Open questions and recent results in the theory of geometrized first-order jet bundles

Vladimir Balan, Panayiotis C. Stavrinos and Kostadin Trenčevski

Abstract

Within the framework of first order jet-Generalized Lagrange Spaces, the present survey article provides a survey presenting the work of a Romanian research group in the field of $d-$geometric structures on the first order jet space $J^1(T, M)$. Recent advances and actual open questions regarding these basic distinguished structures - which extend the corresponding generalized Lagrange, Lagrange, Finsler and Riemann $d-$structures of the tangent bundle framework, are described.

Mathematics Subject Classification: 58A20, 53C60, 53C05, 53C17, 53C22, 53C80, 70S05, 55R10, 49S05, 70H05, 35J50, 35J55.
Key words: jet space, Lagrange space, Finsler space, nonlinear connection, Cartan connection, Einstein equations, Maxwell equations, geodesic, Jacobi field, Lorentz equations, quadratic Lagrangians, multisymplectic manifold.

1. Basic structures and $d-$geometric objects on $J^1(T, M)$

The study of the Riemann - Finsler - Lagrange - Generalized Lagrange structures

$$\{\mathcal{R}^n\} \subset \{\mathcal{F}^n\} \subset \{\mathcal{L}^n\} \subset \{\mathcal{GL}^n\}, \tag{1.1}$$

defined on the tangent bundle $\xi_T = (TM, \pi, M)$ of a real differentiable manifold M ([35, 36, 37, 23, 21]) and further, defined on osculating spaces of higher order ([34, 35, 33])

$$\xi_{O,k} = (Osc^{(k)}(M) = i^*_{\{0\}\times M} J^k(\mathbb{R}, M), \pi_k, M \equiv \{0\} \times M),$$

($k \geq 1$) has flourished in the last decades. The main common feature relies on the presence of the distinguished ($d-$) geometrized approach, characterized by the existence of a nonlinear connection. This permits to build $d-$tensor fields, i.e. tensor fields living on the total space of the bundle, whose components linearly change in terms of the Jacobian matrix of the base manifold coordinate-change.

Later, the $d-$framework was naturally extended to the fibration of first-order jets

Proceedings of The 3-rd International Colloquium "Mathematics in Engineering and Numerical Physics" October 7-9 , 2004, Bucharest, Romania, pp. 16-22.

$$\xi = (E = J^1(T, M), \pi, T \times M), \quad \dim T = m \geq 1,$$

with roots in the works of G.S.Asanov and S.F.Ponomarenko ([3]-[6]), further developed by Neagu and Udriste ([42]-[50]) and by other authors ([9]-[13], [18], [39]). This $d-$*jet* (also called *geometrized jet*) framework provides an alternative to the osculator extensions $\xi_{O,k}$, while still generalizing the structures (1.1) on the tangent case ξ_T - recaptured as the autonomous mono-parametric flat subcase of ξ.

The basic object of the geometrized jet framework is the (Ehresmann) *non-linear connection* on E ([27]) which determines the local adapted basis of $\mathcal{X}(E)$ and of its dual, essential in locally expressing the d-tensor fields on E.

The similar chain to (1.1) on $J^1(T, M)$ of main geometric structures

$$\{\mathcal{JR}^n\} \subset \{\mathcal{JF}^n\} \subset \{\mathcal{JL}^n\} \subset \{\mathcal{JGL}^n\}, \tag{1.2}$$

was rigorously defined in [14] for the case when the bundle admits a nondegenerate (0,2) vertical $d-$tensor field. As subcases, the structures $\mathcal{JGL}^n$ and $\mathcal{JL}^n$ were first introduced and developed respectively by Neagu and Udriste in [43], [48] and [51].

The foundations of the $d-$jet approach, pointing out fundamental non-linear connections and Berwald and Cartan-type $d-$linear connections with applications in relativity, were provided in [51]. A primer of $d-$geometric objects on $J^1(T, M)$ - including harmonic maps induced by multi-time sprays and discussing non-linear connections and lifts of vector fields, can be found in [49, 52, 65]. The study of the basic $d-$geometric objects in natural coordinates and of splittable metrics was provided in [61]. Further, in [50, 65] were described geometrical aspects related to $\mathcal{JL}^n$ for Kronecker reducible affine second-order Lagrangians

$$L = G(Y, Y) + B(Y) + U,$$

where $G = \frac{1}{2}Hess_Y(L) = g_T^{-1} \otimes g_M$, B is an 1-form, U is a function on $T \times M$ and g_T is a given multi-time dependent vertical metric; the autonomous subcase was developed in [47].

The equations which produce naturally a nonlinear connection - the basic concept of the $d-$jet approach, were shown to represent extended harmonicity equations for the jets ([69]). The attempt to find a nonlinear connection naturally induced by the metric structure by using the variational principle was successfull just in the particular Kronecker case ([43, 7, 65]), but the general case is still untractable ([30, 54]). However, alternative methods of providing a nonlinear connection were pointed out, both for the splittable General Lagrange ([43, 65]) and the Finsler ([7]) $d-$jet structures.

The complete characterization of the class of non-linear connections produced by certain geometric objects (metrics or Lagrangians) or just the vertical metric on $J^1(T, M)$ represents still an open problem (see [43, 47] and [9]).

An open problem remains to define and characterize a $d-$almost tangent structure on TE, essential in developing the $d-$submanifold theory endowed with induced structures.

As well, the geometries of the (pseudo)-Riemannian metric space $(J^1(T, M), G)$ endowed with the metric given by a $d-$Sasaki-type lift provided by g_T and g_M and of the dual space are subject of present research.

2. Einstein and Maxwell equations in $d-$jet framework

Among the $d-$metric structures on E were emphasized and extensively studied by Neagu ([43]) the splittable jet-Generalized Lagrange metrics; for these metrics the Cartan and Berwald d-linear connections were explicitely constructed and the attached Bianchi identities were derived ([45]). In this ansatz, the generalized Einstein and Maxwell equations without sources were developed, for $\mathcal{JGL}^n$ in ([48]), and for $\mathcal{JL}^n$ of Kronecker-reductive type in ([47, 43, 10, 8, 9]).

Related to symmetries exhibited by the Einstein equations, Killing fields were discussed in [16], while Einstein models involving different energy-momentum tensor fields were examined in [10].

Regarding Maxwell equations, the possible extension of the Hodge operator to the $d-$jet framework and conservation laws for the generalized electromagnetic field theory in Miron-Tatoiu sense were discussed in [15]. The complete set of Maxwell equations of $\mathcal{JL}^n$ were derived in ([8, 10, 15]).

The Einstein and Maxwell equations were obtained in the subcases of conformal metric and relativistic optics Kronecker metric ([46, 43]), which proved to be successfully applied in the Finslerian framework ([22]).

An open problem is to examine the applicability in physics of the developed models, and to construct alternative ones based primarily on nonlinear and $d-$linear connection which might prove their usefulness in GR.

Significant progress in applying the $d-$framework to multisymplectic Lagrangian and Hamiltonian formalisms of first-order field theory was achieved in [65] - which introduces a jet-covariant Hamiltonian theory. As well, the usefulness of such models for economy was revealed in [59].

3. Paths and $d-$geodesics in $d-$jet framework

The notions of (h- and v-)paths, and of (h- and v-) geodesics in $J^1(T, M)$ extend the corresponding definitions of the tangent subcase provided in [37, 57]. The same holds for stationary curves, which rely on the notion of force/acceleration $d-$vector fields. An analytic description of stationary curves, paths and geodesics on $J^1(T, M)$ in the Cartan-Kronecker case was provided in [11, 10].

For specific Lagrangians, the Euler-Lagrange equations were shown to provide in the 1-parametric autonomous case the classical Lorentz equations ([11, 10]). As well, analytic solutions for the path and geodesic equations, with numerical simulations were obtained in [17, 18].

The equations of geodesics and of Jacobi fields of the $d-$jet framework were obtained in ([19, 20]). These equations extend the Finslerian particular case ([21, 28]) and adjust a previous extension attempt ([1]). An interesting survey of applications of geodesic surfaces is included in [59].

Remains as open problem to provide physical models which involve non-torsionless linear connections ([13]), where the depending on torsion force-term in the adapted equations of $d-$Jacobi fields gets physical meaning ([19, 20, 24]). As well, the role played by the nonlinear connection (which is present in the force term) in the Morse theory of geodesics remains an open problem.

4. PDEs and Dynamical systems vs. the $d-$ jet framework

The solutions of certain ODEs and PDEs were shown ([48, 66]) to be solutions of certain classes of harmonic maps between certain jet-Generalized Lagrange spaces.

Within this approach, the solution of the Poincaré problem was provided in the geometrized jet-framework ([64]), and in [26, 68] was proved the existence of periodical solutions of multi-time Hamilton equations via periodical extremals of the dual action, when the Hamiltonian satisfies suitable conditions.

As well, structures of Finsler, Lagrange and Hamilton type were studied, in relation with the quadratic Lagrangians on $J^1(T, M)$ attached to SODEs and control systems ([60]). The equivalence between Euler-Lagrange equations and Hamilton equations in multi-time framework was studied in [26] and the action that produces multi-time Hamilton equations was described, emphasizing the role of the polysymplectic structure ([70]).

As subject of research, remains to reveal the physical meaning of the extended Legendre transform, and to point out the physical relevance of the extended laws of conservation of energy ([69, 70, 15]).

Acknowledgement. The present work was partially supported by the Grants CNCSIS MEN A1107/2005 and A1478/2004. Special thanks are addressed to Professor Constantin Udrişte for constructive discussions.

References

[1] M. Anastasiei, I. Bucătaru, *Jacobi fields in generalized Lagrange spaces*, Rev. Roum. Math. Pures Appl., 42 (1997), 9-10, 689-695.

[2] M. Anastasiei, H. Shimada, *Beil metrics associated to a Finsler space*, BJGA, 3 (1998), 2, 1-16.

[3] G.S.Asanov, *Finsler Geometry, Relativity and Gauge Geometry*, D.Reidel, Dordrecht, 1985.

[4] G.S.Asanov, *Jet extension of Finslerian gauge approach derivation of generalized Einstein-Yang-Mills equations*, Sem. Mec., No. 16, Univ. of Timisoara, 1988, 1-47.

[5] G.S.Asanov, S.F.Ponomarenko, *Finslerian Fiberings over Space-Time, Associated Gauge Fields and Connections* (Russian), Nauka Eds., Acad.Sci.Mold.Rep., 1989.

[6] G.S.Asanov, S.F.Ponomarenko, *Finslerian jet and higher-order gauge approach stochastic space-time. Distribution function*, Sem. Mec., No. 21, Univ. of Timisoara, 1989, 1-51.

[7] V. Balan, *Finslerian non-linear connections and paths on first-order jet bundles*, Proc. of The 7-th Int. Conf. of Tensor Society, 23-28 August 2004, Timişoara, Romania, Tensor N.S. (2005), 171-179.

[8] V. Balan, *Generalized Maxwell and Lorentz equations on first-order geometrized jet spaces*, Proc. of the Int. Conf. in Geom. and Topology Cluj-Napoca, Romania, 1-5 October 2002, Cluj Univ. Press 2003, 25-37.

[9] V. Balan, *Lagrangian structures in geometrized jet framework*, Proc. of The Conf. of Applied Diff. Geom. – General Relativity, Thessaloniki, June 26-30 2002, Thessaloniki, Greece, BSG Proc. 11, Geometry Balkan Press 2004, Bucharest, 25-39.

[10] V. Balan, *Lorentz-type equations in first-order jet spaces endowed with nonlinear connection*, Proc. of The First French-Romanian Colloq. of Numerical Physics (MENP-1), October 30-31, 2000, Bucharest, Romania, Geometry Balkan Press 2002, 105-114.

[11] V. Balan, *Notable curves in geometrized $J^1(T, M)$ jet framework*, BJGA 8 (2003), 2, 1-10.

[12] V. Balan, *On gedesics and deviations of geodesics in fibered Finslerian approach*, Math. Rep., Rom., Academy Eds., 4.6 (1994), 4, 415-422.

[13] V.Balan, *Synge-Beil and Riemann-Jacobi jet structures with applications to physics*, IJMMS, Hindawi Publ., 27 (2003), 1693-1702.

[14] V. Balan, *Variational problems in the geometrized first-order jet framework*, Proc. of the Int. Workshop on Global Analysis, April 15 - 17, 2004 Ankara, Turkey, AIP Conf. Proc. #729, USA, 2004, 91-98.
[15] V. Balan, P.C.Stavrinos, *Extended Maxwell equations with sources in geometrized jet framework*, AGG 21 (3) (2004), 263-276.
[16] V. Balan, P.C.Stavrinos, *Killing symmetries of Einstein equations in geometrized fibered framework*, Proc. of "The Second Int. Coll. of Mathematics in Engg. and Numerical Physics" (MENP-2), April 22-26 2002, Bucharest, Romania, BSG Proc. 8, Geometry Balkan Press 2002, 1-8.
[17] V.Balan, K.Trenčevski, *Equations of deviations of geodesics. Special cases in the fibered Finslerian case*, Tensor N.S., 56 (1995), 184-192.
[18] V. Balan, K. Trenčevski, *Stationary Lorentz curves in geometrized jet spaces. Numerical simulation and analytic solutions*, AGG 20 2 (2003), 197-210.
[19] V. Balan, N.Voicu, *Distinguished geodesics and Jacobi fields on first order jet spaces*, CEJM (Central European Journal of Mathematics), electronic, http://www.cesj.com) 2 (2004), 2, 1-10.
[20] V. Balan, N.Voicu, *Note on geodesics in distinguished jet framework*, Analele Univ. Bucuresti, LIII (2004), 1, 63-72.
[21] D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, 2000.
[22] R.G. Beil, *Notes on a new Finsler metric function*, BJGA, 2, 1 (1997), 1-7.
[23] A. Bejancu, *Finsler Geometry and Applications*, E.Horwood, 1990.
[24] N. Brinzei Voicu, *On metrical linear connections with torsion in Riemannian Geometry*, Proc. of The 3-rd Int. Coll. "Math. in Engg. and Numerical Physics" (MENP-3), 7-9 October 2004, Bucharest, Romania, BSG Proc. ?, Geometry Balkan Press 2005, ?-?.
[25] I. Comic, *Horizontal and vertical geodesics in the Riemannian space*, Mat. Vestnik, 42 (1990), 3-4, 139-153.
[26] I. Duca, A.-M. Teleman and C. Udrişte, *Poisson-gradient dynamical systems with convex potential*, Proc. of The 3-rd Int. Coll. "Math. in Engg. and Numerical Physics" (MENP-3), 7-9 October 2004, Bucharest, Romania, BSG Proc. ?, Geometry Balkan Press 2005, ?-?.
[27] Ch. Ehresmann, *Extension du calcul des jets aux jets non holonomes*, C.R. Acad Sci, Paris 239, 1762-1764 (1954).
[28] B.T.M. Hassan, *Sprays ans Jacobi fields in Finsler geometry*, An. Univ. Timişoara, Ser. Şt. Mat., vol. XIX, fasc. 2, 1981, 129-139.
[29] L. Kozma, *On osculation of geodesics of a Finsler pair connection*, Tensor N.S., 54(1993), 22-27.
[30] D. Krupka, *A geometric theory of ordinary first order variational problems in fibered manifolds*, I., II, J. Math. Anal. Appl. 49 (1975), 180-206, 469-476.
[31] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisha Press, Kyoto, 1986.
[32] J. Milnor, *Morse Theory*, Ann. of Math. Stud., Princeton Univ. Press, 1963.
[33] R. Miron, *The Geometry of Higher Order Finsler Spaces*, Hadronic Press, 1998.
[34] R. Miron, *The Geometry of Higher Order Lagrange Spaces. Applications in Mathematics and Physics*, Kluwer Acad. Publishers, FTPH, 1997.
[35] R. Miron, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publishers, 1994.
[36] R. Miron, M. Anastasiei, *The Geometry of Vector Bundles. Theory and Applications*, Kluwer, Dordrecht 1994.
[37] R. Miron, M.Anastasiei, *Vector Bundles. Lagrange Spaces. Applications to Relativity*, Geometry Balkan Press, 1996.
[38] R. Miron, D. Hrimiuc, H. Shimada, S.V. Sabău, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Acad. Publishers, FTPh. 198, 2001.

[39] R. Miron, M. S. Kirkovits, M. Anastasiei, *A geometrical model for variational problems of multiple integrals*, Proc. of Conf. on Diff. Geom. and Appl., Dubrovnik, Yugoslavia, June 26-July 3, (1988).

[40] R. Miron, M. Tatoiu-Radivoiovici, *A Lagrangian theory of electromagnetism*, Rep. Math. Phys., 27 (1989), 49-84.

[41] R. Miron, R.K. Tavakol, V. Balan, I. Roxburgh, *Geometry of Space-Time and Generalized Lagrange Gauge theory*, Publicationes Mathematicae, Debrecen, 42 (1993), 3-4, 215-224.

[42] M. Neagu, *Canonical nonlinear connections on jet bundles of first order*, arXiv:math.DG/0111163 v1 14 Nov 2001.

[43] M. Neagu, *Generalized metrical multi-time Lagrangian geometry of physical fields*, Forum Mathematicum, 15 (2003), 1, 63-92;
http://www.degruyter.de/journals/forum/2003/pdf/15_63.pdf;
http://xxx.lanl.gov/abs/math.DG/0011003, 2000.

[44] M. Neagu, *Generalized metrical multi-time Lagrangian model for General Relativity and electromagnetism*, to appear.

[45] M. Neagu, *Ricci and Bianchi identities for h-normal Γ-linear connections on* $J^1(T,M)$, IJMMS 34 (2003), 2177-2191.

[46] M. Neagu, *The generalized metrical multi-time Lagrange space of relativistic geometrical optics*, http://xxx.lanl.gov/abs/math.DG/0011262, (2000).

[47] M. Neagu, *The geometry of autonomous metrical multi-time Lagrange space of electrodynamics*, IJMMS, Hindawi Publ., 29.1 (2002), 7-15;
http://xxx.lanl.gov/abs/math.DG/0010091, (2000).

[48] M. Neagu, C. Udrişte, *From PDE systems and metrics to geometric multi-time field theories*, Sem. Mec. 79, West Univ. of Timisoara, 2001,
http://xxx.lanl.gov/abs/math.DG/0009071, 2000.

[49] M. Neagu, C. Udrişte, *Multi-time dependent sprays and harmonic maps on* $J^1(T,M)$, http://xxx.lanl.gov/abs/math.DG/0009049, 2000.

[50] M. Neagu, C. Udrişte, *The geometry of metrical multi-time Lagrange spaces*, http://xxx.lanl.gov/abs/math.DG/0009071, 2000.

[51] M. Neagu, *The Riemann-Lagrange geometry of 1-jet spaces*, Ph.D. Thesis, Sci. Adv. C.Udrişte, Univ. Politehnica of Bucharest, 2001.

[52] M. Neagu, C. Udrişte and A. Oană, *Multi-time sprays and h-traceless maps on* $J^1(T,M)$, BJGA 10, 2 (2005), 76-92.

[53] H.Rund, *The Geometry of Finsler Spaces*, Springer Verlag, Berlin, 1959.

[54] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge Univ. Press, 1989.

[55] Z. Shen, *Differental Geometry of Sprays and Finsler Spaces*, Kluwer, 2001.

[56] Z. Shen, *Geometric meanings of curvatures in Finsler geometry*, Rend. Circ. Mat. Palermo, Serie II, Suppl. 66 (2001), 165-178.

[57] P.C. Stavrinos, H. Kawaguchi, *Deviation of geodesics in the gravitational field of Finslerian Space-Time*, Memoirs of Shonan Inst. of Technol. 27 (1993), 1, 35-40.

[58] J.L. Synge, *On the deviation of geodesics and null geodesics*, Ann. of Math., 35 (1934), 4, 705-713.

[59] C. Udrişte, M. Ferrara, D. Opriş, *Economic Geometric Dynamics*, Geometry Balkan Press, Bucharest 2004.

[60] C. Udrişte, *Finsler-Lagrange-Hamilton structures associated to control systems*, Proc. of Conf. on Finsler and Lagrange Geometries, Univ. Al.I.Cuza, Iaşi, 26-31 August, 2001, (Eds: M. Anastasiei and P.L.Antonelli), Kluwer Academic Publishers, 2003, 233-243.

[61] C. Udrişte, *From integral manifolds and metrics to potential maps*, Atti dell'Accademia Peloritana dei Percolanti, Classe 1 di Scienze Fis. Mat. e Nat., LXXXI-LXXXII, A01006 (2003-04), 1-14.

[62] C. Udrişte, *Geodesic motion in a gyroscopic field of forces*, Proc. of The 7-th Int. Conf. of Tensor Society, 23-28 August 2004, Timişoara, Romania, Tensor N.S., to appear.

[63] C. Udrişte, *Geometric Dynamics*, Mathematics and Its Applications, 513, Kluwer Academic Publishers, 2000.
[64] C. Udrişte, *Geometric Dynamics*, Southeast Asian Bulletin of Mathematics, Springer-Verlag, 24, 1 (2000), 313-322.
[65] C. Udrişte, *Nonclassical Lagrangian dynamics and potential maps*, Conf. in Mathematics in Honour of Prof. Radu Roşca on the Occasion of his Ninetieth Birthday, Katholieke Univ. Brussel, Katholieke Univ. Leuven, Belgium, Dec. 11-16, 1999; arXiv: math.DS/0007060, v1, 2000.
[66] C. Udrişte, *Solutions of PDEs as potential maps using first order Lagrangians*, Balkan Journal of Geometry and Its Applications, Vol.6, No.1 (2001), 93-108.
[67] C. Udrişte, *Tools of geometric dynamics*, Bull. Instit. Geodinamica, Romanian Acad., v. 14, fasc. 4 (2003), 1-26.
[68] C. Udrişte, I. Duca, *Periodical solutions of multi-time Hamilton equations*, An. Univ. Buc., 55, 1 (2005), 179-188.
[69] C. Udrişte, M. Neagu, *Geometrical interpretation of solutions of certain PDEs*, BJGA 4, 4 (1999), 145-152.
[70] C. Udrişte, A. M. Teleman, *Hamiltonian approaches of field theory*, IJMMS, 57 (2004), 3045-3056.

Vladimir Balan
University Politehnica of Bucharest, Department Mathematics I,
313 Splaiul Independenţei,
RO-060042 Bucharest, Romania
e-mail: vbalan@mathem.pub.ro

Panayiotis C.Stavrinos
University of Athens, Department of Mathematics,
Panepistemiopolis, Athens 15784, Greece
e-mail: pstavrin@atlas.uoa.gr

Kostadin Trenčevski
Inst. of Math., St.Cyril and Methodius University,
1000 Skopje, Macedonia
e-mail: kostatre@iunona.pmf.ukim.edu.mk