Functions of Markov chains

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Abstract

A function of a homogeneous Markov chain preserves the Chapman-Kolmogorov property if and only if the transition matrix of this Markov chain satisfies some conditions. In this paper we give a theorem which describes the structure of a matrix for which the conditions mentioned above are fulfilled.

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1 Introduction

Let \((x_n)_{n \in \mathbb{N}}\) be a homogeneous Markov chain on \((\Omega, \mathcal{K}, P)\) with the state space \(X = \{1, 2, \ldots, m\}\), \(Y = \{1, \ldots, p\}\), \(p < m\), a set and \(\varphi : X \to Y\) a surjection. The problem is to find the conditions under which the stochastic process \((\varphi \circ x_n)_{n \in \mathbb{N}}\) has the Chapman-Kolmogorov property, i.e. for any \(n_1 < n_2 < n_3\) and any \(i, j \in \{1, \ldots, p\}\)

\[
P(\varphi \circ x_{n_3} = i \mid \varphi \circ x_{n_1} = j) = \sum_{\alpha=1}^{p} P(\varphi \circ x_{n_3} = i \mid \varphi \circ x_{n_2} = \alpha) P(\varphi \circ x_{n_2} = \alpha \mid \varphi \circ x_{n_1} = j).
\]

In the particular case when \([\mu]Q = [\mu]\), where \(\mu = P \circ x_0^{-1}\) is the initial distribution of the Markov chain \((x_n)_{n \in \mathbb{N}}\), \(Q = (P(x_{n+1} = j \mid x_n = i))_{1 \leq i,j \leq m}, \ n \geq 0\), is its transition matrix and \([\mu] = (\mu(\{1\}) \ldots \mu(\{m\})), \ \mu(\{i\}) > 0\) for any \(i = 1, \ldots, m\), the stochastic process \((\varphi \circ x_n)_{n \in \mathbb{N}}\) has the Chapman-Kolmogorov property if and only if \(\varphi(Q^n) = \varphi(Q)^n, \ n \geq 1\) (for the definition of \(\varphi(Q)\) see section 2). In a previous paper (see [1]) we proved a theorem that gives conditions under which the relations \(\varphi(Q^n) = \varphi(Q)^n, \ n \geq 1\), hold. In the present paper we give a stronger result, this is a theorem that describes the structure of a matrix \(Q\) which satisfies the \((\mu, \varphi)\)-condition, i.e. \(Q\) is stochastic, \([\mu]Q = [\mu]\) and \(\varphi(Q^n) = \varphi(Q)^n, \ n \geq 1\).

2 The \((\mu, \varphi)\)-condition

As in the previous section, we consider that \((x_n)_{n \in \mathbb{N}}\) is a homogeneous Markov chain on \((\Omega, \mathcal{K}, P)\) with the state space \(X = \{1, 2, \ldots, m\}\), \(\mu = P \circ x_0^{-1}\) its initial distribution and \(Q = (P(x_{n+1} = j | x_n = i))_{1 \leq i, j \leq m}\) its transition matrix. We denote \([\mu] = (\mu(\{i\}) \ldots \mu(\{m\}))\) and we suppose that \(\mu(\{i\}) > 0\) for any \(i = 1, \ldots, m\).

Let \(Y = \{1, \ldots, p\}\), \(p < m\), and let \(\varphi : X \to Y\) be a surjection such that for \(i \in Y\), \(\varphi^{-1}(i) = \{t_{i-1} + 1, t_{i-1} + 2, \ldots, t_i\}\), where \(0 = t_0 < t_1 < \ldots < t_p = p\).

We denote \([\mu^\varphi] = (\mu^\varphi_{i,j})_{1 \leq i \leq p, 1 \leq j \leq m}\), where for each \(i \in Y\),

\[
\mu^\varphi_{i,j} = \frac{\mu(\{j\})}{\sum_{s=1}^{t_i-t_{i-1}-1} \mu(\{t_{i-1} + s\})} \quad \text{if } j \in \varphi^{-1}(i) \quad \text{and} \quad \mu^\varphi_{i,j} = 0 \quad \text{if } j \notin \varphi^{-1}(i),
\]

\[\{I_\varphi\} = (\epsilon^\varphi_{i,j})_{1 \leq i \leq m, 1 \leq j \leq p}, \quad \text{where for each } j \in Y, \epsilon^\varphi_{i,j} = 1 \text{ if } i \in \varphi^{-1}(j) \text{ and } \epsilon^\varphi_{i,j} = 0 \text{ if } i \notin \varphi^{-1}(j) \quad \text{and}
\]

\[\varphi(A) = [\mu^\varphi] \cdot A \cdot [I_\varphi]
\]

if \(A\) is a square matrix of order \(m\).

**Proposition.** If \(\mu\) and \(\varphi\) are as above and \([\mu]Q = [\mu]\), then the stochastic process \((\varphi \circ x_n)_{n \in \mathbb{N}}\) has the Chapman-Kolmogorov property if and only if \(\varphi(Q^n) = \varphi(Q)^n\), \(n \geq 1\).

**Proof.** Let \(\Pi_n = (P(x_{\nu+n} = i | x_\nu = j))_{1 \leq i, j \leq m}\) and \(\Phi_n = (P(\varphi \circ x_{\nu+n} = i | \varphi \circ x_\nu = j))_{1 \leq i, j \leq p} (\nu \in \mathbb{N})\).

\[
P(\varphi \circ x_{\nu+n} = i | \varphi \circ x_\nu = j) = \frac{P(\varphi \circ x_{\nu+n} = i, \varphi \circ x_\nu = j)}{P(\varphi \circ x_\nu = j)} = \sum_{\alpha=1}^{t_i-t_{i-1}-1} \sum_{\beta=1}^{t_j-t_{j-1}-1} P(x_{\nu+n} = t_{i-1} + \alpha, x_\nu = t_{j-1} + \beta) \\
\sum_{\beta=1}^{t_j-t_{j-1}} \mu(\{t_{j-1} + \beta\}) \left( \sum_{\alpha=1}^{t_i-t_{i-1}-1} P(x_{\nu+n} = t_{i-1} + \alpha, x_\nu = t_{j-1} + \beta) \right)
\]

\[
(\mu(\{i\}) = P(x_n = i), \ n \geq 0). \text{ So we have } \Phi_n = \varphi(\Pi_n), \ n \geq 1.
\]

Because \((x_n)_{n \in \mathbb{N}}\) is a Markov chain, it has the Chapman-Kolmokorov property and this property can be written \(\Pi_{n_1+n_2} = \Pi_{n_1} \cdot \Pi_{n_2}\), \(n_1, n_2 \geq 1\), or, in an equivalent form, \(\Pi_n = \Pi^n\), \(n \geq 1\). Since \(\Pi_1 = Q\), we have \(\Pi_n = Q^n\).

\(\varphi \circ x_n\) has the Chapman-Kolmogorov property if and only if \(\Phi_{n_1+n_2} = \Phi_{n_1} \cdot \Phi_{n_2}\), \(n_1, n_2 \geq 1\), or \(\Phi_n = \Phi^n\), \(n \geq 1\), but it can be written \(\varphi(\Pi_n) = \varphi(Q^n), \ n \geq 1\), or \(\varphi(Q^n) = \varphi(Q)^n, \ n \geq 1\). Q.E.D.

**Remark.** If \(\varphi(Q^n) = \varphi(Q)^n, \ n \geq 1\), then the spectrum of \(Q\) contains the spectrum of \(\varphi(Q)\).

Indeed, if \(\varphi(Q^n) = \varphi(Q)^n, \ n \geq 1\), and \(P\) is a polynomial such that \(P(Q) = 0\), then \(P(\varphi(Q)) = 0\). If \(P_Q\) is the minimal polynomial of \(Q\) and \(P_Q(\varphi(Q))\) the minimal
polynomial of $\varphi(Q)$, then $P_{\varphi(Q)}$ is a divisor of $P_Q$. Consequently the spectrum of $Q$ contains the spectrum of $\varphi(Q)$.

**Definition.** If $Q$ is a stochastic matrix, $[\mu]Q = [\mu]$ and $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$, then we say that $Q$ satisfies the $(\mu, \varphi)$-condition.

We want to know which is the structure of a matrix $Q$ that satisfies the $(\mu, \varphi)$-condition, provided that $\mu$ and $\varphi$ are as in the beginning of this section. The theorem which we will prove in the following section will bring us very near to this goal.

3 The structure of a stochastic matrix which satisfies the $(\mu, \varphi)$-condition

If $A$ is a square matrix and $\lambda$ is an eigenvalue of $A$, then we denote by $m_A(\lambda)$ the multiplicity order of $\lambda$ as a root of the characteristic polynomial of $A$.

If $u, v \in \mathcal{M}_{m \times 1}(\mathbb{R})$ we use the notation $\varphi u = [\mu] \cdot u$ and $^tv = v \cdot [I_\varphi]$.

**Theorem.** Let $\mu$ be a distribution on $X$ with $\mu(\{i\}) > 0$, $i \in X$, and $\varphi : X \to Y$ a surjection as above. Then the matrix $Q$ satisfies the $(\mu, \varphi)$-condition, all the eigenvalues of $Q$ are real and $m_{\varphi(Q)}(\lambda) = m_Q(\lambda)$ for any eigenvalue $\lambda$ of $\varphi(Q)$ if and only if $Q$ satisfies the following conditions:

1) $Q = \sum_{i=1}^{k} \left( \lambda_i \sum_{j=n_{i-1}+1}^{n_i} u_j^t v_j + \sum_{j=n_{i-1}+1}^{n_i} \varepsilon_j u_j^t v_{j+1} \right), \quad \lambda_1 = 1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$, $u_j, v_j \in \mathcal{M}_{m \times 1}(\mathbb{R})$ and $\varepsilon_j = 0, \varepsilon_j \in \{0, 1\}$ for any $j = 2, \ldots, m$, $0 = n_0 < n_1 < \ldots < n_k = m$.

2) $^tv_1 = \delta_{ij}, i, j \in \{1, \ldots, m\}$.

3) $^tv_1 = (1 \ldots 1)$.

4) There is $l \in \{1, \ldots, k\}$ such that $p = n_l$.

5) If $i \geq l + 1$, then $\sum_{j=n_{i-1}+1}^{n_i} \varphi u_j^t v_j = 0$ and

$\sum_{j=n_{i-1}+1}^{n_i-1} \varepsilon_j \varepsilon_{j+1} \ldots \varepsilon_{j+q-1} \varphi u_j^t v_{j+q} = 0$ for any $q = 1, \ldots, n_i - n_{i-1} - 1$.

6) The matrix $Q$ has only positive entries.

**Proof.** First we suppose that the matrix $Q$ satisfies the $(\mu, \varphi)$-conditions, all the eigenvalues of $Q$ are real and $m_{\varphi(Q)}(\lambda) = m_Q(\lambda)$ for any eigenvalue $\lambda$ of $\varphi(Q)$.

Let $\{\lambda_1, \ldots, \lambda_k\}$ be the spectrum of $Q$ and $P_Q(\lambda) = (\lambda - \lambda_1)^{s_1} \ldots (\lambda - \lambda_k)^{s_k}$ the minimal polynomial of $Q$. If $\frac{1}{P_Q(\lambda)} = a_1(\lambda) + \ldots + a_k(\lambda)$, where $a_1(\lambda), \ldots, a_k(\lambda)$ are polynomials, then

$$1 = S_1(\lambda) + \ldots + S_k(\lambda).$$

(3.1)

where $S_i(\lambda) = a_i(\lambda)(\lambda - \lambda_1)^{s_1} \ldots (\lambda - \lambda_{i-1})^{s_{i-1}} (\lambda - \lambda_{i+1})^{s_{i+1}} \ldots (\lambda - \lambda_k)^{s_k}$.

From relation (3.1) we get

$$I_n = S_1(Q) + \ldots + S_k(Q).$$

(3.2)
Let us suppose that $\varepsilon$ that the block which corresponds to $\lambda_i$ null. Since all the matrices $Q$ and $U, V \in \mathcal{M}_n(\mathbb{R})$ such that $U \cdot V = I_n$ and $Q = U \cdot J \cdot V$. Then

$$I_n = U \cdot [S_1(J) + \ldots + S_k(J)] \cdot V.$$ 

Because $U \cdot V = I_n$, we have also $V \cdot U = I_n$ and then from (3.3) we obtain

$$I_n = S_1(J) + \ldots + S_k(J).$$

Let $J = B(\lambda_1) \oplus \ldots \oplus B(\lambda_k)$, where for every $t = 1, \ldots, k$, $B(\lambda_t) = C_1(\lambda_t) \oplus \ldots \oplus C_r(\lambda_t)$ and $C_\alpha(\lambda_t)$ is a Jordan cell of order $a_\alpha$ and $a_1 + \ldots + a_\alpha = m_Q(\lambda_t)$. We put $m_t = m_Q(\lambda_t)$. Since $s_t = \max\{a_1, \ldots, a_\alpha\}$, it follows that $(B(\lambda_t) - \lambda_t I_m)^{s_t} = 0$. Because $(J - \lambda I_n)^{s_t} = (B(\lambda_1) - \lambda_1 I_m)^{s_t} \oplus \ldots \oplus (B(\lambda_k) - \lambda_k I_m)^{s_t}$, the block which corresponds to $\lambda_i$ in the matrix $(J - \lambda I_n)^{s_t}$ is null. The matrix $S_i(J)$ is made up from blocks which have the same orders as the blocks of the matrix $J$ and all those blocks (of $S_i(J)$) which correspond to the eigenvalues $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_k$ are null. Since all the matrices $S_1(J), \ldots, S_{i-1}(J), S_{i+1}(J), \ldots, S_k(J)$ have the block which corresponds to $\lambda_i$ null, taking into consideration the relation (3.4) it follows that the block which corresponds to $\lambda_i$ in the matrix $S_i(J)$ is $I_{m_t}$, i.e. $S_i(J) = O_{m_1} \oplus \ldots \oplus O_{m_{i-1}} \oplus I_{m_t} \oplus O_{m_{i+1}} \oplus \ldots \oplus O_{m_k} (O_m$ is the null matrix of order $m$).

If we put $t e_j = (0 \ldots 0 1 0 \ldots 0)$, where 1 fills the $j$ position, and $n_0 = 0$, $n_j = m_1 + \ldots + m_j$, $j = 1, \ldots, k$, $(n_k = m)$, then we have $S_i(J) = \sum_{j=n_{i-1}+1}^{n_i} e_j t e_j$.

From relation (3.2) we get $Q = QS_1(Q) + \ldots + QS_k(Q)$ and then $Q = \lambda_1 S_1(Q) + (Q - \lambda_1 I_m) S_1(Q) + \ldots + \lambda_k S_k(Q) + (Q - \lambda_k I_m) S_k(Q)$.

If $U = [u_1 \ldots u_n]$ and $t V = [v_1 \ldots v_n]$ ($u_1, \ldots, u_n$ are the columns of $U$ and $t v_1, \ldots, t v_n$ are the rows of $V$), then

$$S_i(Q) = U \cdot S_i(J) \cdot V = \sum_{j=n_{i-1}+1}^{n_i} u_j t v_j$$

and

$$(Q - \lambda_t I_m) S_i(Q) = \sum_{j=n_{i-1}+1}^{n_i} (Q - \lambda_t I_m) u_j t v_j = \sum_{j=n_{i-1}+1}^{n_i-1} \varepsilon_j u_j t v_{j+1},$$

where $\varepsilon_j \in \{0, 1\}$ for $j = n_{i-1} + 1, \ldots, n_i - 1$. So we obtain

$$Q = \sum_{i=1}^k \left( \lambda_i \sum_{j=n_{i-1}+1}^{n_i} u_j t v_j + \sum_{j=n_{i-1}+1}^{n_i} \varepsilon_j u_j t v_{j+1} \right).$$

Since $Q$ is a stochastic matrix, one of its eigenvalues $\lambda_1, \ldots, \lambda_k$ is equal to 1. Let us suppose that $\lambda_1 = 1$. Because all the eigenvalues of $Q$ are real, it follows that $\lambda_2, \ldots, \lambda_k \in \mathbb{R}$. We have $u_j$, $v_j \in \mathcal{M}_{m \times 1}(\mathbb{R})$, $\varepsilon_j \in \{0, 1\}$ for any $j$, and $0 = n_0 <
$n_1 < \ldots < n_k = m$. So $Q$ has the same form as in 1), but we have to show that $\varepsilon_1 = 0$. This fact will be proved below.

From $V \cdot U = I_n$ we obtain $^t v_i u_j = \delta_{ij}, i, j \in \{1, \ldots, m\}$ and therefore the condition 2) is fulfilled.

$u_1$ is a right eigenvector for the eigenvalue 1 and $^t v_1$ is a left eigenvector for the same eigenvalue. Because a right eigenvector for the eigenvalue 1 is $^t (1 \ldots 1)$, $^t v_1 u_1 = 1$, $[\mu]Q = [\mu]$ and $[\mu]^t (1 \ldots 1) = 1$, we can take $^t u_1 = (1 \ldots 1)$ and $^t v_1 = [\mu]$. Now the condition 3) is satisfied.

We have $^t v_1 Q = ^t v_1 + \varepsilon_1 ^t v_2$, $^t v_1 = [\mu]$ and $[\mu]Q = [\mu]$. It results that $\varepsilon_1 ^t v_2 = 0$ and because $^t v_2 \neq 0$, we have $\varepsilon_1 = 0$. So the condition 1) is completely satisfied.

The remark made in the previous section tell us that the spectrum of $\varphi(Q)$. Because $\varphi(Q)$ is a stochastic matrix, $\lambda_1 = 1$ belongs to the spectrum of $\varphi(Q)$. Let $\{\lambda_1, \ldots, \lambda_l\}$ be the spectrum of $\varphi(Q)$. We have $m_{\varphi(Q)}(\lambda_1) + \ldots + m_{\varphi(Q)}(\lambda_l) = p$ ($p$ is the order of $\varphi(Q)$) and because $m_{\varphi(Q)}(\lambda_l)$, $i = 1, \ldots, l$, we get $m_{\varphi(Q)}(\lambda_1) + \ldots + m_{\varphi(Q)}(\lambda_l) = p$, this is $n_l = m_1 + \ldots + m_l = p$, $(m_l = m_Q(\lambda_l))$ and so the condition 4) is fulfilled.

We have

$$\varphi(Q) = \sum_{i=1}^{k} (\lambda_i S_i(\varphi(Q)) + (\varphi(Q) - \lambda_i I_p) S_i(\varphi(Q)))$$

(because $\varphi(P(Q)) = P(\varphi(Q))$ for any polynomial $P$).

Since for $i \geq l+1$ the minimal polynomial of $\varphi(Q)$ is a divisor of $S_i$, $S_i(\varphi(Q)) = 0$ and $(\varphi(Q) - \lambda_i I_p) S_i(\varphi(Q)) = 0$.

On the other hand,

$$S_i(\varphi(Q)) = \varphi(S_i(Q)) = \varphi \left( \sum_{j=n_{i-1}+1}^{n_i} u_j ^t v_j \right) = \sum_{j=n_{i-1}+1}^{n_i} \varphi u_j ^t v_j$$

and

$$(\varphi(Q) - \lambda_i I_p) S_i(\varphi(Q)) = \varphi \left( ((Q - \lambda_i I_m) S_i(Q))^{(p)} \right) = \sum_{j=n_{i-1}+1}^{n_{i-q}} \varepsilon_j \varepsilon_{j+1} \ldots \varepsilon_{j+q-1} \varphi u_j ^t v_j \varepsilon_{j+q},$$

and so we obtain the condition 5). The condition 6) is obvious satisfied because $Q$ is a stochastic matrix.

We suppose now that the conditions 1), ... 6) are satisfied.

Since $^t v_i u_i = 0$ for any $i = 2, \ldots, m$ and $^t u_1 = (1 \ldots 1)$, the sum of all the terms in the same row of a matrix $u_i$. $^t v_i$ or $u_i. ^t v_{i+1}$ with $i \geq 2$ is equal to 0. Because the sum of all the terms in the same row of the matrix $u_1$. $^t v_1$ is equal to 1 ($u_1 = (1 \ldots 1)$, $^t v_1 = [\mu]$) and the matrix $Q$ has only positive terms, $Q$ is a stochastic matrix.

From $^t v_1 u_1 = 1$ and $^t v_i u_i = 0$ for $i \geq 2$, we obtain $^t v_1 Q = ^t v_1 + \varepsilon_1 ^t v_2$. Because $^t v_1 = [\mu]$ and $\varepsilon_1 = 0$ we have $[\mu]Q = [\mu]$.

Let us denote $B_i = \sum_{j=n_{i-1}+1}^{n_i} u_j ^t v_j$, $N_i = \sum_{j=n_{i-1}+1}^{n_i-1} \varepsilon_j u_j ^t v_{j+1}$ and $A_i = \lambda_i B_i + N_i$, $i = 1, \ldots, k$. From 2) we obtain for any $i, j \in \{1, \ldots, k\}$: $B_i. B_j = \delta_{ij} B_i$, $N_i. N_j = 0$ for $i \neq j$, $N_i = 0$ where $q_i = n_i - n_{i-1}$, $N_i. B_j = B_j \cdot N_i = 0$ for $i \neq j$ and $N_i \cdot B_i = B_i \cdot N_i = N_i$. Then
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\[ Q^n = \sum_{i=1}^{k} A_i^n = \sum_{i=1}^{k} (\lambda_i^n B_i + \sum_{q=1}^{\min(n,q_i-1)} C_i^n \lambda_i^{n-q} N_i^q). \]

We see that \( N_i^q = \sum_{j=n_i-1+1}^{n_i-q} \varepsilon_j \varepsilon_{j+1} \ldots \varepsilon_{j+q-1} u_j^t v_{j+q} \) and then the condition 5) implies \( \varphi(B_i) = 0 \) and \( \varphi(N_i^q) = 0 \) for \( i \geq l + 1 \). We have

\[ \varphi(Q^n) = \sum_{i=1}^{l} (\lambda_i^n \varphi(B_i) + \sum_{q=1}^{\min(n,q_i-1)} C_i^n \lambda_i^{n-q} \varphi(N_i^q)). \]

We will show that \( \varphi(Q^n) \) has the same form. Since \( UV = I_m \),
\[ \sum_{i=1}^{m} u_i^t v_i = I_m \]
and consequently \( \varphi\left(\sum_{i=1}^{m} u_i^t v_i\right) = \varphi(I_m) \). Taking into consideration the conditions 4) and 5),
\[ \sum_{i=1}^{p} \psi u_i^t v_i^\varphi = I_p \]
(\( I_p \) is the unit matrix of order \( p \)). This relation implies \( u_i^t v_i^\varphi = \delta_{ij} \), \( i, j \in \{1, \ldots, p\} \), and we have \( \varphi(Q) = \sum_{i=1}^{l} (\lambda_i \varphi(B_i) + \varphi(N_i)) \),
\[ \varphi(B_i) = \sum_{j=n_i-1+1}^{n_i-1} \varepsilon_j \varphi u_j^t v_j^\varphi, \]
\[ \varphi(N_i) = \sum_{j=n_i-1+1}^{n_i-1} \varepsilon_j \varphi u_j^t v_{j+1}^\varphi. \]
Therefore we can follow the same way as we followed when we found the form of \( Q^n \) and we get
\[ \varphi(Q^n) = \sum_{i=1}^{l} (\lambda_i^n \varphi(B_i) + \sum_{q=1}^{\min(n,q_i-1)} C_i^n \lambda_i^{n-q} \varphi(N_i^q)). \]
But \( \varphi(N_i^q) = \varphi(N_i^q) \) and then \( \varphi(Q^n) = \varphi(Q^n) \).

The condition 1) gives in fact a Jordan form of the matrix \( Q \) and we see that \( \lambda_1, \ldots, \lambda_k \) are the eigenvalues of this matrix and all are real. Using the same argument we get \( m_Q(\lambda_i) = n_i - n_{i-1} \).

Since \( \varphi(Q) = \sum_{i=1}^{l} \left( \lambda_i \sum_{j=n_i-1+1}^{n_i} \varphi u_j^t v_j^\varphi \sum_{j=n_i-1+1}^{n_i-1} \varepsilon_j \varphi u_j^t v_{j+1}^\varphi \right) \) and \( v_i^\varphi \varphi u_j = \delta_{ij} \),
\( i, j \in \{1, \ldots, p\} \), here we have a Jordan form of the matrix \( \varphi(Q) \) and we can see that \( m_{\varphi(Q)}(\lambda_i) = n_i - n_{i-1} = m_Q(\lambda_i), i = 1, \ldots, l \). Q.E.D.

**Example.** Let \( X = \{1, 2, 3, 4\} \), \( \mu \) such that \( [\mu] = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix} \), \( Y = \{1, 2\} \)
and $\varphi : X \to Y$, $\varphi(1) = \varphi(2) = 1$, $\varphi(3) = \varphi(4) = 2$. Then

\[
[\mu^\varphi] = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\quad \text{and} \quad
[I^\varphi] = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix}.
\]

Let

\[
t_u^1 = (1 \ 1 \ 1 \ 1), \ t_u^2 = (1 \ 0 \ 0 \ -1),
\]

\[
t_u^3 = \left(\frac{1}{2} \ -\frac{1}{2} \ \frac{1}{2} \ -\frac{1}{2}\right), \ t_u^4 = (3 \ -1 \ -1 \ -1),
\]

\[
t_v^1 = \left(\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4}\right), \ t_v^2 = (0 \ 1 \ 0 \ -1),
\]

\[
t_v^3 = (0 \ -1 \ 1 \ 0), \ t_v^4 = \left(\frac{1}{4} \ -\frac{1}{4} \ -\frac{1}{4} \ \frac{1}{4}\right).
\]

Consider the matrix

\[
Q = u_1^tv_1 + \lambda_2 u_2^tv_2 + \lambda_3 (u_3^tv_3 + u_4^tv_4) + u_3^tv_4.
\]

We have $n_1 = 1$, $n_2 = 2$, $n_3 = 4$, $l = 3$, $p = n_2$. If we put $\lambda_2 = \frac{3}{8}$ and $\lambda_3 = \frac{1}{4}$, then we get

\[
Q = \begin{pmatrix}
9 & 3 & 1 & 3 \\
16 & 16 & 16 & 16 \\
1 & 9 & 5 & 1 \\
16 & 16 & 16 & 16 \\
5 & 1 & 5 & 5 \\
16 & 16 & 16 & 16 \\
1 & 3 & 5 & 7 \\
16 & 16 & 16 & 16
\end{pmatrix}
\]

and all the conditions 1),...,6) are fulfilled. Consequently we have $[\mu]Q = [\mu]$ and $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$.

References


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