On a Gauss-Kuzmin-type problem for a new continued fraction expansion with explicit invariant measure

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Abstract

We study a new continued fraction expansion of reals in the unit interval. Using the ergodic behaviour of a homogeneous random system with complete connections associated with this expansion we obtain a Gauss-Kuzmin-type theorem.

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1. Introduction

The Gauss-Kuzmin theorem is one of the most important results in the metrical theory of regular continued fractions (see [5], [6]).

From the time of Gauss, a great number of Gauss-Kuzmin-type theorems followed. The Gauss-Kuzmin problem has been generalized in various directions for other continued fraction expansions. We remark that the Gauss transformation has strong ties with chaos theory [1], [2].

In this paper we consider another expansion of reals in the unit interval $I$ different from the regular continued fraction-expansion. Using the approach of dependence with complete connections, our aim is to prove a Gauss-Kuzmin-type theorem for this new expansion. In Section 2 we describe this expansion and the associated transformation. In Section 3 we introduce a homogeneous random system with complete connections associated with this expansion. In Section 4 the ergodic behaviour of this random system allows us to obtain a convergence rate result.
2. Another expansion and some examples

Write \( x \in [0,1) \) as

\[
\frac{2^{-a_1}}{1 + \frac{2^{-a_2}}{1+ \frac{\ddots}{\ddots}}} = [a_1, a_2, \ldots],
\]

where \( a_n \) are natural numbers. First, it is clear that every irrational \( x \in [0,1) \) has a unique expansion of the type of (1). Second, we note that some particular cases of this type of continued fractions have been studied before. For example, by setting \( q = 1/2 \) and \( a_n = n \), the right-hand side of (1) gives the well-known continued fraction of Rogers and Ramanujan

\[
\frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ddots}}}.
\]

Another example is the result due to Davison [3]. Let \( a_n = F_n \), where \( F_n \) is the \( n \)-th Fibonacci number. Davison showed that

\[
\frac{2^{-F_1}}{1 + \frac{2^{-F_2}}{1 + \frac{2^{-F_3}}{1 + \ddots}}} = \frac{1}{2} \sum_{n \geq 1} 2^{-[n\phi]},
\]

where \( \phi \) is the Golden Ratio and \([\cdot]\) denotes the entire function.

Define the transformation \( T : [0,1) \rightarrow [0,1) \) as follows

\[
T(x) = \begin{cases} 
0, & x = 0 \\
[a_2, a_3, \ldots], & x = [a_1, a_2, \ldots].
\end{cases}
\]

It follows from (1) and (2) that for \( x \neq 0 \) we have

\[
x = \frac{2^{-a_1}}{1 + T(x)}.
\]

Consequently, using (3) we can write the transformation \( T \) of \([0,1)\) as

\[
T(x) = 2^{[\log(1/x)/\log 2]} - 1, \quad x \neq 0,
\]

where \( \{u\} \) denotes the fractionary part of a real \( u \).

One should think of \( a_n \) as the incomplete quotients or digits of \( x \). For \( x \neq 0 \), we get

\[
a_1(x) = [\log(1/x)/\log 2], \quad a_n(x) = a_1(T^{(n-1)}(x)), \quad n \in \mathbb{N}^*, \ n \geq 2.
\]
3. Preliminary results

Let \( \mu \) be a non-atomic probability measure on \( \mathcal{B} (= \) the \( \sigma \)-algebra of Borel subsets of \( I = [0,1] \)) and define
\[
F_n(x) = \mu(T^n < x), \ x \in I, \ n \in \mathbb{N},
\]
with \( F_0(x) = \mu([0,x)) \).

**Proposition 1.** For each \( n \in \mathbb{N} \), \( F_n \) satisfies the following Gauss-type relation
\[
F_{n+1}(x) = \sum_{k \geq 0} \left( F_n(\alpha_k) - F_n\left(\frac{\alpha_k}{x+1}\right)\right), \ x \in I,
\]
where \( \alpha = 1/2 \).

**Proof.** Since \( T^n = \frac{2^{-a_{n+1}}}{T^{n+1} + 1} \) it follows that
\[
F_{n+1}(x) = \mu(T^{n+1} < x) = \sum_{k \geq 0} \mu\left(\frac{\alpha_k}{x+1} < T^n < \alpha_k\right) = \\
= \sum_{k \geq 0} \left( F_n(\alpha_k) - F_n\left(\frac{\alpha_k}{x+1}\right)\right).
\]

We now assume that \( F'_n \) exists and is bounded (\( \mu \) has bounded density). By induction we have that \( F'_n, n \in \mathbb{N}^* \), exist and are bounded as well. This allows us to differentiate (4) term by term, obtaining
\[
F'_{n+1}(x) = \sum_{k \geq 0} \frac{\alpha_k}{(x+1)^2} F'_n\left(\frac{\alpha_k}{x+1}\right).
\]

Let us introduce the function \( f_n \) in (5) where
\[
f_n(x) = \frac{F'_n(x)}{\rho(x)}, \ n \in \mathbb{N},
\]
with \( \rho(x) = \frac{1}{(x+1)(x+2)} \), \( x \in I \). Then (5) becomes
\[
f_{n+1}(x) = \sum_{k \geq 0} p_k(x)f_n\left(\frac{\alpha_k}{x+1}\right),
\]
where
\[
p_k(x) = \frac{\alpha_k^{k+1}(x+1)(x+2)}{(\alpha_k+1)(\alpha_k+2)(x+1)(x+2)}.
\]

**Proposition 2.** The function \( P(x, k) = p_k(x) \) defines a transition probability function from \((I, \mathcal{B})\) to \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\).

**Proof.** We have to verify that \( \sum_{k \geq 0} P(x, k) = 1 \) for all \( x \in I \).
On a Gauss-Kuzmin-type problem

\[ P(x, k) = \alpha^{k+1} + \frac{\Delta_k}{\alpha^k + x + 1} - \frac{\Delta_{k+1}}{\alpha^{k+1} + x + 1} \]

with \( \Delta_k = \alpha^k - \alpha^{2k} \), it follows that

\[
\sum_{k \geq 0} P(x, k) = \lim_{k \to \infty} \left( \frac{\alpha(1 - \alpha^{k+1})}{1 - \alpha} - \frac{\Delta_{k+1}}{\alpha^{k+1} + x + 1} \right) = 1.
\]

Proposition 2 allows us to consider the random system with complete connections (RSCC) (see [4] Section 1.1)

(7) \( ((I, B), (N, P(N)), u, P) \)

where \( I, B, P(N) \) and \( P \) are defined previously, while \( u : I \times N \to I \) is given by

\[ u(x, k) = u_k(x) = \frac{\alpha^k}{x + 1}. \]

Further, we denote by \( U \) the associated Markov operator of the RSCC (7) with the transition probability function

\[ Q(x, B) = \sum_{\{k \in N | u_k(x) \in B\}} p_k(x), \quad x \in I, \quad B \in B. \]

Then \( Q^n(\cdot, \cdot) \) will denote the \( n \)-step transition probability function of the same Markov chain (see [4]).

The ergodic behaviour of the RSCC (7) allows us to find the limiting distribution function \( F = F_\infty \) and the invariant measure \( Q_\infty \) induced by \( F \). To study the ergodicity of the RSCC (7), let us consider the norm \( \| \cdot \|_L \) defined on \( L(I) = \text{the space of Lipschitz real-valued functions defined on } I \) by

\[ \|f\|_L = \sup_{x \in I} |f(x)| + \sup_{x' \neq x''} \frac{|f(x') - f(x'')|}{|x' - x''|}, \quad f \in L(I). \]

Proposition 3. The RSCC (7) is uniformly ergodic.

Proof. We have

\[ \frac{d}{dx} u(x, k) = -\alpha^k \frac{k}{(x+1)^2}, \]

\[ \frac{d}{dx} P(x, k) = -\frac{\Delta_k}{(\alpha^k + x + 1)^2} + \frac{\Delta_{k+1}}{(\alpha^{k+1} + x + 1)^2}, \]

for all \( x \in I \) and \( k \in N \), so that \( \sup_{x \in I} \left| \frac{d}{dx} u(x, k) \right| = \alpha^k \) and \( \sup_{x \in I} \left| \frac{d}{dx} P(x, k) \right| < \infty \).

Hence the requirements of definition of an RSCC with contraction are fulfilled (see Definition 3.1.15 in [4]). By Theorem 3.1.6 in [4]), it follows that the Markov chain associated with this RSCC with contraction is a Doeblin-Fortet chain and its transition operator is a Doeblin-Fortet operator. To prove the regularity of \( U \) w.r.t. \( L(I) \) let us define recursively \( x_{n+1} = (x_n + 2)^{-1}, \quad n \in N, \) with \( x_0 = x \). A criterion of regularity is expressed in Theorem 3.2.13 in [4], in terms of the supports \( \sum_n(x) \) of
the \( n \) step transition probability functions \( Q^n(x, \cdot), n \in \mathbb{N}^* \). Clearly \( x_{n+1} \in \sum_1(x_n) \) and therefore Lemma 3.2.14 in [4] and an induction argument lead to the conclusion that \( x_n \in \sum_n(x), n \in \mathbb{N}^* \). But \( \lim_{n \to \infty} x_n = \sqrt{2} - 1 \) for any \( x \in I \). Hence

\[
d \left( \sum_n(x), \sqrt{2} + 1 \right) \leq |x_n - \sqrt{2} - 1| \to 0 \quad \text{as} \quad n \to \infty,
\]

where \( d(x, y) = |x - y|, \forall x, y \in I \). Finally, the regularity of \( U \) w.r.t. \( L(I) \) follows from Theorem 3.2.13 in [4]. Moreover, \( Q^n(\cdot, \cdot) \) converges uniformly to a probability measure \( Q^\infty \) and that there exist two positive constants \( q < 1 \) and \( K \) such that

\[
||U^n f - U^\infty f||_L \leq K q^n ||f||_L, \quad \forall n \in \mathbb{N}^*, \quad \forall f \in L(I),
\]

where

\[
U^n f(\cdot) = \int_I f(y)Q^n(\cdot, dy), \quad U^\infty f(\cdot) = \int_I f(y)Q^\infty(dy).
\]

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\[4\]

**Proposition 4.** The invariant probability measure \( Q^\infty \) of the transformation \( T \) has the density \( \rho(x) = \frac{1}{(x + 1)(x + 2)}, x \in I \), with the normalizing factor \( \frac{1}{\log(4/3)} \).

**Proof.** On account of the uniqueness of \( Q^\infty \) we have to show that \( \int_I Q(x, B)Q^\infty(dx) = Q^\infty(B) \) for any \( B \in \mathcal{B} \). Since the intervals \([0, u) \subset I\) generate \( \mathcal{B} \) it is sufficient to check the above equation just for \( B = [0, u), 0 < u \leq 1 \). We have

\[
Q(x, [0, u)) = \sum_{k: \log_{\alpha}(x + 1) + 1} P(x, k).
\]

Then

\[
\frac{1}{\log(4/3)} \int_I Q(x, [0, u))\rho(x)dx = \frac{1}{\log(4/3)} \int_I \sum_{k: \log_{\alpha}(x + 1) + 1} \left( \frac{1}{\alpha^{k+1} + x + 1} - \frac{1}{\alpha^k + x + 1} \right) dx = \frac{1}{\log(4/3)} \log_2 \frac{2(u + 1)}{u + 2} = Q^\infty([0, u)).
\]

Similarly we can treat the case \( \alpha < u \leq 1 \) and obtain the desired result.

\[4\]

**4. Main result**

Now we are able to find the limiting distribution function

\[
F(x) = F_\infty(x) = \lim_{n \to \infty} \mu(T^n < x)
\]

and obtain a convergence rate result.
**Theorem 1 (A Gauss-Kuzmin-Type Theorem).** If \( \mu \) has a Riemann integrable density, then

\[
F(x) = \frac{1}{\log(4/3)} \log \frac{2(x + 1)}{x + 2}, \quad x \in I.
\]

If the density of \( \mu \) is a Lipschitz function, then there exist two positive constants \( q < 1 \) and \( K \) such that for all \( x \in I, n \in \mathbb{N}^* \)

\[
\mu(T^n < x) = \frac{1}{\log(4/3)} (1 + \theta q^n) \log \frac{2(x + 1)}{x + 2},
\]

where \( \theta = \theta(\mu, n, x) \), with \( |\theta| \leq K \).

Proof. By virtue of (9) we have

\[
U^\infty f_0 = \int_I f_0(x) Q^\infty(dx) = \frac{1}{\log(4/3)}, \quad f_0 \in L(I).
\]

Taking into account (8) there exist two constants \( q < 1 \) and \( K \) such that

\[
||U^n f_0 - U^\infty f_0|| \leq K q^n ||f_0||_{L}, \quad \forall n \in \mathbb{N}^*.
\]

As \( L(I) \) is a dense subset of \( C(I) (= \) the metric space of real continuous functions defined on \( I \) with the supremum norm \( ||f|| = \sup_{x \in I} |f(x)| \)) we have

\[
(10) \quad \lim_{n \to \infty} ||(U^n - U^\infty) f_0|| = 0
\]

for all \( f_0 \in C(I) \). Therefore (10) remains valid for measurable \( f_0 \) that are \( Q^\infty \)-almost surely continuous, that is, for Riemann-integrable \( f_0 \). We have

\[
F(x) = \lim_{n \to \infty} \mu(T^n < x) = \lim_{n \to \infty} \int_0^x U^n f_0(u) \rho(u) du =
\]

\[
= \frac{1}{\log(4/3)} \int_0^x \rho(u) du = \frac{1}{\log(4/3)} \log \frac{2(x + 1)}{x + 2}.
\]

\[\square\]

**Remark.** The study of optimality of the convergence rate remains an open question. A Wirsing-type approach [8] to the Perron-Frobenius operator of the associated transformation under its invariant measure, allows us to obtain a near-optimal solution to this problem and will appear in [7].

**References**


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