Mean shapes, image fusion and scene reconstruction

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Abstract

Averaging on shape spaces is applied to digital image averaging and scene reconstruction. The method of averaging images due do K.V. Mardia, also known as image fusion, is based on a reconstruction of the grey level around a certain mean shape of a configuration that is present in all images. For mean shapes here we use extrinsic means on shape spaces associated with certain equivariant embeddings of such shape spaces. We illustrate our methodology with a simple example of image fusion.

Key words and phrases: shape spaces and their equivariant embeddings, gray level digital imaging, landmarks and image warping.

AMS subject classification: Primary 62H35; secondary53C40

1 Introduction

In this paper we present in a slightly more general context, a method of digital image fusion, essentially due to K.V. Mardia [5]. While some of the results are new, this paper is for the most of it a survey paper that was given at a Geometry Conference in Thessaloniki, Greece in June 2002. The paper is written in an expository style, that should be accessible to geometers, giving them the opportunity to further the research in this applied math subject.
In section 2, we recall the notion of pseudogroup of transformations of, and introduce the G-shape spaces. In particular, when G is the pseudoroup of direct similarities of $\mathbb{R}^m$ we recover the similarity shape spaces introduced by D. J. Kendall in [13]. In the case when G is the pseudoroup of affine transformations of $\mathbb{R}^m$ we recover the affine shape spaces introduced by Sparr [32]. Finally in the case when G is the pseudogroup of projective transformations of $\mathbb{R}^m$ we recover the projective shape spaces introduced in [29]. The shape spaces are orbifolds or in some cases manifolds with a nice geometrical structure of symmetric spaces. Since in general it is useful

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to average distributions of such shapes, in section 3 we introduce the Fréchet mean (sets) associated with a distance ρ on the shape space as were introduced by Fréchet ([8]). The Fréchet mean associated with the distance induced on a manifold M by an embedding in an Euclidan space was called extrinsic mean in [1]. In particular since planar similarity shape spaces have a natural structure of complex projective spaces, in this section we consider the Veronese-Whitney embedding of a complex projective space and we compute the extrinsic mean associated with such an embedding. In the particular case of the empirical distribution on a finite set of projective points, this extrinsic mean turns out to correspond with the so called Procrustes mean shape as defined by Kent in [15].
In section 4 we introduce the reader to high level analysis of gray level digital images. In this context we consider landmarks in a scene pictured in a digital image, and the configuration of points in $\mathbb{R}^2$ that are associated with an ordered set of landmarks. Then we define the operation of cropping from a digital image, and of random image picturing a given scene. To a given configuration a scene's landmarks in a random image we associate a random shape, and with such a shape we associate its extrinsic mean. In addition for random images we consider transformations, and we define the mean image of a random image, in terms of the mean shape of the associated random shape and with the group of transformations. In particular the mean image of a random sample of images of a scene is considered by taking mean image w.r.t. the empirical distribution for the images in the sample. The section concludes with an example of image averaging and with an example of planar scene reconstruction.

2 G-shape spaces of k-ads

Definition 2.1 A family G of pairs (U, f_U), where U is an open subset of $\mathbb{R}^m$ and f_U is a diffeomorphism from U onto the subset $f_U(U)$ of $\mathbb{R}^m$, such that
i. for any element of G, U, f_U, its inverse $f_U(U), f_U^{-1})$ is in G
ii. for any pairs $(U, f_U), (V, f_V) \in \mathcal{G}$, the pair $((f_U)^{-1}(V), f_U|(f_U)^{-1}(V))$ is in G, and the pair $((f_U)^{-1}(V), f_V \circ f_U)$ is in G
is said to be a pseudogroup of transformations of $\mathbb{R}^m$.

Any action of a Lie group on $\mathbb{R}^m$, generates a pseudogroup of transformations on $\mathbb{R}^m$. A typical example of pseudogroup of transformations on $\mathbb{R}^m$, that is not generated by such a Lie group action is given by the pairs (U, f_U), where f_U is the restriction of a map f defined by $y = f(x)$, with

$$y^j = \frac{\sum_i a_i^j x^i}{\sum_i a_0^j x^i}, \forall j = 1, ..., m \tag{1}$$

is such that $\det((a_i^j)_{i,j=0,...,m}) \neq 0$.
The general study of configurations of a system of k points in $\mathbb{R}^m$, where the effects of a pseudogroup G of transformations have been filtered out, was in initiated in [24], and was motivated by applications involving the pseudogroup generated by the

direct similarities of $\mathbb{R}^m$ (see [5]), or by affine transformations of $\mathbb{R}^m$ ([32]), or the pseudogroup of projective transformations of $\mathbb{R}^m$ ([9]). The G - shape of a subset K of $\mathbb{R}^m$, is the set of all orbits of K under transformations in G. The simplest type of object which can be studied consists of a labelled set of k points, called landmarks. The resulting orbifold quotient of $(\mathbb{R}^m)^k/\mathcal{G}$ is called the $\mathcal{G}$-shape space of k-ads, and is labelled $\mathcal{G}\Sigma_m^k$. If the pseudogroup action is generated by a locally free group action of a Lie group G of dimension g, the dimension of the $\mathcal{G}$-shape space of k-ads is $mk-g$.
EXAMPLE 2.1. G is the pseudogroup of transformations generated by direct similarities of $\mathbb{R}^m$. A k-ad is represented as a $k \times m$ matrix X, say. Then for any $c \in \mathbb{R}^m$, any for any special orthogonal matrix G, and for any scalar $r > 0$, X has the same shape as $rXG + 1_k c^T$. If we consider only matrices X, with $rank(X) > 1$. the resulting $\mathcal{G}$-shape space of k-ads, is Kendall's shape space Σ_m^k (see [13], [16]). In particular when k =2, Σ_2^k, the dimension of the group of direct similarities of the Euclidean plane is 4, therefore the similarity shape space has dimension $2k-4$. D. J. Kendall showed that actually Σ_2^k is diffeomorphic to the complex projective space $\mathbb{C}P^{k-2}$.
EXAMPLE 2.2. G is the pseudogroup of transformations generated by affine transformations of $\mathbb{R}^m$. Since the affine group has dimension m^2+m, the resulting $\mathcal{G}$-shape space of k-ads, also called the affine shape space is $m(k-m-1)$ dimensional. Sparr showed that if one considers only k-ads in general position, the corresponding affine shape space is diffeomorphic to the real Grassmann manifold $G_m(k-1)$ of m-dimensional linear subspaces of $\mathbb{R}^{k-1}$ (see [11], [12]).
EXAMPLE 2.3. G is the pseudogroup of projective transformations in $\mathbb{R}^m$, given in (1). Motivated by various practical applications, a landmark based investigation of planar shapes via central projections has been initiated by Maybank and Beardsley in [26] and Lenz and Meer in [17]. For applications of projective geometry see also [27], [28], or [10]. The projective shape space has been considered in Goodall and Mardia in [9] as a space of projective invariants. If we consider the projective transformations of $\mathbb{R}^m$ as restrictions of the action of the projective general linear group PGL(m) on $\mathbb{R}P^m$, to an affine open subset, it follows that the corresponding shape space has dimension $m(k-m-2)$. In [22] it was shown that the projective shape space $P\Sigma_m^k$ of k-ads of points in $\mathbb{R}P^m$, with the first $m+2$ landmarks in general position was identified with $(\mathbb{R}P^m)^{k-m-2}$.

3 Equivariant embeddings of shape spaces and extrinsic mean shapes of k-ads

For the standard definitions and results of large sample theory needed in this section we send the reader to a classical textbook on this subject, such as [7].
The coordinates of landmarks in a k-ad are usually digitized either manually or automatically from images, using imaging software. Nevertheless such registration procedures are subject to errors, and for this reason from a number of registrations, an average shape of the k-ad, that should be close to the true shape of the configuration is sought. Subsequently one may assume that the registered observations yield a sample of shapes $(X_1, ..., X_n)$ from a given probability distribution Q and the sample

mean shape is a consistent estimator of the true mean shape of Q (for details see [1]). However the previous statement needs to be considered with caution, since the mean shape space of a distribution of k-ads is not obviously defined, as the $\mathcal{G}$-shape space of k-ads is not a linear space, therefore the usual averaging formula is meaningless in this case. The mean of a distribution Q on $\mathcal{G}\Sigma_m^k$ can be nevertheless defined if we assume that this space is a metric space with a distance ρ. For example, if we identify $P\Sigma_1^4$ with the unit circle $\mathbb{R}P^1$, ρ can be selected to be the arc length.
Through distance ρ, it is possible to define a measure of spread following the general treatment by Fréchet (see [8] or Kendall et al. in [14]). Given a probability measure Q on $\mathcal{G}\Sigma_m^k$, we define for $\psi \in \mathcal{G}\Sigma_m^k$, the functional

$$F_Q(\psi) = E[\rho^2(x,\psi)] = \int_{P\Sigma_m^k} \rho^2(x,\psi)Q(dx). \tag{2}$$

Then the total Fréchet variance $t\Sigma_{\mathcal{F}}$ is defined as

$$t\Sigma_{\mathcal{F}} = \inf_{\psi \in P\Sigma_m^k} F_Q(\psi)$$

where the prefix t to Σ stands for total and suffix $\mathcal{F}$ stands for Fréchet. Assume there is a unique $\psi \in \mathcal{G}\Sigma_m^k$, such that $t\Sigma_{\mathcal{F}} = F_Q(\psi)$; such a ψ is said to be the Fréchet mean shape of Q.,

$$\psi := \mu_{\mathcal{F}}.$$

Note for unlike with probability distributions on $\mathbb{R}^p$, the minimizer ψ may not be unique.
Let $(X_1, ..., X_n)$ be a sample from a probability measure Q on $\mathcal{G}\Sigma_m^k$ and let $\widehat{Q}_n$ be the empirical probability measure

$$\widehat{Q}_n = \frac{1}{n}(\delta_{X_1} + ... + \delta_{X_n}). \tag{3}$$

The Fréchet sample mean set is the set $\widehat{\mu}_{\mathcal{F}} = \{\widehat{\psi} \in \mathcal{G}\Sigma_m^k, F_{\widehat{Q}_n}(\widehat{\psi}) = t\Sigma_{\mathcal{F}}(\widehat{Q}_n)\}$. If $\widehat{\mu}_{\mathcal{F}}$ has a unique element, this element is called the Fréchet sample mean and is labelled $\overline{X}_{\mathcal{F}}$. The strong consistency of the Fréchet sample mean set on a compact metric space is due to Ziezold [33] which implies that if $\mu_{\mathcal{F}}$ exists, then any measurable choice $\widehat{\psi}$ from $\widehat{\mu}_{\mathcal{F}}$ is a strongly consistent estimator of $\mu_{\mathcal{F}}$.
In case $\mathcal{G}\Sigma_m^k$ has a manifold structure, a distance ρ on it, can be obtained via any embedding j into an Euclidean space, by pulling back the restriction of the Euclidean distance. The Fréchet (sample) mean is in this case extrinsic (sample) mean and it is labelled $\mu_{j,E}$ or simply μ_E. In particular if the $\mathcal{G}$-shape space happens to have a structure of homogeneous space, an equivariant is preferred. Shape spaces in examples 1 to 3 in section 3 are symmetric. In particular in the case of similarity shape spaces, one should use equivariant embeddings of complex projective spaces similarity shape averaging. A list of equivariant embeddings of $\mathbb{C}P^{k-2}$ can be found in [15]. Of particular interest is the so called Veronese-Whitney quadratic embedding

$$j([z]) = zz^*, z \in \mathbb{C}^{k-1}, ||z|| = 1 \tag{4}$$

In [1] it was shown that if $[Z]$, $\|Z\| = 1$ is a random variable valued in $\mathbb{C}P^{k-2}$, then the extrinsic population mean exists if the largest eigenvalue of $E(ZZ^*)$ is simple (i.e., has multiplicity one) and in this case $\mu_E = [\mu]$, where μ is an eigenvector of $E(ZZ^*)$ corresponding to the largest eigenvalue, with $\|\mu\| = 1$.
In the case of affine shape spaces, one should use equivariant embeddings of real Grassmann manifolds for shape averaging. Such embeddings have been studied by Dimitric in [4]. Dimitric showed that an equivariant embedding of $G_m(k-1)$ can be defined as follows: let $Sym(k-1)$ be the set of $(k-1) \times (k-1)$ symmetric matrices endowed with the canonical Euclidean square norm $\|A\|^2 = Tr(AA^t)$; then the embedding of $G_m(k-1)$ into $Sym(k-1)$ is obtained by identifying each $m-j$-dimensional vector subspace L with the matrix p_L of orthogonal projection into L. Dimitric (1996) proved that j is an equivariant embedding that has parallel second fundamental form and embeds the Grassmannian minimally into a hypersphere. This is an extension of the Veronese-Whitney embedding of projective spaces considered in [1].
Assume a probability distribution Q of affine shapes of configurations in general position is nonfocal w.r.t. this embedding j. In this case, in [30] it was shown that the mean $\mu_{j(Q)}$ of the corresponding distribution $j(Q)$ of $(k-1) \times (k-1)$ symmetric matrices of rank m, has the eigenvalues $\lambda_1 \geq ... \geq \lambda_{k-1}$ such that $\lambda_m > \lambda_{m+1}$. The extrinsic mean of Q is the vector subspace spanned by unit eigenvectors corresponding to the first m eigenvalues of $\mu_{j(Q)}$. Assume $(\pi_1, ...\pi_n)$ is a sample of size n of m-dimensional vector subspaces $\pi_1, ...\pi_n$ of $\mathbb{R}^{k-1}$, and the subspace π_r is spanned by the orthonormal unit vectors $\{x_{r,a}\}_{a=1,...,m-j}$ and set $x_r = (x_{r,a})_{a=1,...,m-j}$. The extrinsic sample mean is of this sample, when it exists, is the $m-j$-vector subspace $\overline{\pi}$ generated by the unit eigenvectors corresponding to the first largest m eigenvalues of $\sum_r x_x x_r^t$.
Finally in the case of projective shape spaces of k-ads, one should use equivariant embeddings of products of a number of $q = k - m - 2$ copies of $\mathbb{R}P^m$. First, we consider the case of $q = 1$ and the embedding j of $\mathbb{R}P^m$ into $S(m+1)$, the space of symmetric matrices (Kent, 1992) given for the directional representation $[x] = \{\pm x, \| x \| = 1\}$, by

$$j([x]) = xx^T. \tag{5}$$

Here, the Euclidian norm of a matrix $A \in S(m+1)$ is given by $\| A \|^2 = \mathrm{tr} AA^T$, i.e. A is an $(m+1) \times (m+1)$ symmetric matrix and if $A = j([x])$ with $\| x \| = 1$ then $\| A \| = 1$. Moreover, if $[X_r]$, $(\|X_r\| = 1)$, $r = 1, ..., n$, is a random sample from a probability measure Q on $\mathbb{R}P^m$ and the extrinsic mean μ_E of Q exists, then the extrinsic sample mean $\overline{[X]}_E$ is a strongly consistent estimator of $\mu_E(Q)$. Note that when it exists $\overline{[X]}_E$ is given by $\overline{[X]}_E = [\mathbf{m}]$, where $\mathbf{m}$ is a unit eigenvector of $\frac{1}{n}\sum_{n=1}^{n} X_r X_r^T$, corresponding to the largest eigenvalue.
For $q > 1$ $(k > m+3)$, we can define an embedding j_k of $P\Sigma_m^k = (\mathbb{R}P^m)^q$ in $(S(m+1))^q$ in terms of j:

$$j_k([x_1], ..., [x_q]) = (j[x_1], ..., j[x_q]) \tag{6}$$

where $x_s \in \mathbb{R}^{m+1}, \| x_s \| = 1; s = 1, ..., q$. Again, it can be shown that if the largest eigenvalues of each of the q matrix components of $E(j_k(Q))$ is simple then the extrinsic mean $\mu_{E,j_k(Q)}$ exists and is given by

$$\mu_E := \mu_{E,j_k(Q)} = ([\gamma_1(m+1)], ..., [\gamma_q(m+1)]) \tag{7}$$

where $\gamma_s(m+1)$ is a unit eigenvector corresponding to the largest eigenvalue of the sth component of $E(j_k(Q))$ i.e., of $E(X_s X_s^T)$. If $X_r, r = 1, ..., n$ is a random sample from Q, then in the directional representation

$$X_r = ([X_{r,1}], ..., [X_{r,q}]), \| X_{r,s} \| = 1; s = 1, ..., q, \tag{8}$$

where $X_{r,s}$ is a $(m+1) \times 1$ vector, and X_r is $(m+1) \times q$ matrix. Note that $X_{r,s}$, the sth directional component X_r determined up to a sign, for fixed s, $s = 1, \ldots, q$. The matrix of sums of squares and products of $X_{r,s}$ is given by

$$J_s = \Sigma_{r=1}^{n} X_{r,s} X_{r,s}^T, \tag{9}$$

which is a well defined $(m+1) \times (m+1)$ matrix. Let $d_s(a)$ and $g_s(a)$ be the eigenvalues in increasing order and the corresponding unit eigenvector of J_s, $a = 1, ..., m+1$. Then the extrinsic sample mean in this case is

$$\overline{X}_{n,E} = ([g_1(m+1)], ..., [g_q(m+1)]). \tag{10}$$

4 Image fusion and scene reconstruction

Gray level images are generated by matrices of integers. Their entries (corresponding to grey levels at pixels) range on a scale from 0 to 255, and reflect the brightness of the scene pictured. The combined effect of the the brightness of these pixels, as displayed on the screen, is the way we perceive a continuous digital image. Therefore it is the reflectance of objects, and not the content, that dictates the appearance of a scene in a digital image. On the other hand, most of the time, we are only interested in the content of a scene displayed, so considering only low level image analysis, is like "not seeing the forest because of the trees".
Beside illumination, the appearance of a scene is mostly influenced by pose. Most often we classify objects by what we commonly agree to be their "frontal" view.

Definition 4.1 A scene is almost flat with a frontal view if the projection of the camera aperture on the average plane of the scene, falls somewhere in the middle of the scene.

Frontal views, can be recognized in the case of images of scenes containing clear features, which can be regarded as parallel lines, such as images of buildings, since in frontal views parallelism is almost preserved. However, quite often, ground conditions may not allow the photographer to take a frontal view of the scene, so that images taken from side views, in which the projection of the aperture of the camera on the mean plane of the scene pictured falls somewhere outside the central area of the picture. It is natural to attempt to transform a side view into a frontal view, without having to distort the scene. This is possible if the scene is perfectly flat, and we ignore

errors of registration, rounding errors, etc.
Such a transformation, should take in particular line segments to line segments, and it is known that the only local transformations of $\mathbb{R}^2$ with this properties are the projective transformations given by the equations:

$$x' = \frac{a_1x + a_2y + a_3}{c_1x + c_2y + c_3},$$
$$y' = \frac{b_1x + b_2y + b_3}{c_1x + c_2y + c_3}. \tag{11}$$

The vector $(b_1, b_2, b_3)^t$ in equation (1) is uniquely determined up to a multiplicative constant. Given two quadruples of points in general position $(p_1, p_2, p_3, p_4), (q_1, q_2, q_3, q_4) \in (\mathbb{R}^2)^4$, there is a unique projective transformation α, such that

$$\alpha(p_j) = q_j, \forall j = 1, .., 4.$$

We will use this expression in our method of reconstruction of a planar scene from side views.
A procedure mostly used in the media, and in medical image registration, is image averaging, through appropriate warping, or fusing of images . For a population of grey level frontal images, Mardia suggested in [18] a general algorithm (see also ([5], p.271)) to get estimators for the "mean" of a population of frontal images . We extend this algorithm to the following setting given in [24]:
Assume $\mathcal{D}_r$ is a family of diffeomorphisms depending on $4r$ parameters, defined in a neighborhood of the rectangle, support of the image, with the property that for any two configurations $(x_1, ..., x_r), (y_1, ..., y_r)$ of points in general position, there is a unique diffeomorphism $\alpha \in D_r$ defined on a neighborhood of $\{x_1, ..., x_r\}$, such that $\alpha(x_j) = y_j, \forall j = 1, ..., r$, and this family of diffeomorphisms is continuous as a function of $(x_1, ..., x_r), (y_1, ..., y_r)$.

Definition 4.2 Assume $\mathcal{I}$ is a population of grey level images of almost flat scenes, and $X = (X_{1I}, ..., X_{kI})$ is a configuration of landmarks seen in these images . Let $(X_1, ..., X_k)$ be a configuration of points for the $\mathcal{G}$-mean shape of X, and let $\alpha_I \in \mathcal{D}_r$ be such that

$$\alpha_I(X_{sI}) = X_s, \forall s = 1, ..., r.$$

Then the $\mathcal{GD}$-mean image is the gray level image :

$$g_\mu(i, j) = E(g(\alpha_I(i, j))), \tag{12}$$

where the expected value is w.r.t. the probability measure Q and $\alpha \in \mathcal{D}_r$ carries the configuration $(x_{1I}, ..., x_{rI})$ to the first r landmarks of a fixed representative of $E(\mathcal{G}\sigma(X_{1I}, ..., X_{kI}))$. The $\mathcal{GD}$-sample mean image of a sample $(I_1, X), ..., (I_n, X))$ is the $\mathcal{GD}$-mean image of the empirical distribution.

If $\mathcal{G}$ is the group of direct similarities in the Euclidean plane, and D is a family of diffeomorphisms, pairs of TPS's with small displacement, we get a TPS image average algorithm in [5]. This algorithm also applies to fusing frontal images of faces.

In Computer Vision, there are certain complex estimation problems in relation with the errors made at landmark location and such problems for example are discussed by Faugeras and Luong ([6], Ch. 6). The estimation techniques discussed there are related to the binocular view. However, image data collected involves multiple views, so the distributions of linear and planar projective shapes of random configurations of points recorded from side view images of the same scene is important. Faugeras and Luong fused multiple close views of images obtained from a camera rotating around an axis. In this section we consider the general case of "closer" multiple views. Often, ground conditions may not allow the photographer to take a frontal view of the scene and the resulting images are side views. For such views, projective shape analysis is the appropriate tool as shown in [25].
The approach to scene reconstruction in that paper is as follows: first represent the partial views of a scene, under an angle bounded from bellow, as images on a larger canvas of a given size. Two views that partially overlap, are transformed via transformations that bring in coincidence a given configuration of landmarks in their overlapping region to convenient representative of their extrinsic mean projective shape. For each of the resulting individual image, ignore the neighboring background pixels of the canvas. The alignment of more views is done along a representative of the extrinsic mean projective shape configuration of a the common group of landmarks in a fixed image, that may be regarded as a frontal image.

Definition 4.3 A reconstructed scene is at each point of the canvas given by the mean gray level of the transformed images that cover that point. For a given sample of multiple partial views, the estimate of the reconstructed larger scene at a pixel is obtained using the sample mean gray level for each of the contributed transformed images that share that pixel .

For actual data, the multiple views in general do not share a common region, so that the practical algorithm consists in starting with a frontal view that is overlapping with the largest number of other views, and selecting convenient subsets of views, and transformations to estimate the reconstructed scene.

5 Examples

In a first example of image fusion using the warping algorithm in the previous section based on extrinsic mean similarity shapes, we consider two faces in 1 and their fused image in reffusion. The fused image was obtained using a code that takes into account the definition of a fused image and a rounding step needed by the corresponding algorithm to adjust for the discrete pixel location. In the figure 3 are displayed the nine views of the northern face of large building in Atlanta. This face of the building can be approximated with a plane. While this side is planar at the lower floors, the upper central part of the building is set back, compared to its lateral parts that are protruding outward, which makes the recovery more complicated.
One of the views, which is brighter shares some common region with each of the

Figure 1: Frontal images of two human faces.

Figure 2: The two images above fused into one image .

other views, and this central view was used for alignment of the others and as a frontal view in the fusing process. We used seventeen landmarks, projective mean shapes and projective transformations to estimate the reconstructed scene which is displayed in figure 4. In spite of the 3D and registration error effects, the building can be recognized from this reconstruction.

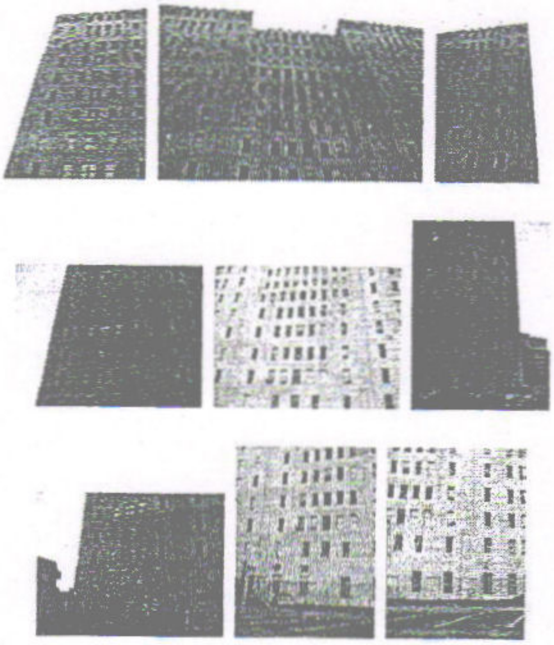

Figure 3: Partial views of a side of a building in Atlanta.

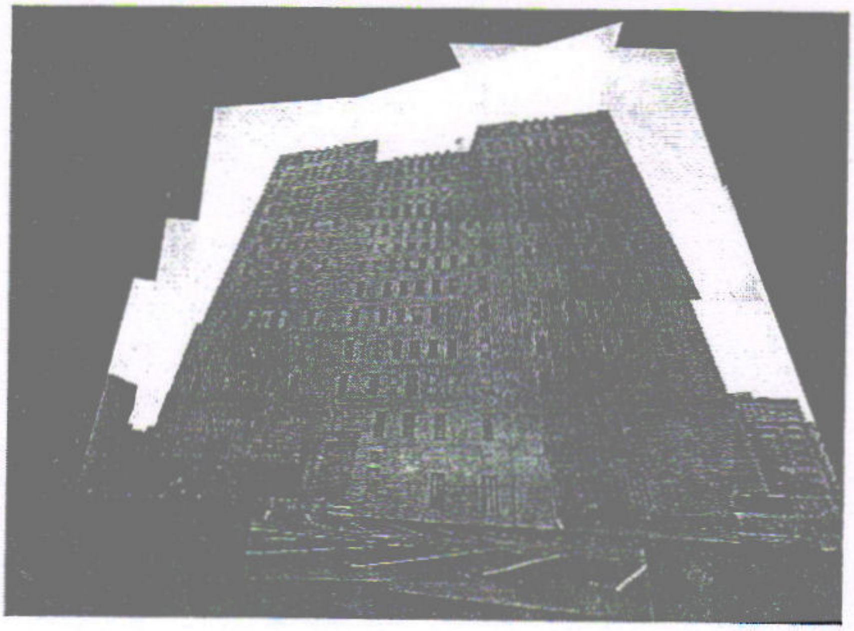

Figure 4: Estimate of the reconstructed scene from views in Fig. 3.

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