alpha2)}{C}{}^a{}_{bc})$, $(\alpha=0,1,2)$, in the adapted basis $\left(\delta_a,\delta_{1a},\dot\partial_{2a}\right)$ we can prove:

Lemma 10.1.
1°. Each of following geometrical object fields

$$d(dx^a)-dx^b\wedge\underset{(\alpha)}{\omega}{}^a{}_b,\ d\left(\delta y^{(\beta)a}\right)-\delta y^{(\beta)b}\wedge\underset{(\alpha)}{\omega}{}^a{}_b,\qquad(\alpha=0,1,2,\beta=1,2),$$

are d-vector fields.
2°. The geometrical object fields

$$d\underset{(\alpha)}{\omega}{}^a{}_b-\underset{(\alpha)}{\omega}{}^c{}_b\wedge\underset{(\alpha)}{\omega}{}^a{}_c,\qquad(\alpha=0,1,2),$$

are d-tensor fields, with respect to indices a and b.

Using the previous Lemma we can prove, by a straightforward calculus, a fundamental result in the geometry of T^2M.

Theorem 10.1. *For any N-linear connection D, with the coefficients* $D\Gamma(N) = \left(\underset{(\alpha 0)}{L}{}^{a}{}_{bc}, \underset{(\alpha 1)}{C}{}^{a}{}_{bc}, \underset{(\alpha 2)}{C}{}^{a}{}_{bc}\right), (\alpha = 0,1,2)$, *the following structure equations hold good:*

$$\begin{aligned} d\left(dx^a\right) - dx^b \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} &= -\underset{(\alpha)}{\overset{(0)}{\Omega}}{}^{a}, \\ d\left(\delta y^{(1)a}\right) - \delta y^{(1)b} \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} &= -\underset{(\alpha)}{\overset{(1)}{\Omega}}{}^{a}, \\ d\left(\delta y^{(2)a}\right) - \delta y^{(2)b} \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} &= -\underset{(\alpha)}{\overset{(2)}{\Omega}}{}^{a}, \qquad (\alpha = 0,1,2), \end{aligned}$$

and

$$d\underset{(\alpha)}{\omega}{}^{a}{}_{b} - \underset{(\alpha)}{\omega}{}^{f}{}_{b} \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{f} = -\underset{(\alpha)}{\Omega}{}^{a}{}_{b}, \qquad (\alpha = 0,1,2),$$

where $\underset{(\alpha)}{\overset{(0)}{\Omega}}{}^{a}, \underset{(\alpha)}{\overset{(1)}{\Omega}}{}^{a}, \underset{(\alpha)}{\overset{(2)}{\Omega}}{}^{a}, (\alpha = 0,1,2)$ *are the 2-forms of torsion*

$$\begin{aligned} \underset{(\alpha)}{\overset{(0)}{\Omega}}{}^{a} = {} & \tfrac{1}{2}\underset{(0)}{\overset{\alpha}{T}}{}^{a}{}_{bc} dx^b \wedge dx^c + \\ & + \underset{(\alpha 1)}{C}{}^{a}{}_{bc} dx^b \wedge \delta y^{(1)c} + \underset{(\alpha 2)}{C}{}^{a}{}_{bc} dx^b \wedge \delta y^{(2)c}, \end{aligned}$$

$$\begin{aligned} \underset{(\alpha)}{\overset{(1)}{\Omega}}{}^{a} = {} & \tfrac{1}{2}\underset{(01)}{R}{}^{a}{}_{bc} dx^b \wedge dx^c + \\ & + \underset{(11)}{\overset{\alpha}{P}}{}^{a}{}_{bc} dx^b \wedge \delta y^{(1)c} + \underset{(21)}{P}{}^{a}{}_{bc} dx^b \wedge \delta y^{(2)c} + \\ & + \tfrac{1}{2}\underset{(1)}{\overset{\alpha}{S}}{}^{a}{}_{bc} \delta y^{(1)b} \wedge \delta y^{(1)c} + \underset{(\alpha 2)}{C}{}^{a}{}_{bc} \delta y^{(1)b} \wedge \delta y^{(2)c}, \end{aligned}$$

$$\begin{aligned} \underset{(\alpha)}{\overset{(2)}{\Omega}}{}^{a} = {} & \tfrac{1}{2}\underset{(02)}{R}{}^{a}{}_{bc} dx^b \wedge dx^c + \\ & + \underset{(12)}{P}{}^{a}{}_{bc} dx^b \wedge \delta y^{(1)c} + \underset{(22)}{\overset{\alpha}{P}}{}^{a}{}_{bc} dx^b \wedge \delta y^{(2)c} + \\ & + \tfrac{1}{2}\underset{(12)}{R}{}^{a}{}_{bc} \delta y^{(1)b} \wedge \delta y^{(1)c} + \underset{(22)}{\overset{\alpha}{Q}}{}^{a}{}_{bc} \delta y^{(1)b} \wedge \delta y^{(2)c} + \tfrac{1}{2}\underset{(2)}{\overset{\alpha}{S}}{}^{a}{}_{bc} \delta y^{(2)b} \wedge \delta y^{(2)c}, \end{aligned}$$

and where $\underset{(\alpha)}{\Omega}{}^{a}{}_{b}, (\alpha = 0,1,2)$, *are the 2-forms of curvature*

$$\begin{aligned} \underset{(\alpha)}{\Omega}{}^{a}{}_{b} = {} & \tfrac{1}{2}\underset{(0\alpha)}{R}{}_{b}{}^{a}{}_{cd} dx^c \wedge dx^d + \\ & + \underset{(1\alpha)}{P}{}_{b}{}^{a}{}_{cd} dx^c \wedge \delta y^{(1)d} + \underset{(2\alpha)}{P}{}_{b}{}^{a}{}_{cd} dx^c \wedge \delta y^{(2)d} + \\ & + \tfrac{1}{2}\underset{(1\alpha)}{S}{}_{b}{}^{a}{}_{cd} \delta y^{(1)c} \wedge \delta y^{(1)d} + \underset{(2\alpha)}{Q}{}_{b}{}^{a}{}_{cd} \delta y^{(1)c} \wedge \delta y^{(2)d} + \tfrac{1}{2}\underset{(2\alpha)}{S}{}_{b}{}^{a}{}_{cd} \delta y^{(2)c} \wedge \delta y^{(2)d}. \end{aligned}$$

Remarks 1°. The theorem 10.1 is extremly important in a theory of submanifolds embedded in the total space T^2M of the bundle $\left(T^2M, \pi^2, M\right)$.

2°. For any $JN-$linear connection JD with coefficients $JD\Gamma(N) = (L^a{}_{bc}, \underset{(1)}{C}{}^a{}_{bc}, \underset{(2)}{C}{}^a{}_{bc})$ we have

$$\underset{(0)}{\overset{(0)}{\Omega}}{}^a = \underset{(1)}{\overset{(0)}{\Omega}}{}^a = \underset{(2)}{\overset{(0)}{\Omega}}{}^a = \overset{(0)}{\Omega}{}^a, \underset{(0)}{\overset{(1)}{\Omega}}{}^a = \underset{(1)}{\overset{(1)}{\Omega}}{}^a = \underset{(2)}{\overset{(1)}{\Omega}}{}^a = \overset{(1)}{\Omega}{}^a,$$
$$\underset{(0)}{\overset{(2)}{\Omega}}{}^a = \underset{(1)}{\overset{(2)}{\Omega}}{}^a = \underset{(2)}{\overset{(2)}{\Omega}}{}^a = \overset{(2)}{\Omega}{}^a, \underset{(0)}{\Omega}{}^a{}_b = \underset{(1)}{\Omega}{}^a{}_b = \underset{(2)}{\Omega}{}^a{}_b = \Omega^a{}_b.$$

Then, by the relations (3.1) the structure equations for the $JN-$linear connection are easy to write, [1].

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