IS – LM model with pure money financing

Constantin Udriște and Andrea Niglia

Abstract

Section 1 analyzes the Hopf bifurcation phenomena in an IS-LM flow with pure money financing. Section 2 studies the geometric dynamics associated to an IS-LM flow and to the Euclidean structure of the space.

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1 IS-LM flow with pure money financing

Let us consider the following ingredients:

(1) independent economic variables: the income \( Y \), the rate of interest \( R \), the money stock \( M \);

(2) the economic functions: the investment \( I(Y, R) \), the disposable income \( Y^D = Y - T(Y) \), the saving \( S(Y^D) \), the taxation \( T(Y) \), the money demand \( L(Y, R) \);

(3) economic constants: the adjustment speed of income reacting to excess demand \( \alpha > 0 \), the adjustment speed of excess demand for money \( \beta > 0 \), the economic barrier (government spending) \( G \);

(4) economic constraints:

\[ I_R < 0, \quad S_{Y^D} > 0, \quad I_Y > 0, \quad 0 < T_Y < 1, \quad L_Y > 0, \quad L_R < 0. \]

The dynamic IS-LM with pure money financing of the government budget deficit is given by the flow

\[
\frac{dY}{dt} = \alpha(I(Y, R) + G - S(Y^D) - T(Y))
\]

\[
\frac{dR}{dt} = \beta(L(Y, R) - M)
\]

\[
\frac{dM}{dt} = G - T(Y).
\]

Suppose this ODEs system has a unique equilibrium point \( (Y_1, R_1, M_1) \) situated in the first orthant. To examine the possible emergence of cycles, we use the Jacobian matrix.
\[
A = \begin{pmatrix}
\alpha(I_Y - \sigma_Y) & \alpha I_R & 0 \\
\beta L_Y & \beta L_R & -\beta \\
-T_Y & 0 & 0
\end{pmatrix},
\]
where \( \sigma_Y = S_Y D (1 - T_Y) + T_Y \), and all the partial derivatives are taken at the equilibrium point. The characteristic equation is
\[
\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0,
\]
where
\[
a_1 = -(\alpha(I_Y - \sigma_Y) + \beta L_R) \\
a_2 = \alpha \beta ((I_Y - \sigma_Y)L_R - I_R L_Y) \\
a_3 = -\alpha \beta T_Y I_R.
\]
Taking into consideration the signs of derivatives we obtain \( a_2, a_3 > 0 \). The sign of \( a_1 \) remains uncertain. Supposing \( I_Y - \sigma_Y > 0 \), but sufficiently small, we ensure the condition \( a_1 > 0 \).

If we want to apply the Hopf bifurcation theorem, we need one negative real root and a pair of pure imaginary roots of characteristic equation. Since all the coefficients are positive, it remains to check that \( a_1 a_2 - a_3 = 0 \), i.e.,
\[
\alpha \beta \{ -[\alpha(I_Y - \sigma_Y) + \beta L_R][(I_Y - \sigma_Y)L_R - I_R L_Y] + T_Y I_R \} = 0.
\]

Let us consider \( \alpha = \text{const} \) and \( \beta \) parameter. Then the solution
\[
\beta_0 = \frac{1}{L_R} \left\{ \frac{T_Y I_R}{(I_Y - \sigma_Y)L_R - I_R L_Y} - \alpha(I_Y - \sigma_Y) \right\} > 0
\]
is the critical value of \( \beta \). Since \( a_1 a_2 - a_3 \) changes sign as \( \beta \) passes through \( \beta_0 \), the real part of the complex roots, \( u(\beta) \), also changes sign as it passes through zero. Also, we can check
\[
\frac{du}{d\beta}(\beta_0) \neq 0.
\]
This completes the proof of the existence of a closed orbit for values of the parameter in a neighborhood of \( \beta_0 \).

## 2 IS-LM geometric dynamics

Introducing \( x^1 = Y, x^2 = R, x^3 = M, \) and \( X^1 = \alpha(I(Y, R) + G - S(Y D) - T(Y)), \)
\( X^2 = \beta(L(Y, R) - M), X^3 = G - T(Y), \) we obtain the Euler-Lagrange prolongation
\[
\frac{d^2 x^i}{dt^2} = \frac{\partial f}{\partial x^i} + \sum_j \left( \frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i} \right) \frac{dx^j}{dt}, \quad i, j = 1, 2, 3,
\]
where
\[
f = \frac{1}{2} \delta_{ij} X^i X^j.
\]
is the density of energy determined by the Euclidean structure \( \delta_{ij} \) and the vector field \( (X^1, X^2, X^3) \).
The economic constraints impose the signs of the coefficients in the differential system (1):

\[
\begin{align*}
\frac{\partial X^1}{\partial x^2} - \frac{\partial X^2}{\partial x^1} &= \alpha I_R - \beta L_Y < 0 \\
\frac{\partial X^2}{\partial x^3} - \frac{\partial X^3}{\partial x^2} &= -\beta < 0 \\
\frac{\partial X^3}{\partial x^1} - \frac{\partial X^1}{\partial x^3} &= -T_Y \in (-1,0).
\end{align*}
\]

In this way, the gyroscopic force in the right hand side of (1) is specified.

The solutions of the second order differential system (1) are horizontal pregeodesics of the Riemann-Jacobi-Lagrange manifold \((R^3 \setminus \mathcal{E}, g_{ij}, N^i_j)\) determined by the following ingredients:

- **Lagrangian** 
  \(L = \frac{1}{2} \delta_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} - \delta_{ij} X^i \frac{dx^j}{dt} + f\)
- **Hamiltonian** 
  \(H = \frac{1}{2} \delta_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} - f\)
- **Jacobi metric** 
  \(g_{ij} = (H + f)\delta_{ij}\)
- **non-linear connection** 
  \(N^i_j = \Gamma^i_{jk} y^k - F^i_j\)

The geometric dynamics is like a geostrophic wind associated to the IS-LM flow.

**Open problem.** Find the economic interpretation of trajectories in the IS-LM geometric dynamics that are different from IS-LM field lines.

**References**


Constantin Udriște  
Department of Mathematics I, Splaiul Independenței 313,  
RO-060042 Bucharest, Romania.  
email: udriste@mathem.pub.ro

Andrea Niglia  
University of Messina, Department V. Pareto,  
75 Via dei Verdi, IT-98122 Messina, Italy.