Bifurcation and stability in a system derived from the Lorenz system

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Abstract

A nonlinear three-dimensional differential system derived from the Lorenz system is analyzed. Insights on bifurcation and stability are presented. The system possesses a subcritical Hopf bifurcation. For $b = \frac{3}{5}a$ the point S_+ loses stability. As far as we know, not many systems derived from the Lorenz system have been studied.

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Key words: Nonlinear systems, stability, Hopf bifurcation.

1 Introduction

The nonlinear differential three-dimensional system

$$(1) \qquad \begin{cases} x' = a\,(y - x) \\ y' = 4ax - axz \\ z' = xy - bz \end{cases}$$

with a, b real numbers, $a \neq 0$, is derived from the Lorenz system. It help us to understand better the family of Lorenz systems. As far as we know, not many systems derived from Lorenz was studied. Originally, the Lorenz system was a model of convection. In the last decades the nonlinear system was intensively studied because the nonlinear phenomena are met in many areas, from engineering to human brain [10, 1] and heart disease. Chaos is a phenomenon closely related to nonlinear systems. In [8] it is studied the Duffing oscillator modified and are found the conditions to transition to chaos.

2 Stability analysis

As we said above, not many systems derived from the Lorenz system have been studied. We record here two of them: *Lü* system [4] and Chen system [9]. Good

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notes on these systems are reported in [3, 5, 11]. Of the two system, the *Lü* system is the most similar to our system (1). Due to this fact we recall it here. The *Lü* system is described by the following three nonlinear differential equations:

$$\begin{cases} x' = a\,(y - x) \\ y' = -xz + cy \\ z' = xy - bz \end{cases}$$

Although system (1) and *Lü* system are similar, having the same number of linear and nonlinear terms but differing in some terms, they exhibit different dynamics, fig.1 a)-b). The flow of system (1) is drawn bellow together with *Lü* attractor.

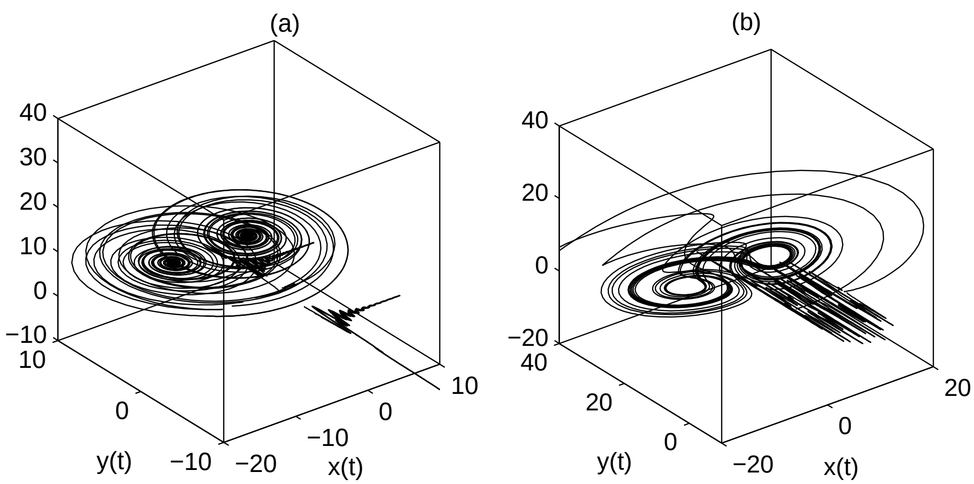

Figure 1: a) The attractor of the system (1) for the parametric vector (a,b)=(2,5) b) The attractor of the *Lü* system for the parametric vector (a,b,c)=(36,3,20)

In the following we briefly describe some basic properties of the system (1).

Solving the three equations $x' = y' = z' = 0$ we get that system (1) has three isolated equilibria $O(0,0,0), S_+(2\sqrt{b}, 2\sqrt{b}, 4), S_-(-2\sqrt{b}, -2\sqrt{b}, 4)$ if $b > 0$ and it has only one isolated equilibrium $O(0,0,0)$ for $b < 0$.

For $b = 0$ all points in the form $(0,0,m)$, m real, are non-isolated equilibria of the system. In the following consider $b \neq 0$. For system (1) the divergence is:

$$div(f) = \frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} = -a - b.$$

Hence, for all real a and b such that $a + b > 0$, the system (1) is dissipative, with an exponential contraction rate:

$$\frac{dV}{dt} = e^{-a-b}.$$

That is, a volume element V_0 is contracted by the flow into a volume element $V_0 e^{(-a-b)t}$ in time t. This means that each volume containing the system trajectory shrinks to zero as t tends to infinity at an exponential rate e^{-a-b} . Therefore, all system orbits are ultimately confined to a specific limit set of zero volume, and the system asymptotic motion settles onto an attractor [5]. The system (1) is conservative if and only if $a + b = 0$.

Proposition 1. *The equilibrium $O(0,0,0)$ is unstable for any $a \neq 0$.*

Proof. Indeed, the Jacobian matrix of the system at $O(0,0,0)$ is $J_0 = \begin{pmatrix} -a & a & 0 \\ 4a & 0 & 0 \\ 0 & 0 & -b \end{pmatrix}$, with characteristic polynomial

$$(\lambda + b)\left(\lambda^2 + a\lambda - 4a^2\right) = 0.$$

This leads to the eigenvalues: $\lambda_1 = -b$, $\lambda_2 = -\frac{1}{2}a - \frac{1}{2}a\sqrt{17}$, $\lambda_3 = -\frac{1}{2}a + \frac{1}{2}a\sqrt{17}$. Now it is clear that if $a > 0 \Rightarrow \lambda_3 > 0$ and if $a < 0 \Rightarrow \lambda_2 > 0$. Consequently the equilibrium point $O(0,0,0)$ is unstable for any $a \neq 0$.

In the following consider the stability of the system at $S_+(2\sqrt{b}, 2\sqrt{b}, 4), S_-(-2\sqrt{b}, -2\sqrt{b}, 4)$.

Because the system is invariant under the transformation $(x, y, z) \to (-x, -y, z)$, one only needs to consider the stability of system (1) at $S_+(2\sqrt{b}, 2\sqrt{b}, 4)$. Under the linear transformation $(x, y, z) \to (X, Y, Z)$

$$\left\{ \begin{array}{l} x = X + 2\sqrt{b} \\ y = Y + 2\sqrt{b} \\ z = Z + 4 \end{array} \right. \tag{2}$$

the system (1) becomes

$$\left\{ \begin{array}{l} X' = a\,(Y - X) \\ Y' = -2a\sqrt{b}Z - aXZ \\ Z' = 2\sqrt{b}\,(X + Y) - bZ + XY \end{array} \right. \tag{3}$$

Then, we have to consider the stability of system (3) at $O(0,0,0)$. The Jacobian matrix of system (3) at $O(0,0,0)$ is:

$$J_+ = \begin{pmatrix} -a & a & 0 \\ 0 & 0 & -2a\sqrt{b} \\ 2\sqrt{b} & 2\sqrt{b} & -b \end{pmatrix}.$$

and the characteristic equation is : (*) $\lambda^3 + \lambda^2(a+b) + 5ab\lambda + 8a^2b = 0$.

Using Routh-Hurwitz conditions, this equation has all roots with negative real parts if and only if $A > 0, C > 0, AB - C > 0$ where $A = a + b, B = 5ab, C = 8a^2b$, namely:

$$\left\{ \begin{array}{l} a + b > 0 \\ b > 0 \\ a\,(5b - 3a) > 0 \end{array} \right.$$

Consequently, we have the theorem:

Theorem 1. The equilibrium points $S_+(2\sqrt{b}, 2\sqrt{b}, 4), S_-(-2\sqrt{b}, -2\sqrt{b}, 4)$ are asymptotically stable if and only if $\left(a + b > 0, b > 0, 5ab - 3a^2 > 0\right)$.

Proposition 2. Equation (*) has purely imaginary roots if and only if $b > 0, b = \frac{3}{5}a$. In this case the solutions of equation (*) are $\lambda_1 = \frac{-8}{5}a, \lambda_{2,3} = \pm ia\sqrt{3}$.

Proof. If $\lambda_{2,3} = \pm i\omega$ are the complex solutions and λ_1 the real solution of (*) then, from $\lambda_1+\lambda_2+\lambda_3 = -(a+b) \Rightarrow \lambda_1 = -(a+b)$. This easily leads to $b > 0, b = \frac{3}{5}a$ and $\lambda_1 = \frac{-8}{5}a, \lambda_{2,3} = \pm ia\sqrt{3}$.

In the following we will prove that the system (1) displays a Hopf bifurcation at the point S_+. For $b := b_c = \frac{3}{5}a$ the point S_+ loses its stability and the bifurcation is subcritical, i.e appears an unstable limit cycle.

Theorem 2. If $b = \frac{3}{5}a$, equation (*) has a negative solution $\lambda_1 = \frac{-8}{5}a < 0$ together with a pair of purely imaginary roots $\lambda_{2,3} = \pm ia\sqrt{3}$ such that $\mathrm{Re}(\lambda_b'(b_c)) \neq 0$, therefore the system (1) displays a Hopf bifurcation at the point S_+ and the bifurcation is subcritical.

Proof. If $b = \frac{3}{5}a$ the equation (*) is transformed into

$$\left(\lambda + \frac{8}{5}a\right)\left(\lambda^2 + 3a^2\right) = 0 \tag{4}$$

with solutions $\lambda_1 = \frac{-8}{5}a$ and $\lambda_{2,3} = \pm ia\sqrt{3}$.

From (*) results that:

$\lambda_b' = -\frac{5a\lambda+8a^2+\lambda^2}{5ab+2\lambda(a+b)+3\lambda^2}$, so that

$\lambda_b'(b_c) = -\frac{5a\lambda+8a^2+\lambda^2}{5ab+2\lambda(a+b)+3\lambda^2} = -\frac{25a\lambda+40a^2+5\lambda^2}{16a\lambda+15a^2+15\lambda^2}$ with $\lambda = \pm ia\sqrt{3}$.

Then,
$\mathrm{Re}\left(\lambda_b'(b_c)\right) = -\frac{75}{278} < 0$ and
$\mathrm{Im}\left(\lambda_b'(b_c)\right) = \frac{575}{834}\sqrt{3} > 0$.

Consequently, the system (3) displays a Hopf bifurcation at $O(0,0,0)$, so the system (1) displays a Hopf bifurcation at the point S_+.

The fact that the Hopf bifurcation is subcritical results from the following theorem.

Theorem 3. If $b = \frac{3}{5}a, b > 0$ the point $S_+(2\sqrt{b}, 2\sqrt{b}, 4)$ of the system (1) is unstable for any $a > 0$.

Proof. In this case the system (3) becomes:

$$\begin{cases} X' = a(Y - X) \\ Y' = -XZa - 2Za\sqrt{\frac{3}{5}a} \\ Z' = XY - \frac{3}{5}Za + 2\left(X+Y\right)\sqrt{\frac{3}{5}a} \end{cases} \tag{5}$$

and $\lambda_1 = \frac{-8}{5}a < 0$, $\lambda_{2,3} = \pm ia\sqrt{3}$ with $a > 0$ are the eigenvalues of the Jacobian matrix of this system. Then, the eigenvectors corresponding to $\lambda_1 = \frac{-8}{5}a$, $\lambda_2 = ia\sqrt{3}$, $\lambda_3 = -ia\sqrt{3}$ are, respectively,

$$v_1 = \begin{pmatrix} -\frac{25}{12}\sqrt{\frac{3}{5}a} \\ \frac{5}{4}\sqrt{\frac{3}{5}a} \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{5}a} + \frac{1}{2}i\sqrt{\frac{1}{5}a} \\ 2i\sqrt{\frac{1}{5}a} \\ 1 \end{pmatrix} \text{ and}$$

$$v_3 = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{5}a} - \frac{1}{2}i\sqrt{\frac{1}{5}a} \\ -2i\sqrt{\frac{1}{5}a} \\ 1 \end{pmatrix}$$

Then, the vectors $w = \frac{v_2+v_3}{2} = \left(\frac{1}{2}\sqrt{\frac{3}{5}a}, 0, 1\right)$, $w' = \frac{v_2-v_3}{2i} = \left(\frac{1}{2}\sqrt{\frac{1}{5}a}, 2\sqrt{\frac{1}{5}a}, 0\right)$ and v_1 provide us the following linear transformation of the system (5):

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{5}a} & \frac{1}{2}\sqrt{\frac{1}{5}a} & -\frac{25}{12}\sqrt{\frac{3}{5}a} \\ 0 & 2\sqrt{\frac{1}{5}a} & \frac{5}{4}\sqrt{\frac{3}{5}a} \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} \tag{6}$$

or,equivalently,

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} \frac{48}{139}\sqrt{\frac{5}{3a}} & -\frac{12}{139}\sqrt{\frac{5}{3a}} & \frac{115}{139} \\ \frac{30}{139}\sqrt{\frac{5}{a}} & \frac{62}{139}\sqrt{\frac{5}{a}} & -\frac{15}{139}\sqrt{3} \\ -\frac{48}{139}\sqrt{\frac{5}{3a}} & \frac{12}{139}\sqrt{\frac{5}{3a}} & \frac{24}{139} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{7}$$

namely,

$$\begin{cases} X_1 = \frac{115}{139}Z + \frac{48}{139}\sqrt{\frac{5}{3a}}X - \frac{12}{139}\sqrt{\frac{5}{3a}}Y \\ \\ Y_1 = -\frac{15}{139}Z\sqrt{3} + \frac{30}{139}\sqrt{\frac{5}{a}}X + \frac{62}{139}\sqrt{\frac{5}{a}}Y \\ \\ Z_1 = \frac{24}{139}Z - \frac{48}{139}\sqrt{\frac{5}{3a}}X + \frac{12}{139}\sqrt{\frac{5}{3a}}Y \end{cases}$$

After some calculations, the system (5) is transformed into:

$$(8)\quad \begin{cases} X_1' = aY_1\sqrt{3} + \frac{25}{139}aX_1Y_1\sqrt{3} + \frac{193}{1112}aX_1Z_1 - \frac{1907}{3336}aY_1Z_1\sqrt{3} + \\ \qquad + \frac{6}{139}aX_1^2 + \frac{23}{139}aY_1^2 - \frac{3275}{2224}aZ_1^2 \\ Y_1' = -aX_1\sqrt{3} - \frac{40}{139}aX_1Y_1 + \frac{2221}{3336}aX_1Z_1\sqrt{3} + \frac{7}{1112}aY_1Z_1 - \\ \qquad - \frac{31}{139}aX_1^2\sqrt{3} - \frac{3}{139}aY_1^2\sqrt{3} + \frac{7325}{6672}aZ_1^2\sqrt{3} \\ Z_1' = -\frac{8}{5}aZ_1 + \frac{14}{695}aX_1Y_1\sqrt{3} + \frac{28}{139}aX_1Z_1 - \frac{19}{139}aY_1Z_1\sqrt{3} - \\ \qquad - \frac{6}{139}aX_1^2 + \frac{24}{695}aY_1^2 - \frac{25}{278}aZ_1^2 \end{cases}$$

Then, the 2-dimensional local center manifold of system (8) near the origin is the set :

$W^c_{loc}(O_1) = \left\{(X_1, Y_1, Z_1) \in \mathbb{R}^3 \mid Z_1 = h(X_1, Y_1), |X_1| + |Y_1| \ll 1\right\}$ where $h(0,0) = \frac{\partial h}{\partial X_1}(0,0) = \frac{\partial h}{\partial Y_1}(0,0) = 0.$

With the substitution $Z_1 = h(X_1, Y_1)$ in (8), the vector field on the center manifold is:

$$(9)\quad \begin{cases} X_1' = aY_1\sqrt{3} + \frac{25}{139}aX_1Y_1\sqrt{3} + \frac{193}{1112}aX_1h - \frac{1907}{3336}aY_1h\sqrt{3} + \\ \qquad + \frac{6}{139}aX_1^2 + \frac{23}{139}aY_1^2 - \frac{3275}{2224}ah^2 \\ Y_1' = -aX_1\sqrt{3} - \frac{40}{139}aX_1Y_1 + \frac{2221}{3336}aX_1h\sqrt{3} + \frac{7}{1112}aY_1h - \\ \qquad - \frac{6}{139}aX_1^2 + \frac{24}{695}aY_1^2 - \frac{25}{278}ah^2 \end{cases}$$

Assume that $Z_1 = h(X_1, Y_1) = a_{11}X_1^2 + a_{12}X_1Y_1 + a_{22}Y_1^2 + ...$
Substituting $X_1 = w + u, Y_1 = i(w - u)$, system(9) becomes

$$w' = iaw\sqrt{3} + \frac{93}{1112}ahu + \frac{25}{278}ahw + \frac{29}{139}auw - \frac{73}{100}ah^2 + \frac{23}{278}au^2 - \frac{57}{278}aw^2 - \frac{1}{25}iahu\sqrt{3} - $$

$$- \frac{86}{139}iahw\sqrt{3} + \frac{34}{139}iauw\sqrt{3} - \frac{27}{50}iah^2\sqrt{3} + \frac{3}{278}iau^2\sqrt{3} + \frac{53}{278}iaw^2\sqrt{3}$$

$$u' = -iau\sqrt{3} + \frac{25}{278}ahu + \frac{2}{25}ahw + \frac{29}{139}auw - \frac{73}{100}ah^2 - \frac{57}{278}au^2 + \frac{23}{278}aw^2 + \frac{86}{139}iahu\sqrt{3} + $$

$$+ \frac{1}{25}iahw\sqrt{3} - \frac{34}{139}iauw\sqrt{3} + \frac{27}{50}iah^2\sqrt{3} - \frac{53}{278}iau^2\sqrt{3} - \frac{3}{278}iaw^2\sqrt{3}$$

where $u = \overline{w}$.
From
$Z_1 = a_{11}X_1^2 + a_{12}X_1Y_1 + a_{22}Y_1^2 + O\left(|w|^3\right),$

in complex variables Z_1 is in the form

$$Z_1 = N_{11}w^2 + N_{12}wu + N_{22}u^2 + O\left(|w|^3\right) \tag{10}$$

with $Z_1' = 2N_{11}w'w + N_{12}\left(w'u + wu'\right) + 2N_{22}u'u$
Using the above terms w' and u', after some calculations, this leads to

$$Z_1' = 2iaw^2N_{11}\sqrt{3} - 2iau^2N_{22}\sqrt{3} + O\left(|w|^3\right) \tag{11}$$

On the other hand, from (8) we have:

$Z_1' = -\frac{8}{5}aZ_1 + \frac{14}{695}aX_1Y_1\sqrt{3} + \frac{28}{139}aX_1Z_1 - \frac{19}{139}aY_1Z_1\sqrt{3}-$

$-\frac{6}{139}aX_1^2 + \frac{24}{695}aY_1^2 - \frac{25}{278}aZ_1^2$

and then, also after some calculations, we have that

(**) $Z_1' = -\frac{8}{5}au^2N_{22} - \frac{8}{5}aw^2N_{11} - \frac{8}{5}auwN_{12} - \frac{12}{695}auw - \frac{14}{695}iau^2\sqrt{3} - \frac{54}{695}au^2+$

$+\frac{14}{695}iaw^2\sqrt{3} - \frac{54}{695}aw^2 + O\left(|w|^3\right)$

Equating coefficients of w^2, wu, u^2 in (11) and (**) one can find:

$N_{11} = -\frac{1}{500} + \frac{1}{50}i$, $N_{12} = -\frac{3}{278}$ and $N_{22} = -\frac{1}{500}i\sqrt{3} - \frac{1}{50}$
and
$h = w^2\left(\frac{1}{50}i - \frac{1}{500}\right) - \frac{3}{278}uw - u^2\left(\frac{1}{500}i\sqrt{3} + \frac{1}{50}\right)$

Then, the dynamics on the center manifold is governed by the equation [6]:

$w' = iaw\sqrt{3} + \frac{93}{1112}ahu + \frac{25}{278}ahw + \frac{29}{139}auw - \frac{73}{100}ah^2 + \frac{23}{278}au^2 - \frac{57}{278}aw^2 - \frac{1}{25}iahu\sqrt{3}-$
$-\frac{86}{139}iahw\sqrt{3} + \frac{34}{139}iauw\sqrt{3} - \frac{27}{50}iah^2\sqrt{3} + \frac{3}{278}iau^2\sqrt{3} + \frac{53}{278}iaw^2\sqrt{3},$
that is:
$w' = iaw\sqrt{3} + \left(\frac{53}{278}i\sqrt{3} - \frac{57}{278}\right)aw^2 + \left(\frac{23}{278} + \frac{3}{278}i\sqrt{3}\right)au^2 + \left(\frac{29}{139} + \frac{34}{139}i\sqrt{3}\right)auw +$
$\left(\frac{1}{125} + \frac{7}{100}i\sqrt{3}\right)auw^2 + +O\left(|w|^3\right)$
because we are interested only in coefficients of w^2, wu, u^2, w^2u. Note by:
$g_{20} = 2\left(\frac{53}{278}i\sqrt{3} - \frac{57}{278}\right)a$, $g_{11} = \left(\frac{29}{139} + \frac{34}{139}i\sqrt{3}\right)a$
$g_{02} = 2\left(\frac{23}{278} + \frac{3}{278}i\sqrt{3}\right)a$, $g_{21} = 2\left(\frac{1}{125} + \frac{7}{100}i\sqrt{3}\right)a$

Now we are able to use the first Lyapunov coefficient to prove that the point S_+ is unstable and the bifurcation is subcritical , a method related in [2]. The first Lyapunov coefficient is defined as:

$\ell_1(0) = \frac{\mathrm{Re}(C(0))}{\sqrt{3}a}$ where

$C(0) = \frac{i}{2\sqrt{3}a}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2} \simeq \left(1.8377 \times 10^{-2} - 0.1157i\right)a$

Hence, $\ell_1 > 0$. Consequently, the point $O(0,0,0)$ of the system (3) is unstable, so the point $S_+(2\sqrt{b}, 2\sqrt{b}, 4)$ of the system (1) is unstable and the Hopf bifurcation is subcritical.

3 Conclusions

Using a rigorous mathematical analysis based on symbolic and numerical computations we have studied a relative new system. There are obtained some insights on stability and bifurcation. The system possesses a Hopf bifurcation and the bifurcation is subcritical. Surely, there is still a lot of work, and this paper is a step in analyzing this system.

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