Multidimensional residual spectral capacities

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Abstract

In this paper we define and analyse the concepts of multidimensional residual spectral capacities (S-spectral capacities) and the S-decomposable systems of operators (see preliminaries). For a commuting S-decomposable system of operators $a = (a_1, a_2, \dots, a_n) \subset B(X)$, there exists a S-spectral capacity E and $E(F)$ is a spectral maximal space of a ($F \subset C^n$ closed).

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Preliminaries.

Let X be a Banach space, let $S(X)$ be the family of the closed linear subspaces of X, let $S \subset \mathbf{C}^n$ be a compact set and let F_S be the family of closed sets $F \subset \mathbf{C}^n$ that have the property: either $F \cap S = \emptyset$ or $F \supset S$.

We shall call *S-spectral capacity* an application $E : F_S \to S(X)$ that meets the properties:

1. $E(\emptyset) = \{0\}$, $E(\mathbf{C}^n) = X$;
2. $E\left(\cap_{i=1}^{\infty} F_i\right) = \cap_{i=1}^{\infty} E(F_i)$ for any series $\{F_i\}_{i\in\mathbf{N}} \subset F_S$;
3. for any open finite S-covering $\{G_S\} \cup \{G_j\}_{j=1}^{m}$ of $\mathbf{C}^n$ we have

$$X = E(\bar{G}_S) + \sum_{j=1}^{m} E(\bar{G}_S).$$

A commuting system of operators $a = (a_1, a_2, \dots, a_n) \subset B(X)$ is said to be S-decomposable if there exists a S-spectral capacity such that

4. $a_j E(F) \subset E(F)$ for any $F \in F_S$ and for any j;
5. $\sigma(a, E(F)) \subset F$ for any $F \in F_S$,

In case $S = \emptyset$, the S-spectral capacity is said to be a spectral capacity, and the system is decomposable.

If $a = (a_1, a_2, \dots, a_n) \subset B(X)$ is a commuting system of operators and $\sigma(a, X)$ is the system's Taylor spectrum reported to X, we shall denote by $U(\sigma(a, X))$ the algebra of the seeds of analytic functions defined in a neighbourhood of $\sigma(a, X)$. It is known that there exists a homomorphism from $U(\sigma(a, X))$ to $B(X)$ so that $1 \to 1_X$

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and $z_i \to a_i$, $(i = 1, 2, \dots, n)$ where 1 means the seed associated to the function $z \to 1$ and z_i the seed associated to the coordinate function [1]. We shall further make use of the following result, proved in [1].

Proposition 1. *Let Y, Z be two Banach spaces, $\tau : Y \to Z$ a continuous homomorphism and let $b = (b_1, b_2, \dots, b_n) \subset B(Y)$, $c = (c_1, c_2, \dots, c_n) \subset B(Z)$ be two systems of operators that commute such that $\tau b_i = c_i \tau$, for any $i = 1, 2, \dots, n$. If $f \in U(\sigma(b, Y)) \cup \sigma(c, Z))$ then we also have $\tau f(b) = f(c)\tau$.*

Proposition 2. *Let $a = (a_1, a_2, \dots, a_n) \subset B(X)$ and $\sigma(a, X) = \sigma_1 \cup \sigma_2$ with $\sigma_1 \cap \sigma_2 = \emptyset$, σ_1, σ_2 closed. If $X = X_1 \oplus X_2$ is the direct sum decomposition according to* ([1, *Theorem* 4.9]), *where $\sigma(a, X_1) = \sigma_1$, $\sigma(a, X_2) = \sigma_2$, then X_1, X_2 are spectral maximal spaces of a.*

Proof. Let Y be a closed subspace of X invariant to a such that $\sigma(a, y) \subset \sigma(a, X_1)$. We mark with p_2 the projection of X on X_2, with b_i the restriction of a_i at Y, $b_i = a_i|Y$, with c_i the restriction of a_i at X_2, $c_i = a_i|X_2$, and with τ the restriction of p_2 at Y, $\tau = p_2|Y$. Since p_2 commutes with a_i $(i = 1, 2, \dots, n)$ ([1], 4.9) we have

$$\tau b_i = x_i \tau.$$

By setting $b = (b_1, b_2, \dots, b_n)$, $c = (c_1, c_2, \dots, c_n)$, we have

$$\sigma(b, Y) \cap \sigma(c, X_2) = \emptyset.$$

Let now f be the seed of the analytic function equal with 1 in a neighbourhood of $\sigma(b, Y)$, and equal with 0 in a neighbourhood of $\sigma(c, X_2)$. According to Proposition 1 we obtain $p_2 I_Y = 0$ (since $f(b) = I_Y f(c) = 0$) for $Y \subset X_1$; consequently X_1 is a spectral maximal space of a, and the same can be similarly proved for X_2.

Theorem 3. *Let $a = (a_1, a_2, \dots, a_n) \subset B(X)$ be a S-decomposable system and E a spectral S-capacity of a. Then $E(F)$ is a spectral maximal space of a ($F \subset \mathbf{C}^n$ closed).*

Proof. Let Y be an invariant closed subspace of X to a with $\sigma(a, Y) \subset F$ for a certain closed set $F \subset \mathbf{C}^n$. Let $F \supset S$. Then there exists an open S-covering of C^n $\{G_S, G\}$ such that $G_S \supset S$ and $\bar{G} \cap F = \emptyset$, and

$$X = E(\bar{G}_S) + E(\bar{G}).$$

According to a isomorphism theorem, the quotient space $X/E(\bar{G}_S)$ is isomorphic with

$$E(\bar{G})/E(\bar{G}_S) \cap E(\bar{G}) = E(\bar{G}_S)/E(\bar{G}_S \cap \bar{G}).$$

Taylor's theorem concerning the inclusion of the spectra ([1], Lemma 1.2) yields

$$\sigma(a, E(\bar{G}))/E(\bar{G}_S \cap \bar{G}) \subset \sigma(a, E(\bar{G}_S \cap \bar{G})) \cup \sigma(a, E(\bar{G})) \subset (\bar{G}_S \cap \bar{G}) \cup \bar{G} = \bar{G},$$

meaning

$$\sigma(a, X/E(\bar{G}_S)) \subset \bar{G}.$$

Denoting by φ the canonical map of X on $X/E(\bar{G}_S)$, by b_i the restriction of a_i to Y, by c_i the operator induced by a_i in $Z = X/E(\bar{G}_S)$ and by τ the restriction of φ to Y, we shall put $b = (b_1, b_2, \dots, b_n)$, $c = (c_1, c_2, \dots, c_n)$. This implies

$$\sigma(b, Y) \cap \sigma(c, Z) \subset F \cap G = \emptyset.$$

If f is the embryo of the analytic function equal to 1 on $\sigma(b, Y)$ and to 0 on $\sigma(c, Z)$ then $f(b) = l_Y$ and $f(c) = 0$. By applying Proposition 1 we obtain $\varphi \cdot l_Y = 0$ hence $Y \subset E(\bar{G}_S)$. Since G_S is arbitrary with the property $G_S \supset F$ we infer that $Y \subset \cap\{E(\bar{G}_S), G_S \supset F\} = E(F)$. When $F \cap S = \emptyset$ we proceed similarly.

Corollary 4. *Let $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ be a S-decomposable system. Then a admits a single S-spectral capacity E.*

Proof. Let E and E_1 be two spectral S-capacities of a. Then, according to the preceeding theorem $E(F)$ and $E_1(F)$ are spectral maximal spaces of a and from the inclusions

$$\sigma(a, E(F)) \subset F, \quad \sigma(a, E_1(F)) \subset F$$

it follows that

$$E(F) \subset E_1(F),\ E_1(F) \subset E(F),$$

hence the two S-spectral capacities coincide.

Remark 5. If E is the S-spectral capacity of the S-decomposable system $a = (a_1, a_2, \ldots, a_n) \subset B(X)$, then $E(F_1 \cup F_2) = E(F_1) \oplus E(F_2)$ if F_1, F_2 are closed and disjoint $F_1, F_2 \in F_S$ meaning E is disjoint additive [11]. Indeed, we have $E(F_1) \subset E(F_1 \cup F_2)$, $(i = 1, 2)$, therefore $E(F_1) \oplus E(F_2) \subset E(F_1 \cup F_2)$; but $E(F_1 \cup F_2) = Y_{F_1} \oplus Y_{F_2}$ (see [1, Theorem 4.9]), where $\sigma(a, Y_{F_i}) \subset F_i$ $(i = 1, 2)$, according $Y_{F_i} \subset E(F_i)$ and $Y_{F_1} \oplus Y_{F_2} = E(F_1) \oplus E(F_2)$.

Proposition 6. *Let $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ be a S-decompo-sable system such that* $\dim S = 0$. *Then a admits the following spectral decomposition: for any open covering $\{G_j\}_1^m$ of $\mathbf{C}^n$ there exists the spectral maximal spaces $\{Y_j\}_1^m$ of a, such that $X = \sum_{j=1}^{m}$ and $\sigma(a, Y_j) \subset G_j$ $(j = 1, 2, \ldots, m)$.*

Proof. Let $\{G_j\}_1^m$ be an open and finite covering of $\mathbf{C}^n$. By putting $G'_j = G_j \cap (\mathbf{C}^n \setminus S)$ and by observing that $\{G_j\}_1^m$ is also a covering of S, it follows that there exists an open covering $\{G''_j\}_1^m$ of S such that $G''_j \subset G_j$, $G''_i \cap G''_j = \emptyset$ $(i \neq j$, $i, j = 1, 2, \ldots, m)$; indeed, this fact is a consequence of [13, Lemma 6.2], because S is totally disconnected, then $\dim S = 0$. Then there will exist a covering

$$\{H_j\}_1^m \cup \{H'_j\}_1^m$$

of $\mathbf{C}^n$ such that $\bar{H}_j \subset G'_j$, $\bar{H}'_j \subset G''_j$ $(j = 1, 2, \ldots, m)$. Let us set $H_S = \bigcup_{j=1}^{m} H'_j$; then

$$\{H_S\} \cup \{H_j\}_1^m$$

is a S-covering of $\mathbf{C}^n$. There will exist the spectral maximal spaces

$$\{Y_S\} \cup \{Y'_j\}_1^m$$

of a such that

$$X = Y_S + \sum_{j=1}^{m} Y_j, \quad \sigma(a, Y_S) \subset H_S, \quad \sigma(a, Y_j) \subset H_j.$$

But $Y_S = Y_S^{(1)} \oplus Y_S^{(2)} \oplus \ldots \oplus Y_S^{(m)}$ with $\sigma(a, Y_S^{(j)}) \subset H'_j$ $(j = 1, 2, \ldots, m)$ according to [1, Theorem 4.9]. It will suffice to show that there exists a spectral maximal space X_j of a such that $Y_S^{(j)} \subset X_j$, $Y_j \subset X_j$ and $\sigma(a, X_j) \subset G_j$ $(j = 1, 2, \ldots, m)$. By setting $F_1^{(j)} = \bar{H}_j \cup \bar{H}'_j$ and $F_1^{(j)} = S \cap (H'_1 \cup H'_2 \cup \ldots \cup H'_{j-1} \cup H'_{j+1} \cup \ldots \cup H'_m)$ we notice that $F_1^{(j)} \cap F_2^{(j)} = \emptyset$ and $F_1^{(j)} \cup F_2^{(j)} \subset S$, hence

$$E(F_1^{(j)} \cup F_2^{(j)}) = Y_1^{(j)} \oplus F_2^{(j)}$$

(according to Proposition 2 and using [1, Theorem 4.9]), the wanted spectral maximal space will be $X_j = Y_1^{(j)}$ $(j = 1, 2, \ldots, m)$. In this sense we have the following

Lemma 7. *Let $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ a decomposable system. Then*

$$\sigma_{a_i}(x) = \pi_i \sigma(a, x)$$

for all $1 \leq i \leq n$ and for any $x \in X$, where π_i is the projection of $\mathbf{C}^n$ on the plane $\mathbf{C}$ corresponding to the index i.

Proof. Let $z \in \sigma(z, x)$ and let us suppose that $\pi_i(z) = z_i \notin \sigma_{a_i}(x)$; then there exists an analytic function $f_i : V_{z_i} \to X$ such that

$$x \equiv (z_i - a_i) f_i(z_i) = (z_1 - a_1)0 + \ldots + (z_i - a_i) f_i(z) + \ldots + (z_n - a_n)0$$

hence $z \in \sigma(a, x)$, a contradiction; hence $\pi_i \sigma(a, x) \subset \sigma_{a_i}(x)$. Conversely, let $F = \sigma(a, x)$; from $x \in X_a(F) = X_{[a](F)}$ and $\sigma(a, X_{[a]}(F)) \subset F$ implies that

$$\sigma_{a_i}(x) \subset \pi_i \sigma(a, X_{[a]}(F)) = \sigma(a_i | X_{[a](F)}) \subset \pi_i F = \pi_i \sigma(a, x)$$

whence follows the equality

$$\sigma_{a_i}(x) = \pi_i \sigma(a, x).$$

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