Leibniz dynamics with time delay

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Abstract

In this paper we show that several dynamical systems with time delay can be described as vector fields associated to smooth functions via a bracket of Leibniz structure. Some examples illustrate the theoretical considerations.

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Key words: Leibniz system with time delay, metriplectic structure, revisited differential system with time delay.

1. Introduction

A Leibniz structure on a smooth manifold M is defined by a tensor field B of type $(2,0)$. The tensor field B and a smooth function h on M, called a Hamiltonian function, define a vector field X_h which generates a differential system, called a Leibniz system. Examples of Leibniz structures are: the simplectic structures, the Poisson and almost Poisson structures etc. If B is skewsymmetric then we have an almost simplectic structure and if B is symmetric then we have an almost metric structure (Section 2). A skewsymmetric tensor field P of type $(2,0)$, a symmetric tensor field g of type $(2,0)$ and a smooth function h define a Leibniz system, which characterizes an almost metriplectic manifold. For a skewsymmetric tensor field P of type $(2,0)$, a symmetric tensor field g of type $(2,0)$ and two smooth functions h_1, h_2 one defines an almost Leibniz structure, which in certain conditions is a Leibniz structure for the function $h = h_1 + h_2$. An example of almost Leibniz system is the revisited rigid body (Section 3).

To define a differential system with time delay on a smooth manifold M it is suitable to consider the product manifold $M \times M$ and a vector field $X \in \mathcal{X}(M \times M)$ such that $X(\pi_1^* f) = 0$, where $f \in C^\infty(M)$ and $\pi_1 : M \times M \to M$ is the projection. A class of such systems is represented by these which are defined by a tensor field of type $(2,0)$ having certain components null. Examples of almost Leibniz structures with time delay are: the rigid body with time delay, the three–wave interaction with time delay etc. In the case when the almost Leibniz structure with time delay is defined by a skewsymmetric tensor field of type $(2,0)$, a symmetric tensor field of type $(2,0)$ on $M \times M$ (having certain components null) and two functions h_1, h_2 with some

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properties, one obtain the revisited differential system with time delay associated to the previous system (Section 4).

The results of the paper allow to approach some dynamics with time delay which are described by vector fields on $M \times M$ having some geometric properties as conservation laws, divergence or rotor null etc. (Section 5).

This paper presents differential systems with time delay defined by almost Leibniz structures, examples of such systems and a numerical simulation. Purposely the authors leave aside the analysis of the dynamics considered since that one needs specific methods to investigate the differential systems with time delay.

2. Leibniz systems

Let M be a smooth manifold and $C^\infty(M)$ be the ring of the smooth functions on it. A **Leibniz bracket** on M is a bilinear map $[\cdot,\cdot] : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which is a derivation on each entry, that is,

$$[fg,h] = [f,h]g + f[g,h], \quad [f,gh] = [f,g]h + g[f,h],$$

for any $f, g, h \in C^\infty(M)$. We say that the pair $(M,[\cdot,\cdot])$ is a **Leibniz manifold**.

Let $(M,[\cdot,\cdot])$ be a Leibniz manifold and $h \in C^\infty(M)$. There exist two vector fields X_h^R and X_h^L on M uniquely characterized by the relations

$$X_h^R(f) := [f,h], \quad X_h^L(f) := -[h,f], \quad \forall f \in C^\infty(M).$$

We will call X_h^R the **Leibniz vector field** associated to the **Hamiltonian function** $h \in C^\infty(M)$ and we denote it always by X_h. The differential system generated by the Leibniz vector field X_h will be called a **Leibniz system** or a **Leibniz dynamics**.

Since $[\cdot,\cdot]$ is a derivation on each argument it only depends on the first derivatives of the functions and thus, we can define a tensor map $B : T^*M \times T^*M \to \mathbb{R}$ by

$$B(df,dg) := [f,g], \text{ for any } f, g \in C^\infty(M).$$

We say that the Leibniz manifold $(M,[\cdot,\cdot])$ is **non degenerate** whenever B is non degenerate.

If $(M,[\cdot,\cdot])$ is non degenerate then we can define a tensor field of type $(0,2)$ on M, $\omega : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$, by

$$\omega(X_f, X_g) := [f,g], \text{ for any } f, g \in C^\infty(M).$$

A function $f \in C^\infty(M)$ such that $[f,g] = 0$ (respectively, $[g,f] = 0$) for any $g \in C^\infty(M)$ is called a **left** (respectively, **right**) **Casimir** of the Leibniz manifold $(M,[\cdot,\cdot])$.

A Leibniz manifold $(M,[\cdot,\cdot])$ where the bracket is antisymmetric, that is,

$$[f,g] = -[g,f], \quad \forall f, g \in C^\infty(M),$$

is called an **almost Poisson manifold**. If $(M,[\cdot,\cdot])$ is an almost Poisson manifold we define the **Jacobiator** of the bracket $[\cdot,\cdot]$ as the map $\mathcal{J} : C^\infty(M) \times C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ given by

$$\mathcal{J}(f,g,h) := \sum_{\substack{cyclic\\(f,g,h)}} [[f,g],h], \text{ for any } f,g,h \in C^\infty(M).$$

An almost Poisson manifold for which the Jacobiator is the zero map is a **Poisson manifold**. If $(M,[\cdot,\cdot])$ is a non degenerate Leibniz manifold for which the tensor field ω is a closed 2–form on M then (M,ω) is a **symplectic manifold**.

We point out a relevant variety of systems described via a Leibniz bracket, [6]. Let $g \in \mathcal{T}_2^0(M)$ be a pseudometric on the smooth manifold M, that is, a symmetric non degenerate tensor field of type $(0,2)$ on M. Let $g^\# : T^*M \to TM$ be the associated sharp vector bundle map. Given any smooth function $h \in C^\infty(M)$ we define its gradient $\nabla h : M \to TM$ as the vector field on M given by $\nabla h = g^\# dh$. In these conditions let $[\cdot,\cdot] : C^\infty(M) \times C^\infty(M) \to \mathbb{R}$ be the Leibniz bracket defined by

$$[f,h] := g(\nabla f, \nabla h), \text{ for any } f,h \in C^\infty(M),$$

that is the **pseudometric bracket** associated to g. It is clearly symmetric and non degenerate. The Leibniz vector field X_h associated to any function $h \in C^\infty(M)$ is such that $X_h = \nabla h$, that is X_h generates a **gradient dynamical system**. In local coordinates the vector field X_h has the components

$$X_h^i = g^{ij}\frac{\partial h}{\partial x^j},$$

where (g^{ij}) are the contravariant components of g and $i,j = 1,2,\ldots,\dim M$.

A problem in dynamics is the study of the interactions between waves of different frequencies with different resonance conditions. A particular case **the three–wave interaction** can be formulated as a gradient dynamical system in $\mathbb{R}^3$, using the Leibniz bracket induced by the constant pseudometric g with the components $g_{ii} = 1/s_i\gamma_i$, $g_{ij} = 0$ for $i \neq j$, where $s_1,s_2,s_3 \in \{-1,1\}$, $\gamma_1,\gamma_2,\gamma_3 \in \mathbb{R}^*$, $\gamma_1+\gamma_2+\gamma_3 = 0$ and the Hamiltonian function $h : \mathbb{R}^3 \to \mathbb{R}$, $h(x^1,x^2,x^3) = x^1x^2x^3$. The Leibniz vector field associated to h generates the gradient dynamical system given by

$$\dot{x}^1 = s_1\gamma_1 x^2x^3,\ \dot{x}^2 = s_2\gamma_2 x^1x^3,\ \dot{x}^3 = s_3\gamma_3 x^1x^2.$$

3. Almost metriplectic systems

Let M be a smooth manifold, P a skewsymmetric tensor field of type $(2,0)$ and g a symmetric tensor field of type $(2,0)$. The map $[\cdot,\cdot] : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ given by

$$[f,h] := P(df,dh) + g(df,dh), \quad \forall f,h \in C^\infty(M),$$

defines a Leibniz bracket on M. The Leibniz vector field X_h associated to the Hamiltonian function $h \in C^\infty(M)$ is such that

$$X_h(f) = P(df,dh) + g(df,dh), \text{ for any } f \in C^\infty(M).$$

In local coordinates X_h has the components

$$X_h^i = P^{ij}\frac{\partial h}{\partial x^j} + g^{ij}\frac{\partial h}{\partial x^j},$$

where $P^{ij} = P(dx^i, dx^j)$, $g^{ij} = g(dx^i, dx^j)$.

If P is a Poisson tensor field and g is a non degenerate tensor field, then $(M, P, g, [\cdot,\cdot])$ is called a **metriplectic manifold of the first kind**.

If P is a tensor field defining a simplectic structure and g is a tensor field defining a Riemannian structure on M, then the corresponding metriplectic manifold (M, P, g) was studied by E. Kähler ([5]), where $[\cdot,\cdot]$ was called an **interior product**).

An example of a metriplectic system is the equation arising from the Landau–Lifschitz model for the magnetization vector field $x = (x^1, x^2, x^3)^T \in \mathcal{X}(\mathbb{R}^3)$ in an external vector field $B = (B^1, B^2, B^3)^T \in \mathcal{X}(\mathbb{R}^3)$,

$$\dot{x} = \gamma x \times B + \frac{\lambda}{\|x\|^2}(x \times (x \times B)),$$

where γ and λ are physical parameters, [1], [6]. The Leibniz bracket describing the dynamical system is

$$[f, h](x) = x \cdot (\nabla f(x) \times \nabla h(x)) + \frac{\lambda}{\gamma\|x\|^2}\,(x \times \nabla f(x)) \cdot (x \times \nabla h(x))\,,$$

where $\times$ denotes the standard cross product in $\mathbb{R}^3$, ∇ is the Euclidean gradient, $f, h \in C^\infty(\mathbb{R}^3)$, $x \in \mathbb{R}^3$ and $h(x) = \gamma B \cdot x$ is the Hamiltonian function.

Let M be a smooth manifold and $P, g \in \mathcal{T}_0^2(M)$ two tensor fields of type $(2, 0)$. Consider the map $[\cdot, (\cdot,\cdot)] : C^\infty(M) \times C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defined by the relation

$$[f, (h_1, h_2)] := P(df, dh_1) + g(df, dh_2), \quad \forall f, h_1, h_2 \in C^\infty(M).$$

Proposition 3.1. *The bracket map $[\cdot, (\cdot,\cdot)]$ satisfies the following properties:*
a) $[fh, (h_1, h_2)] = [f, (h_1, h_2)]h + f[h, (h_1, h_2)]$;
b) $[f, h(h_1, h_2)] = h[f, (h_1, h_2)] + h_1 P(df, dh) + h_2 g(df, dh)$;
c) $[f, l(h, h)] = l[f, (h, h)] + h[f, (l, l)]$,
for any $f, h, l, h_1, h_2 \in C^\infty(M)$.

The bracket $[\cdot, (\cdot,\cdot)]$ is a left derivation called an **almost Leibniz bracket** and the structure $(M, P, g, [\cdot(\cdot,\cdot)]$ is said to be an **almost Leibniz manifold**. The restriction of $[\cdot, (\cdot,\cdot)]$ to $C^\infty(M) \times \triangle_{C^\infty(M)}$, where $\triangle_{C^\infty(M)}$ is the diagonal of $C^\infty(M) \times C^\infty(M)$, defines a Leibniz bracket on (M, P, g), because the bracket $[f, h] := [f, (h, h)]$, $\forall f, h \in C^\infty(M)$ is a derivation on each argument.

If P is a Poisson tensor field and g is a non degenerate symmetric tensor field, then $(M, P, g, [\cdot, (\cdot,\cdot)])$ is called a **metriplectic manifold of the second kind**. Such a structure is considered in [2]. Given $h_1, h_2 \in C^\infty(M)$ the Leibniz vector field associated to (h_1, h_2) is such that

$$X_{(h_1,h_2)}(f) = P(df, dh_1) + g(df, dh_2), \quad \forall f \in C^\infty(M).$$

In local coordinates the corresponding differential system is

$$\dot{x}^i = [x^i, (h_1, h_2)] = P^{ij}\frac{\partial h_1}{\partial x^j} + g^{ij}\frac{\partial h_2}{\partial x^j},\ i, j = 1, \ldots \dim M.$$

Proposition 3.2. *Let $(M, P, g, [\cdot, (\cdot,\cdot)])$ be an almost Leibniz manifold with P skewsymmetric, g symmetric (respectively, a multiplectic manifold of second kind)*

and let $h_1, h_2 \in C^\infty(M)$ be two functions such that $P(df, dh_2) = 0$, $g(df, dh_1) = 0$, for any $f \in C^\infty(M)$. The Hamiltonian function $h = h_1 + h_2$ defines on M an almost metriplectic (respectively, a metriplectic) system of the first kind.

Proposition 3.2 is useful when we consider the revisited differential system of a (almost) Poisson differential system with a Hamiltonian function and a Casimir function. More precisely we have

Proposition 3.3. *For a (almost) Poisson differential system on M given by the tensor field P, with a Hamiltonian function h_1 and a Casimir function h_2, there exists a tensor field g such that $(M, P, g, [\cdot, (\cdot, \cdot)])$ is a metriplectic manifold of the second kind. The differential system associated to this structure is called the* **revisited differential system** *of the initial system.*

For example the rigid body with the Poisson structure

$$P = \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix},$$

the Hamiltonian function $h_1 = \dfrac{1}{2}[a_1(x^1)^2 + a_2(x^2)^2 + a_3(x^3)^2]$ and the Casimir function $h_2 = \dfrac{1}{2}[(x^1)^2 + (x^2)^2 + (x^3)^2]$ has the differential system

$$\dot{x}^1 = (a_2 - a_3)x^2x^3,\ \dot{x}^2 = (a_3 - a_1)x^1x^3,\ \dot{x}^3 = (a_1 - a_2)x^1x^2.$$

A tensor field g defining the revisited differential system has the components

$$\begin{array}{l} g^{11} = -a_2^2(x^2)^2 - a_3^2(x^3)^2,\ g^{22} = -a_1^2(x^1)^2 - a_3^2(x^3)^2,\ g^{33} = -a_1^2(x^1)^2 - a_2^2(x^2)^2, \\ g^{12} = g^{21} = a_1a_2x^1x^2, \quad g^{13} = g^{31} = a_1a_3x^1x^3, \quad g^{23} = g^{32} = a_2a_3x^2x^3. \end{array}$$

The revisited differential system is

$$\begin{aligned} \dot{x}^1 &= (a_2 - a_3)x^2x^3 + a_2(a_1 - a_2)x^1(x^2)^2 + a_3(a_1 - a_3)x^1(x^3)^2, \\ \dot{x}^2 &= (a_3 - a_1)x^1x^3 + a_3(a_2 - a_3)x^2(x^3)^2 + a_1(a_2 - a_1)x^2(x^1)^2, \\ \dot{x}^3 &= (a_1 - a_2)x^1x^2 + a_1(a_3 - a_1)x^3(x^1)^2 + a_2(a_3 - a_1)x^3(x^2)^2. \end{aligned}$$

4. Leibniz systems with time delay

Let M be a n–dimensional smooth manifold, the product manifold $M \times M = \{(\widetilde{x}, x) \mid \widetilde{x} \in M, x \in M\}$ and the canonical projections $\pi_1 : M \times M \to M$, $\pi_2 : M \times M \to M$. A vector field X on $M \times M$ satisfying the condition $X(\pi_1^* f) = 0$, for any $f \in C^\infty(M)$ is given in a local chart by $X(\widetilde{x}, x) = X^i(\widetilde{x}, x)\dfrac{\partial}{\partial x^i}$.

A **differential system with time delay** is a differential system associated to a vector field X on $M \times M$ for which $X(\pi_1^* f) = 0$, $\forall f \in C^\infty(M)$ and it is given in a local chart by

$$\dot{x}^i(t) = X^i(\widetilde{x}(t), x(t)), \quad i = 1, 2, \ldots, n, \tag{4.1}$$

where $\widetilde{x}(t) = x(t - \tau)$, with $\tau > 0$ and the initial condition $x(\theta) = \varphi(\theta)$, $\theta \in [-\tau, 0]$ and $\varphi : [-\tau, 0] \to M$ are smooth maps.

Some systems of differential equations with time delay in $\mathbb{R}^n$ were studied in [3], [4]. For such a system are relevant the geometric properties of the vector field defining that system as first integrals (constants of the motion), Morse functions, almost metriplectic structure etc.

Example 4.1. *The rigid body with time delay in all directions.* Let $a_1, a_2, a_3 \in \mathbb{R}$ be three distinct numbers and the vector field $X \in \mathcal{X}(\mathbb{R}^3 \times \mathbb{R}^3)$ with the components

$$X^1 = a_2 x^2 \widetilde{x}^3 - a_3 x^3 \widetilde{x}^2,\ X^2 = a_3 x^3 \widetilde{x}^1 - a_1 x^1 \widetilde{x}^3,\ X^3 = a_1 x^1 \widetilde{x}^2 - a_2 x^2 \widetilde{x}^1. \tag{4.2}$$

The corresponding differential system with time delay is

$$\begin{aligned} \dot{x}^1(t) &= a_2 x^2(t) x^3(t-\tau) - a_3 x^3(t) x^2(t-\tau), \\ \dot{x}^2(t) &= a_3 x^3(t) x^1(t-\tau) - a_1 x^1(t) x^3(t-\tau), \\ \dot{x}^3(t) &= a_1 x^1(t) x^2(t-\tau) - a_2 x^2(t) x^1(t-\tau), \end{aligned} \tag{4.3}$$

with the initial condition $x^i(\theta) = \varphi^i(\theta)$, $i = 1, 2, 3$, $\theta \in [-\tau, 0]$, $\tau \geq 0$. If $h_1 \in C^\infty(\mathbb{R}^3)$, $h_1(x) = \frac{1}{2}[(x^1)^2 + (x^2)^2 + (x^3)^2]$ and $h_2 \in C^\infty(\mathbb{R}^3)$, $h_2(x) = \frac{1}{2}[a_1(x^1)^2 + a_2(x^2)^2 + a_3(x^3)^2]$, then $X(\pi_2^* h_2) = 0$ and $X(\pi_1^* h_1) = \alpha \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ with $\alpha(\widetilde{x}, x) \neq 0$ for $(\widetilde{x}, x)$ in an open set $D \subset \mathbb{R}^3 \times \mathbb{R}^3$, $\alpha(\widetilde{x}, x) = a_1 x^1 \left(\widetilde{x}^2 x^3 - \widetilde{x}^3 x^2\right) + a_2 x^2 \left(\widetilde{x}^3 x^1 - \widetilde{x}^1 x^3\right) + a_3 x^3 \left(\widetilde{x}^1 x^2 - \widetilde{x}^2 x^1\right)$. h_2 is a first integral for (4.3).

Example 4.2. *The revisited rigid body with time delay.* Let a_1, a_2, a_3 be three distinct real numbers and the vector field $X \in \mathcal{X}(\mathbb{R}^3 \times \mathbb{R}^3)$ with the components

$$\begin{aligned} X^1 &= (a_2 - a_3) x^2 x^3 + a_2(a_1 - a_2) \widetilde{x}^1 \widetilde{x}^2 x^2 + a_3(a_1 - a_3) \widetilde{x}^1 \widetilde{x}^3 x^3, \\ X^2 &= (a_3 - a_1) x^1 x^3 + a_3(a_2 - a_1) \widetilde{x}^2 \widetilde{x}^3 x^3 + a_1(a_2 - a_1) \widetilde{x}^2 \widetilde{x}^1 x^1, \\ X^3 &= (a_1 - a_2) x^1 x^2 + a_1(a_3 - a_1) \widetilde{x}^3 \widetilde{x}^1 x^1 + a_2(a_3 - a_2) \widetilde{x}^3 \widetilde{x}^2 x^2. \end{aligned} \tag{4.4}$$

The differential system associated to X is the differential system with time delay of the revisited rigid body. Let P be the skew symmetric tensor field of type $(2,0)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ given by $P = P^{ij}(x) \dfrac{\partial}{\partial x^i} \otimes \dfrac{\partial}{\partial x^j}$, where

$$\left(P^{ij}(x)\right) = \begin{pmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & x^1 \\ x^2 & -x^1 & 0 \end{pmatrix}$$

and g the symmetric tensor field of type $(2,0)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ given by $g = g^{ij}(\widetilde{x}, x) \dfrac{\partial}{\partial \widetilde{x}^i} \otimes \dfrac{\partial}{\partial x^j}$, where

$$\left(g^{ij}(\widetilde{x}, x)\right) = \begin{pmatrix} -a_2^2 x^2 \widetilde{x}^2 - a_3^2 x^3 \widetilde{x}^3 & a_1 a_2 \widetilde{x}^1 x^2 & a_1 a_3 \widetilde{x}^1 x^3 \\ a_1 a_2 \widetilde{x}^1 x^2 & -a_1^2 x^1 \widetilde{x}^1 - a_3^2 x^3 \widetilde{x}^3 & a_2 a_3 \widetilde{x}^2 x^3 \\ a_1 a_3 \widetilde{x}^1 x^3 & a_2 a_3 \widetilde{x}^2 x^3 & -a_1^2 \widetilde{x}^1 x^1 - a_2^2 \widetilde{x}^2 x^2 \end{pmatrix}.$$

If $h_1(\widetilde{x}) = \dfrac{1}{2}\left[\left(\widetilde{x}^1\right)^2 + \left(\widetilde{x}^2\right)^2 + \left(\widetilde{x}^3\right)^2\right]$, $h_2(x) = \dfrac{1}{2}\left[a_1\left(x^1\right)^2 + a_2(x^2)^2 + a_3(x^3)^2\right]$, then the components (4.4) of X satisfy the relations

$$X^i(\widetilde{x},x)=P^{ij}(x)\frac{\partial h_2}{\partial x^j}+g^{ij}(\widetilde{x},x)\frac{\partial h_1}{\partial \widetilde{x}^j},\ i,j=1,2,3.$$

Let $\mathcal{T}_0^2(M\times M)$ be the modulus of the tensor fields of type $(2,0)$ on the product manifold $M\times M$ and let us denote

$$\mathcal{T}^{02}:=\left\{P\in\mathcal{T}_0^2(M\times M)|P(\pi_1^*f_1,\pi_1^*f_2)=P(\pi_1^*f_1,\pi_2^*f_2)=0,\forall f_1,f_2\in C^\infty(M)\right\},$$

$$\mathcal{T}^{11}:=\left\{g\in\mathcal{T}_0^2(M\times M)|g(\pi_1^*f_1,\pi_1^*f_2)=g(\pi_2^*f_1,\pi_2^*f_2)=0,\forall f_1,f_2\in C^\infty(M)\right\}.$$

Consider $P\in\mathcal{T}^{02}$, $g\in\mathcal{T}^{11}$ and the map $[\cdot,(\cdot,\cdot)]:C^\infty(M)\times C^\infty(M\times M)\times C^\infty(M\times M)\to C^\infty(M\times M)$ defined by the relation

$$[f,(h_1,h_2)]\coloneqq P(d\pi_2^*f,dh_2)+g(d\pi_2^*f,dh_1),\forall f\in C^\infty(M),h_1,h_2\in C^\infty(M\times M).$$

Proposition 4.1. *The bracket map $[\cdot,(\cdot,\cdot)]$ satisfies the following properties:*
a) $[f_1f_2,(h_1,h_2)]=[f_1,(h_1,h_2)]f_2+f_1[f_2,(h_1,h_2)]$;
b) $[f,h(h_1,h_2)]=h[f,(h_1,h_2)]+h_1P(d\pi_2^*f,dh)+h_2g(d\pi_2^*f,dh)$;
c) $[f,l(h,h)]=l[f,(h,h)]+h[f,(l,l)]$,
for any $f_1,f_2\in C^\infty(M)$ and $h,l,h_1,h_2\in C^\infty(M\times M)$.

The bracket $[\cdot,(\cdot,\cdot)]$ is called the **almost Leibniz bracket with time delay** and the structure $(M,P,g,[\cdot,(\cdot,\cdot)])$ is said be an **almost Leibniz manifold with time delay**. For two functions $h_1,h_2\in C^\infty(M\times M)$ the relation $X_{h_1h_2}(f)=[f,(h_1,h_2)]$ defines a vector field such that $X_{h_1h_2}(\pi_1^*f)=0$. In local coordinates

$$X^i_{h_1h_2}=P^{ij}\frac{\partial h_2}{\partial x^j}+g^{ij}\frac{\partial h_1}{\partial\widetilde{x}^j},\quad i,j=1,2,\ldots,n.$$

By a straighforward calculation it results

Proposition 4.2. *If the tensor field $P\in\mathcal{T}^{02}(M\times M)$ is skewsymmetric, the tensor field $g\in\mathcal{T}^{11}(M\times M)$ is symmetric and $h_1,h_2\in C^\infty(M\times M)$ satisfy the conditions $P(d\pi_2^*f,dh_1)=0$, $g(d\pi_2^*f,dh_2)=0$, $\forall f\in C^\infty(M)$, then $[f,(h_1,h_2)]=[f,(h,h)]$, where $h=h_1+h_2$.*

Proposition 4.2 allows the local determination of a tensor field g in terms of derivatives of the functions h_1,h_2.

Proposition 4.3 *Let $h_1,h_2\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$,*

$$D=\left\{(\widetilde{x},x)\in\mathbb{R}^n\times\mathbb{R}^n\mid\frac{\partial h_2}{\partial x^i}\neq 0,\ i=1,2,\ldots,n\right\}$$

and let $P\in\mathcal{T}^{02}(D)$ be a skewsymmetric tensor field such that $P(d\pi_2^*f,d\pi_2^*h_1)=0$, $\forall f\in C^\infty(M)$. There exists a symmetric tensor field $g\in\mathcal{T}^{11}(D)$ with $g(d\pi_2^*f,d\pi_2^*h_2)=0$, $\forall f\in C^\infty(M)$ such that $(D,P,g,[\cdot,(\cdot,\cdot)])$ is a almost Leibniz structure with time delay.

The differential system

$$\dot{x}^i=P^{ij}(\widetilde{x},x)\frac{\partial h_2(\widetilde{x},x)}{\partial x^j}+g^{ij}(\widetilde{x},x)\frac{\partial h_1(\widetilde{x},x)}{\partial\widetilde{x}^j},\ i,j=1,2,\ldots,n,\tag{4.5}$$

is called the **revisited differential system with time delay** associated to the differential system with time delay given by

$$\dot{x}^i = P^{ij}\left(\widetilde{x}, x\right) \frac{\partial h_2\left(\widetilde{x}, x\right)}{\partial x^i}, \quad i, j = 1, 2, \ldots, n,$$

where $\widetilde{x}(t) = x(t-\tau)$, $\tau > 0$.

Example 4.3. Consider the differential system with time delay on $\mathbb{R}^3 \times \mathbb{R}^3$ given by the tensor field P with the components

$$(P^{ij}) = \begin{pmatrix} 0 & x^3 & -\widetilde{x}^2 \\ -x^3 & 0 & x^1 \\ \widetilde{x}^2 & -x^1 & 0 \end{pmatrix}$$

and the function $h_1\left(\widetilde{x}, x\right) = a_1\widetilde{x}^1x^1 + a_2\widetilde{x}^2x^2 + a_3\widetilde{x}^3x^3$, that is

$$\begin{aligned} &\dot{x}^1(t) = a_2x^2(t-\tau)x^3(t) - a_3x^2(t-\tau)x^3(t-\tau), \\ &\dot{x}^2(t) = a_3x^1(t)x^3(t-\tau) - a_1x^1(t-\tau)x^3(t), \\ &\dot{x}^3(t) = a_1x^1(t-\tau)x^2(t-\tau) - a_2x^1(t)x^2(t-\tau). \end{aligned} \tag{4.6}$$

The orbit of that system, for $a_1 = 0.6$, $a_2 = 0.4$, $a_3 = 0.2$, is given in Fig. 1. The function $h_2\left(\widetilde{x}, x\right) = \frac{1}{2}(x^1)^2 + x^2\widetilde{x}^2 + \frac{1}{2}(x^3)^2$ has the property $P(dh_2, df) = 0$, $\forall f \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Based on Proposition 4.3 there exists a tensor field g such that $g(dh_1, df) = 0$, $\forall f \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Its components are

$$(g^{ij}) = \begin{pmatrix} -a_2^2x^2\widetilde{x}^2 - a_3x^3\widetilde{x}^3 & a_1a_2\widetilde{x}^1x^2 & a_1a_3\widetilde{x}^1x^3 \\ a_1a_2\widetilde{x}^1x^2 & -a_1^2x^1\widetilde{x}^1 - a_3x^3\widetilde{x}^3 & a_2a_3\widetilde{x}^2x^3 \\ a_1a_3\widetilde{x}^1x^2 & a_2a_3\widetilde{x}^2x^3 & -a_1^2x^1\widetilde{x}^1 - a_2^2x^2\widetilde{x}^2 \end{pmatrix}.$$

The revisited differential system with time delay associated to the system (4.6) is the following:

$$\begin{aligned} \dot{x}^1(t) &= a_2x^2(t-\tau)x^3(t) - a_3x^2(t-\tau)x^3(t-\tau), \\ \dot{x}^2(t) &= a_3x^1(t)x^3(t-\tau) - a_1x^1(t-\tau)x^3(t) - \\ &\quad -a_1^2x^1(t)x^1(t-\tau)x^2(t) - a_3^2x^2(t)x^3(t)x^3(t-\tau), \\ \dot{x}^3(t) &= a_1x^1(t-\tau)x^2(t-\tau) - a_2x^1(t)x^2(t-\tau). \end{aligned} \tag{4.7}$$

The orbit of that system, for $a_1 = 0.6$, $a_2 = 0.4$, $a_3 = 0.2$ is given in Fig.2.

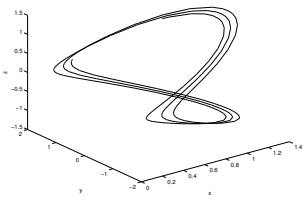

Fig. 1

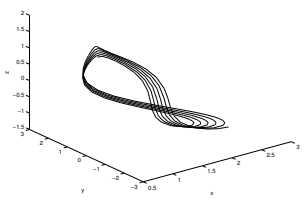

Fig. 2

5. Conclusions

The methods utilized in our paper allow an approach of the differential systems with time delay having some geometrical properties by means of differential geometry. The authors are convinced that several other thing of differential geometry accompany the study of the differential systems with time delay.

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