Critical associated metrics on framed φ-manifolds

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Abstract

Using the induced almost complex and almost contact structures on framed φ-manifolds, defined in [7], we extend some results from the theory of the critical associated metrics on symplectic and on contact manifolds to the case of framed φ-manifolds.

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1 Introduction

Let M be an m-dimensional smooth manifold endowed with a tensor field φ of type (1,1), satisfying the algebraic condition

$$\varphi^3 + \varphi = 0. \tag{1.1}$$

The geometric structure on M defined by φ is called a φ-structure of rank r if the rank r of φ is constant on M and, in this case, M is called a φ-manifold. It follows easily that r is an even number.

If M is a φ-manifold and if there are $m-r$ vector fields ξ_a and $m-r$ differential 1-forms η_a satisfying

$$\varphi^2 = -I + \sum_{a=1}^{m-r} \eta_a \otimes \xi_a, \tag{1.2}$$

$$\eta_a(\xi_b) = \delta_{ab}, \tag{1.3}$$

where $a, b = 1, 2, ..., m-r$, M is said to be globally framed or to have a framed φ-structure. In this case M is called a globally framed φ-manifold or, simply, a framed φ-manifold. From 1.2 and 1.3, one obtains, by some algebraic computations

$$\varphi\xi_a = 0, \quad \eta_a \circ \varphi = 0, \quad \varphi^3 + \varphi = 0. \tag{1.4}$$

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If $m = 2n + 1$ and $rank\ \varphi = 2n$ one obtains an almost contact structure on M.

A framed φ-structure is normal if the tensor field S of type (1,2) defined by

$$S = N_\varphi + \sum_{a=1}^{m-r} d\eta_a \otimes \xi_a, \tag{1.5}$$

vanishes, (see [7]), where

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y], \tag{1.6}$$

with $X, Y \in \chi(M)$, defines the Nijenhuis tensor field, N_φ, of φ.

If g is a (semi-)Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{a=1}^{m-r} \eta_a(X)\eta_a(Y), \tag{1.7}$$

then we say that $(\varphi, \xi_a, \eta_a, g)$ is a metric framed φ-structure and M is called a metric framed φ-manifold.

The metric g is called an associated (semi-)Riemannian metric.

The fundamental 2-form, Ω, of the considered metric framed φ-manifold, M, is defined just like in the case of the almost Hermitian and almost contact metric manifold, by $\Omega(X,Y) = g(X, \varphi Y)$, for any $X, Y \in \chi(M)$.

The framed φ-manifold M with structure tensors $(\varphi, \xi_a, \eta_a, g)$ is called a $\mathcal{C}$-manifold if it is normal, $d\Omega = 0$ and $d\eta_a = 0$, $a = 1, ..., m-r$.

2 A global invariant on the even dimensional framed φ-manifolds

Using the induced almost complex structure, $\widetilde{\varphi} = \varphi + \sum_{a=1}^{s}(\eta_{2a} \otimes \xi_{2a-1} - \eta_{2a-1} \otimes \xi_{2a})$, on a framed φ-manifold, $(M, \varphi, \xi_1, ..., \xi_{2s}, \eta_1, ..., \eta_{2s})$, defined in [7], and the main result in [8], one obtains

Theorem 2.1. *Let M be an even dimensional compact manifold, with $dim\ M = 2m + 2s$. Assume that there exists the global 2-form Ω and the global 1-forms $\{\eta_1, ..., \eta_{2s}\}$, on M, such that the 2-form $\widetilde{\Omega}$, defined by*

$$\widetilde{\Omega} = \Omega + \sum_{a=1}^{s}(\eta_{2a-1} \wedge \eta_{2a}),$$

is closed and of maximal rank. Also assume that M carries a metric framed φ-structure, $(\varphi, \xi_a, \eta_a, g)$, $a = 1, ..., 2s$, with the fundamental 2-form Ω. Let $\mathcal{A}$ be the set of all associated metrics to the framed φ-structures, (φ, ξ_a, η_a), $a = 1, ..., 2s$, on M, with the fundamental 2-form Ω, (note that the 1-forms, $\{\eta_a\}_{a=1}^{2s}$, and the vector fields $\{\xi_a\}_{a=1}^{2s}$ are the same for all this structures). Then

$$I = \int_M (R + R^* + T) d\nu$$

is a constant on the set $\mathcal{A}$, where R and R^* are the scalar and the $*$-scalar curvature of a metric $g \in \mathcal{A}$, and the local expression of T is

$$T = -\frac{1}{2}R^j_{lhk}[2\Omega^{hk} + \sum_{a=1}^{s}(\xi^h_{2a-1}\xi^k_{2a} - \xi^k_{2a-1}\xi^h_{2a})]\sum_{b=1}^{s}(\eta_{2bj}\xi^l_{2b-1} - \eta_{2b-1j}\xi^l_{2b}),$$

where R^j_{lhk} are the local components of the curvature tensor field of the Levi-Civita connection, ∇, of the metric g, Ω_{ij} are the components of the fundamental 2-form Ω and $\Omega^{hk} = g^{hi}\Omega_{ij}g^{jk}$, $g^{ij}g_{jk} = \delta^i_k$, and $\xi_a = \xi^i_a\frac{\partial}{\partial x^i}$, $\eta_a = \eta_{ai}dx^i$, for any $a = 1, ..., 2s$, with respect the local coordinates on M.

Remark 2.2. With the assumptions in the previous theorem, as in the symplectic case or in the contact case, (see [5]), it can be proved that the set $\mathcal{A}$ is infinite dimensional, and $\mathcal{A}$ is totaly geodesic in the set $\mathcal{N}$, of the Riemannian metrics on M with the same volume element. Note that the same results are true in the case of the odd dimensional manifolds considered in Theorem 3.2.

Remark 2.3. If on the Riemannian manifold (M, h) with $dim\ M = 2m + s$, we have a global 2-form, Ω, such that $\Omega^m \neq 0$, the 1-forms $\{\eta_1, ..., \eta_s\}$ and the vector fields $\{\xi_1, ..., \xi_s\}$, such that $\eta_a(\xi_b) = \delta_{ab}$ and $\Omega(X, \xi_a) = 0$, for any $a, b = 1, ..., s$, $X \in \chi(M)$, then, using the same method as in the case of contact manifolds, (see [5]), it can be proved that M carries a metric framed φ-structure, $(\varphi, \xi_a, \eta_a, g)$, such that the fundamental 2-form of g is Ω.

From a result in [8] and from the previous theorem we have

Theorem 2.4. *Let M be a compact even dimensional manifold as in Theorem 2.1. If this manifold carries a normal metric framed φ-structure, then*

$$I = 2\int_M R d\nu$$

is a constant on the set $\mathcal{A}$, where R is the scalar curvature.

Remark 2.5. It is easy to see that an example of a manifold with the properties in the previous theorems is an even dimensional $\mathcal{C}$-manifold with the condition that $\widetilde{\Omega}$ is of maximal rank, where $\widetilde{\Omega}$ is obtained as in Theorem 2.1.

3 Critical associated metrics on the framed φ-manifolds

Using the induced almost complex structure on an even dimensional framed φ-manifold, the results in [2] and the same notations as in Theorem 2.1, one obtains

Theorem 3.1. *Let M be an even dimensional compact manifold, with the same properties as in Theorem 2.1. Then a metric $g \in \mathcal{A}$ is a critical point of*

$$E(g) = \int_M R d\nu, \quad F(g) = \int_M (R - R^* - T) d\nu$$

if and only if the Ricci operator, Q, of g, has the properties

$$Q\varphi X = \varphi QX + \sum_{a=1}^{s}(\eta_{2a}(QX)\xi_{2a-1} - \eta_{2a-1}(QX)\xi_{2a}), \tag{3.1}$$

for $X \in (span\{\xi_a\}_{a=1}^{2s})^{\perp}$, and

$$\begin{cases} \varphi Q\xi_{2a-1} = \varphi^2 Q\xi_{2a}, \\ \eta_{2a-1}(Q\xi_{2a-1}) = \eta_{2a}(Q\xi_{2a}), \\ \eta_{2a}(Q\xi_{2a-1}) = -\eta_{2a-1}(Q\xi_{2a}), \end{cases} \tag{3.2}$$

for any $a = 1, ..., s$.

In the case of an odd dimensional framed φ-manifold we can use a induced almost contact structure also defined in [7] and some results from [3]. One obtains the following

Theorem 3.2. *Let M be a compact odd dimensional manifold, with dim $M = 2m+2s+1$, such that there exists the 2-form, Ω, the 1-forms, $\{\eta_1, ..., \eta_{2s+1}\}$, with the condition*

$$\widetilde{\Omega} = \Omega + \sum_{i=1}^{s}(\eta_{2i-1} \wedge \eta_{2i}) = d\eta_{2s+1}, \tag{3.3}$$

which carries a metric framed φ-structure, $(\varphi, \xi_a, \eta_a, g)$, $a = 1, ..., 2s$, with the fundamental 2-form Ω, and let $\mathcal{A}$ be the set of all associated metrics to framed φ-structures, (φ, ξ_a, η_a), on M, with the fundamental 2-form Ω. Then a metric $g \in \mathcal{A}$ is a critical point of

$$A(g) = \int_M R d\nu$$

in $\mathcal{A}$, if and only if

$$Q\varphi X = \varphi QX + \sum_{a=1}^{s}(\eta_{2a}(QX)\xi_{2a-1} - \eta_{2a-1}(QX)\xi_{2a}), \tag{3.4}$$

for $X \in (span\{\xi_a\}_{a=1}^{2s+1})^{\perp}$, and

$$\begin{cases} \varphi Q\xi_{2a-1} = \varphi^2 Q\xi_{2a}, \\ \eta_{2a-1}(Q\xi_{2a-1}) = \eta_{2a}(Q\xi_{2a}), \\ \eta_{2a}(Q\xi_{2a-1}) = -\eta_{2a-1}(Q\xi_{2a}), \end{cases} \tag{3.5}$$

for any $a = 1, ..., s$, where Q is the Ricci operator corresponding to the metric g.

Using the results in [4] we have an immediate result, concerning the odd dimensional framed φ-manifolds

Theorem 3.3. *Let M be a a compact odd dimensional manifold as in Theorem 3.2. Then a metric $g \in \mathcal{A}$ is a critical point of*

$$I(g) = \int_M (R + R^* + T) d\nu$$

in $\mathcal{A}$, where T is given by

$$T = -\frac{1}{2}R^j_{lhk}[2\Omega^{hk} + \sum_{a=1}^{s}(\xi^h_{2a-1}\xi^k_{2a} - \xi^k_{2a-1}\xi^h_{2a})]\sum_{b=1}^{s}(\eta_{2bj}\xi^l_{2b-1} - \eta_{2b-1j}\xi^l_{2b}),$$

if and only if ξ_{2s+1} is a Killing vector field with respect g.

Finally, from the main result in [1] one obtains

Theorem 3.4. *Let M be a a compact odd dimensional manifold as in Theorem 3.2. Then a metric $g \in \mathcal{A}$ is a critical point of*

$$L(g) = \int_M Ric(\xi_{2s+1})d\nu,$$

in $\mathcal{A}$, if and only if ξ_{2s+1} is a Killing vector field with respect with g.

Remark 3.5. By changing indices, it is easy to see that, in the conditions of the previous theorem, a metric $g \in \mathcal{A}$ is a critical point of $L(g) = \int_M \xi_a d\nu$, $a = 1, ..., 2s+1$, if and only if ξ_a is a Killing vector field with respect with g.

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