The variational problem in Lagrange spaces endowed with (α, β)-metrics

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Abstract

In this paper we study the variational problem of Lagrange spaces with (α, β)-metrics. The results follow the classical ones and some results of Miron R. concerning Lagrange spaces.

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1 Introduction

Let (TM, τ, M) be the tangent bundle of a C^∞-differentiable real, n-dimensional manifold M. If (U, φ) is a local chart on M, then the coordinates of a point $u = (x, y) \in \tau^{-1}(U) \subset TM$ will be denoted by (x, y). Following R.Miron [1], we have the following

Definition 1. a) A *differentiable Lagrangian* on TM is a mapping $L : (x, y) \in TM \to L(x, y) \in \mathbb{R}, \forall u = (x, y) \in TM$, which is of class C^∞ on $\widetilde{TM} = TM \backslash \{0\}$ and is continuous on the null section of the projection $\tau : TM \to M$, such that

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \tag{1.1}$$

is a $(0, 2)-$type symmetric d-tensor field on TM.

b) A differential Lagrangian L on TM is said to be *regular* if

$$\text{rank } ||g_{ij}(x, y)|| = n, \quad \forall (x, y) \in \widetilde{TM}.$$

We will further use its contravariant d-tensor $g^{ij}(x, y)$ given by $g^{ik} g_{kj} = \delta^i_j$.

c) A *Lagrange space* is a pair $L^n = (M, L)$ formed by a smooth real n-dimensional manifold M and a regular differentiable Lagrangian L on M, for which the d-tensor field g_{ij} from (1.1) has constant signature on $\widetilde{TM}$.

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Let $L : TM \to \mathbb{R}$ be a differentiable Lagrangian on the manifold M, which is not necessarily regular. A curve $c : t \in [0, 1] \to (x^i(t)) \in U \subset M$ having the image in a domain of a chart U of M, has the extension to $\widetilde{TM}$ given by $c^* : t \in [0, 1] \to (x^i(t), \frac{dx^i}{dt}(t)) \in \tau^{-1}(U)$.

The integral of action of the Lagrangian L on the curve c is given by the functional

$$I(c) = \int_0^1 L\left(x(t), \frac{dx}{dt}\right) dt. \tag{1.2}$$

Consider the curves $c_\varepsilon : t \in [0, 1] \to (x^i(t) + \varepsilon V^i(t)) \in M$, which have the same endpoints $x^i(0), x^i(1)$ as the curve c, $V^i(0) = V^i(1) = 0$ and ε is a real number, sufficiently small in absolute value, such that $\mathrm{Im} c_\varepsilon \subset U$. The extension of the curve c_ε to TM is

$$c^*_\varepsilon : t \in [0, 1] \to \left(x^i(t) + \varepsilon V^i(t), \frac{dx^i}{dt} + \varepsilon \frac{dV^i}{dt}\right) \in \tau^{-1}(U).$$

The integral of action of the Lagrangian L on the curve c_ε is

$$I(c_\varepsilon) = \int_0^1 L\left(x + \varepsilon V, \frac{dx}{dt} + \varepsilon \frac{dV}{dt}\right) dt.$$

A necessary condition for $I(c)$ to be an extremal value of $I(c_\varepsilon)$ is

$$\left.\frac{dI(c_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0.$$

In order that the functional $I(c)$ be an extremal value of $I(c_\varepsilon)$ it is necessary that c be the solution of the Euler-Lagrange equations

$$E_i(L) \equiv \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = 0, \quad y^i = \frac{dx^i}{dt}.$$

2 The fundamental tensor of a Lagrange space with (α, β)-metric

We consider the functions defined on TM

$$\alpha(x, y) = \sqrt{\gamma_{ij}(x) y^i y^j}, \quad \beta(x) = A_i(x) y^i,$$

where $\gamma_{ij}(x)$ is the metric tensor of a Riemannian space $R^n = (M, \gamma_{ij}(x))$, and $A_i(x)$ is a covector field.

Definition 2. A Lagrange space $L^n = (M, L(x, y))$ is called *with* (α, β)-*metric* if the fundamental function $L(x, y)$ is a function $\hat{L}$, which depends only on $\alpha(x, y)$ and $\beta(x, y)$,

$$L = \hat{L}(\alpha(x, y), \beta(x, y)).$$

We shall use the following notations

$$\partial_i\alpha = \frac{\partial\alpha}{\partial y^i}, \quad \partial_i\beta = \frac{\partial\beta}{\partial y^i}, \quad \partial_i\partial_j\alpha = \frac{\partial^2\alpha}{\partial y^i\partial y^j},$$

$$\hat{L}_\alpha = \frac{\partial\hat{L}}{\partial\alpha},\ \hat{L}_\beta = \frac{\partial\hat{L}}{\partial\beta},\ \hat{L}_{\alpha\alpha} = \frac{\partial^2\hat{L}}{\partial\alpha^2},\ \hat{L}_{\alpha\beta} = \frac{\partial^2\hat{L}}{\partial\alpha\partial\beta}.$$

Proposition 1. *We have the relations*

$$\partial_i\alpha = \alpha^{-1}y_i, \quad \partial_i\partial_j\alpha = \alpha^{-1}\gamma_{ij}(x) - \alpha^{-3}y_iy_j,$$
$$\partial_i\beta = A_i(x), \quad \partial_i\partial_j\beta = 0,$$

where $y_i = \gamma_{ij}(x)y^j$.

We introduce *the moments* of the Lagrangian $L(x,y) = \hat{L}(\alpha(x,y), \beta(x,y))$,

$$p_i = \frac{1}{2}\frac{\partial L}{\partial y^i} = \frac{1}{2}\left(\hat{L}_\alpha\partial_i\alpha + \hat{L}_\beta\partial_i\beta\right)$$

and we get the following result

Proposition 2. *The moments of the Lagrangian* $L(x,y)$ *are given by*

$$p_i = \rho y_i + \rho_1 A_i,$$

where

$$\rho = \frac{1}{2}\alpha^{-1}\hat{L}_\alpha \quad \textit{and} \quad \rho_1 = \frac{1}{2}\hat{L}_\beta. \tag{2.3}$$

The two scalar functions defned in (2.3) are called *the principal invariants of the Lagrange space* L^n.

Proposition 3. *The derivatives of the principal invariants of the Lagrange space* L^n *are given by*

$$\partial_i\rho = \rho_{-2}y_i + \rho_{-1}A_i, \quad \partial_i\rho_1 = \rho_{-1}y_i + \rho_0 A_i,$$

where

$$\rho_{-2} = \frac{1}{2}\alpha^{-2}(\hat{L}_{\alpha\alpha} - \alpha^{-1}\hat{L}_\alpha), \quad \rho_{-1} = \frac{1}{2}\alpha^{-1}\hat{L}_{\alpha\beta}, \quad \rho_0 = \frac{1}{2}\hat{L}_{\beta\beta}.$$

Proposition 4. *The energy*

$$E_L = y_i\frac{\partial L}{\partial y^i} - L$$

of a Lagrangian with (α,β)*-metric is given by*

$$E_L = \alpha\hat{L}_\alpha + \beta\hat{L}_\beta - \hat{L}.$$

We can determine the fundamental tensor g_{ij} of the Lagrange space with (α,β)-metric, as follows

Theorem 1. *The fundamental tensor* g_{ij} *of the Lagrange space with* (α,β)*-metric is*

$$g_{ij} = \rho\gamma_{ij} + c_i c_j, \tag{2.4}$$

where $c_i = q_{-1}y_i + q_0 A_i$ and q_{-1}, q_0 satisfy the equations

$$\rho_0 = (q_0)^2, \quad \rho_{-1} = q_0 q_{-1}, \quad \rho_{-2} = (q_{-1})^2.$$

Theorem 2. *The reciprocal tensor g^{ij} of the fundamental tensor g_{ij} is given by*

$$g^{ij} = \frac{1}{\rho}\gamma^{ij} - \frac{1}{1+c^2}c^i c^j, \tag{2.5}$$

where $c^i = \rho^{-1}\gamma^{ij}c_j$ and $c^i c_i = c^2$.

3 Euler-Lagrange equations in Lagrange spaces with (α, β)-metric

The Euler-Lagrange equations of the Lagrange space L^n endowed with (α, β)-metric are

$$E_i(\hat{L}) \equiv \frac{\partial \hat{L}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial y^i}\right) = 0, \quad y^i = \frac{dx^i}{dt}.$$

Considering the relations

$$\frac{\partial \hat{L}}{\partial x^i} = \hat{L}_\alpha \frac{\partial \alpha}{\partial x^i} + \hat{L}_\beta \frac{\partial \beta}{\partial x^i}, \quad \frac{\partial \hat{L}}{\partial y^i} = \hat{L}_\alpha \frac{\partial \alpha}{\partial y^i} + \hat{L}_\beta \frac{\partial \beta}{\partial y^i}$$

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial y^i}\right) = \frac{d\hat{L}_\alpha}{dt}\frac{\partial \alpha}{\partial y^i} + \frac{d\hat{L}_\beta}{dt}\frac{\partial \beta}{\partial y^i} + \hat{L}_\alpha \frac{d}{dt}\left(\frac{\partial \alpha}{\partial y^i}\right) + \hat{L}_\beta \frac{d}{dt}\left(\frac{\partial \beta}{\partial y^i}\right).$$

by direct calculation we infer

$$E_i(\hat{L}) = \hat{L}_\alpha E_i(\alpha) + \hat{L}_\beta E_i(\beta) - \frac{\partial \alpha}{\partial y^i}\frac{d\hat{L}_\alpha}{dt} - \frac{\partial \beta}{\partial y^i}\frac{d\hat{L}_\beta}{dt}, \quad y^i = \frac{dx^i}{dt},$$

where

$$\frac{d\hat{L}_\alpha}{dt} = \hat{L}_{\alpha\alpha}\frac{d\alpha}{dt} + \hat{L}_{\alpha\beta}\frac{d\beta}{dt}, \quad \frac{d\hat{L}_\beta}{dt} = \hat{L}_{\beta\alpha}\frac{d\alpha}{dt} + \hat{L}_{\beta\beta}\frac{d\beta}{dt}.$$

Then we get

$$\begin{aligned} E_i(\hat{L}) = \quad & \hat{L}_\alpha E_i(\alpha) + \hat{L}_\beta E_i(\beta) - \frac{\partial \alpha}{\partial y^i}\left\{\hat{L}_{\alpha\alpha}\frac{d\alpha}{dt} + \hat{L}_{\alpha\beta}\frac{d\beta}{dt}\right\} - \\ & - \frac{\partial \beta}{\partial y^i}\left\{\hat{L}_{\beta\alpha}\frac{d\alpha}{dt} + \hat{L}_{\beta\beta}\frac{d\beta}{dt}\right\}. \end{aligned}$$

As well, we have

$$E_i(\alpha) = \frac{1}{2\alpha}E_i(\alpha^2) + \frac{1}{2}\frac{\partial \alpha}{\partial y^i}\frac{d\alpha}{dt}, \quad E_i(\beta) = F_{ir}\frac{dx^r}{dt},$$

where

$$F_{ir} = \frac{\partial A_r}{\partial x^i} - \frac{\partial A_i}{\partial x^r}$$

is the electromagnetic tensor field. Finally we have the following relation

$$\begin{aligned} E_i(\hat{L}) = \quad & \rho E_i(\alpha^2) + 2\rho \frac{\partial \alpha}{\partial y^i} \frac{d\alpha}{dt} + 2\rho_1 F_{ir} \frac{dx^r}{dt} - \\ & - \frac{\partial \alpha}{\partial y^i} \left\{ \hat{L}_{\alpha\alpha} \frac{d\alpha}{dt} + \hat{L}_{\alpha\beta} \frac{d\beta}{dt} \right\} - \frac{\partial \beta}{\partial y^i} \left\{ \hat{L}_{\beta\alpha} \frac{d\alpha}{dt} + \hat{L}_{\beta\beta} \frac{d\beta}{dt} \right\}. \end{aligned}$$

Proposition 5. *The Euler-Lagrange equation in the Lagrange space L^n endowed with (α, β)-metric are*

$$E_i(\hat{L}) = 0, \quad y^i = \frac{dx^i}{dt}.$$

Proposition 6. *The following relation holds:*

$$E_i(\alpha^2) = -2\gamma_{ir} \left[\frac{d^2 x^r}{dt^2} + \left\{ {r \atop jk} \right\} \frac{dx^j}{dt} \frac{dx^k}{dt} \right].$$

If we have the natural parametrization of the curve $c : t \in [0,1] \to (x^i(t)) \in M$ relative to the Riemannian metric $\gamma_{ij}(x)$, then $\alpha \left(x, \frac{dx}{dt} \right) = 1$. Then we get:

Theorem 3. *In the canonical parametrization the Euler-Lagrange equations in L^n spaces with (α, β)-metric are*

$$\begin{aligned} E_i(\hat{L}) \equiv \quad & -2\rho\gamma_{ir} \left[\frac{d^2 x^r}{ds^2} + \left\{ {r \atop jk} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} \right] + 2\rho_1 F_{ir} \frac{dx^r}{ds} - \\ & - \frac{d\beta}{ds} \left\{ \hat{L}_{\beta\alpha} \frac{\partial \alpha}{\partial y^i} + \hat{L}_{\beta\beta} \frac{\partial \beta}{\partial y^i} \right\} = 0. \end{aligned} \tag{3.6}$$

Proposition 7. *If the 1-form β is constant on the integral curve c of the Euler-Lagrange equations, then (3.6) rewrite as the Lorentz equations of the space L^n,*

$$\frac{d^2 x^i}{ds^2} + \left\{ {i \atop jk} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = \frac{\rho_1}{\rho} F^i_r \frac{dx^r}{ds},$$

where we denoted $F^i_j = \gamma^{ik} F_{kj}$.

References

[1] Miron R., Anastasiei M., *The Geometry of Lagrange Spaces. Theory and Applications*, Kluwer Acad. Publ., 1994.

[2] Shibata C., *On Finsler space with (α, β)-metrics*, J. of Hokkaido Univ. of Education 35 (1984), 1-16.

[3] Sabau V.S., Shimada H., *Some remarks on Finsler spaces with (α, β) metrics*, Symp. on Finsler Geom. at Hakodate, Sept. 16-19, 1999.

[4] Nicolaescu B., *Lagrange spaces with* (α, β)*-metrics*, Applied Sciences (electronic, http:www.mathem.pub.ro/apps), 1, 1 (2001), 42-47.

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