A chaotic game and its associated fractal

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Abstract

Fractals can be used to approximate the result of an iterative random process. In this paper we study a set of points generated by such a random process. To this purpose we define a fractal for which one we compute the Hausdorff dimension and the study is completed with the associated IFS.

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I. Introduction

Chaotic dynamical systems have received a great attention in last years. The usual definition of chaos is in Devaney's book [1]. Let (X, d) be a metric space and let $f : X \mapsto X$ be a continuous map. The dynamic system f is said to be chaotic if:
i. f is transitive, i.e. for all nonempty open subsets U and V of X, there is $k \in N$ such that $f^k(U) \cap V \neq \emptyset$.
ii. the periodic points of f form a dense subset of X.
iii. f has sensitive dependence on the initial conditions, i.e. there is a real $\delta > 0$ such that for every point $x \in X$ and for every neighborhood W of x there is a point $y \in W$ and $n \in N$ such that $d\left(f^n(x) - f^n(y)\right) > \delta$.
The transitivity is an irreducibility condition, while the second condition is an element of regularity. The sensitivity means that small initial errors lead to large divergences. There are many notions related to chaotic maps. We present briefly the idea of Iterated Function System which will be used further. Let (X, d) be a complete metric space; for every compact set $A \subseteq X$ and $\varepsilon > 0$, we define the ε-collar of A by $A_\varepsilon = \{x \in X \mid \exists y \in A \text{ such that } d(x, y) \leq \varepsilon\}$. The Hausdorff distance between two compact sets A and B is

$$h(A, B) = \inf\{\varepsilon \mid A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\}.$$

The set of compact subsets of X equipped with the Hausdorff distance is a complete metric space, denoted by $(\mathcal{K}(X), h)$. Let $w_1, w_2, ..., w_n$ be contractions on X with contraction factors $c_1, c_2, ..., c_n$, respectively. On the metric space $\mathcal{K}(X)$ we define the Hutchinson operator by

$$W(A) = w_1(A) \cup w_2(A) \cup ... \cup w_n(A),\ \forall A \in \mathcal{K}(X).$$

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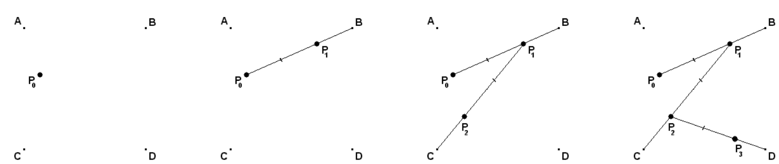
The operator W is a contraction on $(\mathcal{K}(X), h)$, hence it has a fixed point denoted by A_∞; of course, $W(A_\infty) = A_\infty$. The contraction factor of W is $C = \sup\{c_1, c_2, ..., c_n\}$. The iteration of the Hutchinson operator is called the Iterated Function System (IFS) and the fixed point A_∞ is called the attractor of the IFS. The classical fractals (Cantor set, Sierpinski gasket, Sierpinski carpet, Koch curve, etc) are attractors corresponding to adequate Hutchinson operators. Let us now suppose that the contractions $w_1, w_2, ..., w_n$ are similarities (scaling, rotation, translation). It results that the attractor A_∞ is self similar simply because $A_\infty = W(A_\infty) = w_1(A_\infty) \cup w_2(A_\infty) \cup ... \cup w_n(A_\infty)$. Sometimes it is possible to compute the self similarity dimension of A_∞; for instance, if the mappings $w_1, w_2, ..., w_n$ are all one to one, if they have the same contraction factor $C = c_1 = c_2 = ... = c_n$ and if $w_k(A_\infty) \cap w_m(A_\infty) = \emptyset, \forall k \neq m$, then the self similarity of A_∞ (denoted by d) is the solution of the equation $nc^d = 1$, i.e. $d = \frac{\log n}{\log \frac{1}{c}}$. If $c_1, c_2, ..., c_n$ are different, then d satisfies $c_1^d + c_2^d + ... + c_n^d = 1$.

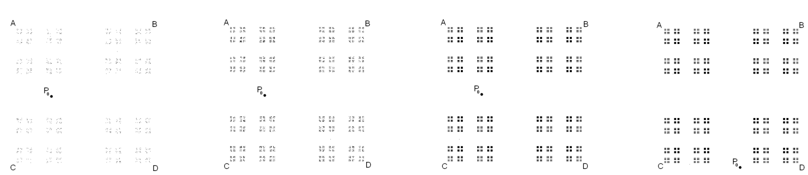
II. A random game

Let us start with the following random game:

We consider a die with four letters: A, B, C, D with equal probabilities and we denote the four vertices of a square by A, B, C, D. We start with an arbitrary point in the plane denoted by P_0. We throw the die and we get a letter $L \in \{A, B, C, D\}$. We generate the new point P_1, which is located on the segment $[P_0L]$ at one third from L. We continue to play with P_1 instead of P_0. In this way we get a sequence of points $P_0, P_1, P_2, ...$ We present below the first 4 steps of this game:



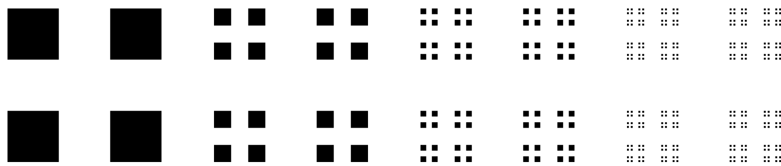
The next pictures present the set of points $\{P_n\}_n$ after the first $1000, 5000$ and 2000000 iterations. We mention that by repeating the experiment with another arbitrary P_0, after a sufficient large number of steps, we get the same stabilized picture.



III. The attractor

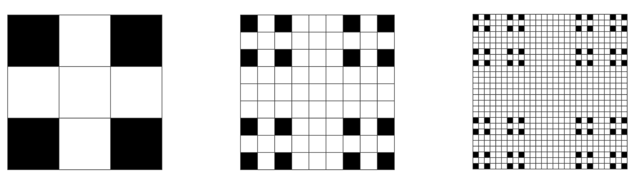
In the plane we consider a square. We divide the square in 9 equal squares by dividing each edge in 3 equal segments. We eliminate "the middle third", i.e., we keep the four squares from the corners. We continue the process and we denote by M_n the set obtained after the n-th iteration. We present below the first 4 steps:

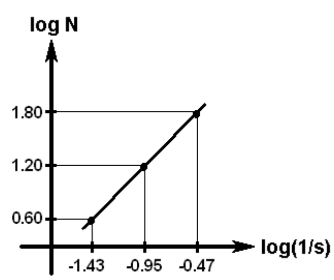
We observe that, by repeating the process infinitely many times we get a fractal which approximates the result of the random game presented in the introduction. We denote this fractal by M_∞.



Proposition *The set M_n consists of 4^n squares, each of them of edge $L_n = \left(\frac{1}{3}\right)^n$. Consequently, the perimeter of M_n is $P_n = 4\left(\frac{4}{3}\right)^n$ and the area $S_n = \left(\frac{4}{9}\right)^n$. It results that the length of the fractal M_∞ is $P_\infty = \lim_{n\to\infty} P_n = \infty$ and its area is $S_\infty = \lim_{n\to\infty} S_n = 0$.*

Usually, it is difficult to compute the Hausdorff dimension for an arbitrary fractal. A very efficient method to approximate it is the Box Counting method, [2], [3], [4]. Let us consider a picture (structure). We cover the structure with a number of square boxes of size s. We count the number of boxes which contain some part of the structure and let $N(s)$ be this number. Clearly, if we increase the number of boxes, or, equivalently, we decrease s to p, we obtain $N(p)$ instead of $N(s)$. Finally, we obtain the following diagram: on the Ox-axis we measure $-\log(s)$ and on the Oy-axis we measure $\log(N(s))$. In this way, we obtain several points for different values of s. The Box Counting Dimension of the structure is defined as the slope of the regression line defined by the points on the diagram. We compute below the Box Counting Dimension of the fractal M_∞.





s	N	log N	log(1/s)
27	4	0.60	-1.43
9	16	1.20	-0.95
3	64	1.80	-0.47

The Box Counting Dimension of M_∞ is $D = \frac{\log(4)}{\log(3)}$. We observe that $D = 2\frac{\log(2)}{\log(3)}$, i.e., it is the double of the Hausdorff dimension of the Cantor set.

IV. The iterated function system associated to M_∞.

The Iterated Function System (see [2]) which generates the fractal M_∞ is given by the following four contractions:

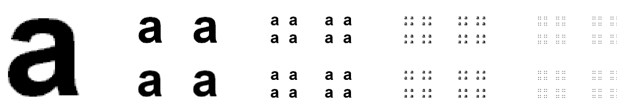
$$w_1(x,y) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} (x,y)^T + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{x}{3} \\ \frac{y}{3} \end{pmatrix}.$$

$$w_2(x,y) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} (x,y)^T + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{x}{3} + \frac{2}{3} \\ \frac{y}{3} \end{pmatrix}.$$

$$w_3(x,y) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} (x,y)^T + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{x}{3} \\ \frac{y}{3} + \frac{2}{3} \end{pmatrix}.$$

$$w_4(x,y) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} (x,y)^T + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{x}{3} + \frac{2}{3} \\ \frac{y}{3} + \frac{2}{3} \end{pmatrix}.$$

If A is a subset of the plane, then the IFS is $W(A) = w_1(A) \cup w_2(A) \cup \cup\, w_3(A) \cup w_4(A)$. The fixed point of the operator W is the fractal M_∞, i.e., $W(M_\infty) = M_\infty$. We give below the first five iterations of the operator W on a subset of the plane:



V. Conclusions

The study of the structure generated by the random process was reduced to the study of a fractal generated by a deterministic process defined by the contraction W. The properties of this fractal have been studied, including the Hausdorff dimension and the associated IFS.

References

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