# HOMOGENEOUS STRUCTURES ON THE LEVICHEV SPACETIMES $IIa$

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**Abstract**

The homogeneous Lorentzian structures and the associated reductive decompositions for the Levichev homogeneous spacetimes of type $IIa$, are determined.

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**Key words:** homogeneous Lorentzian structures, reductive decomposition, Levichev homogeneous spacetimes.

## 1 Introduction

The classical É. Cartan's characterization of Riemannian symmetric spaces as the spaces of parallel curvature was extended by Ambrose and Singer, who gave in [1] a characterization for a connected, simply connected and complete Riemannian manifold to be homogeneous, in terms of a $(1,2)$ tensor field $S$, called by Tricerri and Vanhecke in [5] a *homogeneous Riemannian structure*, which satisfies certain equations (see (1.1) below). In [2] it is defined a *homogeneous pseudo-Riemannian structure* on a pseudo-Riemannian manifold $(M,g)$ as a tensor field $S$ of type $(1,2)$ such that $\nabla$ being the Levi-Civita connection and $R$ its curvature tensor, the connection $\widetilde{\nabla} = \nabla - S$ satisfies the Ambrose-Singer equations

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0. \tag{1.1}$$

In [2] it is proved that *if the pseudo-Riemannian manifold $(M,g)$ is connected, simply connected and geodesically complete then it admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.*

Let $(M,g)$ be a connected, simply connected, and geodesically complete pseudo-Riemannian manifold, and suppose that $S$ is a homogeneous pseudo-Riemannian

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structure on $(M, g)$. We fix a point $o \in M$ and put $\mathfrak{m} = T_o(M)$. If $\widetilde{R}$ is the curvature tensor of the connection $\widetilde{\nabla} = \nabla - S$, we can consider the holonomy algebra $\tilde{\mathfrak{h}}$ of $\widetilde{\nabla}$ as the Lie subalgebra of "skew-symmetric" endomorphisms of $(\mathfrak{m}, g_o)$ generated by the operators $\widetilde{R}_{ZW}$, where $Z, W \in \mathfrak{m}$. Then, according to the Ambrose-Singer construction [1, 5], a Lie bracket is defined in the vector space direct sum $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \mathfrak{m}$ by

$$
\begin{aligned}
&[U, V] = UV - VU, && U, V \in \tilde{\mathfrak{h}},\\
&[U, Z] = U(Z), && U \in \tilde{\mathfrak{h}},\ Z \in \mathfrak{m},\\
&[Z, W] = \widetilde{R}_{ZW} + S_Z W - S_W Z, && Z, W \in \mathfrak{m}\,,
\end{aligned}
\tag{1.2}
$$

and we say that $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ is the *reductive pair* associated to the homogeneous pseudo-Riemannian structure $S$.

Tricerri and Vanhecke [5] have classified the homogeneous Riemannian structures into eight classes, which are defined by the invariant subspaces of certain space $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$. In [3] a similar classification for the pseudo-Riemannian case is given.

On the other hand, Levichev defines in [4] several families of metrics on the Gödel group $G$, obtaining several types of homogeneous Lorentz spaces. The ones of type $IIa$ are connected, simply connected, and geodesically complete. In the present note we determine the homogeneous Lorentzian structures on these homogeneous space-times and their type in Tricerri-Vanhecke's classification, and the associated reductive decompositions.

An extended version of this work will appear elsewhere.

In a forthcoming paper we shall study the groups of isometries corresponding to the above reductive decompositions.

## 2 Homogeneous structures

The Gödel group is the simply connected Lie group $G$ whose Lie algebra $\mathfrak{g}$ has four generators $e_1, e_2, e_3, e_4$, with the only nonvanishing bracket $[e_4, e_1] = e_1$. $G$ admits a realization as $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4)\}$ with multiplication $z = x \cdot y$ obtained from the matrix expression

$$
x \equiv \begin{pmatrix} e^{x_4} & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Consider the subspaces $L_1, L_2, L_3$ of $\mathfrak{g}$ generated respectively by $e_1$; $e_2, e_3$; and $e_1, e_2, e_3$. Then the homogeneous Lorentz group of type $IIa$ is defined by the conditions: $L_2, L_3$ are timelike, and $L_1$ is spacelike (for more details see [4]). Then, for each couple of real numbers $p, q$ with $0 \leq p < 1$, $q > 0$, the left-invariant Lorentzian metric $g_{p,q}$ on $G$ obtained by left translations from the scalar product at the origin with matrix given, with respect to the above basis of $\mathfrak{g}$, by

$$\langle\ ,\ \rangle_{p,q} = \begin{pmatrix} 1 & p & 0 & 0 \\ p & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

is given by

$$g_{p,q} = \begin{pmatrix} e^{-2x_4} & e^{-2x_4}p & 0 & 0 \\ e^{-2x_4}p & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

Then we have

**Theorem 1.** *The homogeneous Lorentzian structures on $(G, g_{p,q})$ are given by*

$$\theta^1 \otimes \theta^1 \wedge \theta^4 + (1-p^2)\,\theta^1 \otimes \theta^2 \wedge \theta^4 + \frac{p}{2}\,(\theta^2 \otimes \theta^1 \wedge \theta^4 + \theta^4 \otimes \theta^1 \wedge \theta^2).$$

*Proof.* We use the conventions $S_{XYZ} = g(S_X Y, Z)$, $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, and $R(X,Y,Z,W) = g(R(Z,W)Y,X)$. In order to determine the homogeneous Lorentzian structures on $(G, g_{p,q})$, one must solve the Ambrose-Singer equations (1.1). The first Ambrose-Singer equation amounts to $S_{XYZ} = -S_{XZY}$ for any homogeneous pseudo-Riemannian structure $S$. One can write the second Ambrose-Singer equation $\widetilde{\nabla} R = 0$ as

$$\begin{aligned} &R_{\nabla_U XYZW} + R_{X\nabla_U YZW} + R_{XY\nabla_U ZW} + R_{XYZ\nabla_U W} \\ &\qquad = S_{UXR(Z,W)Y} - S_{UYR(Z,W)X} + S_{UZR(X,Y)W} - S_{UWR(X,Y)Z}. \end{aligned}$$

Solving, one computes the nonvanishing components of $S$ except for $S_{e_i e_1 e_4}$, $i = 1, \ldots, 4$, for which we must use the third Ambrose-Singer equation $\widetilde{\nabla} S = 0$. This can be written here as

$$S_{\nabla_X YZW} + S_{Y\nabla_X ZW} + S_{YZ\nabla_X W} = S_{S_X YZW} + S_{YS_X ZW} + S_{YZS_X W},$$

for $X, Y, Z, W \in \mathfrak{g}$. Solving, we obtain the rest of nonzero components. Then, with the convention $v \wedge w = v \otimes w - w \otimes v$ for the exterior product, we obtain the expression in the statement. $\square$

As for Tricerri-Vanhecke's decomposition $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$ mentioned in the Introduction, in the present case we deduce

**Corollary 1.** *The homogeneous Lorentzian structures on $(G, g_{p,q})$ belong to*

$$\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3 - \{(\mathcal{S}_1 \oplus \mathcal{S}_2) \cup (\mathcal{S}_1 \oplus \mathcal{S}_3) \cup (\mathcal{S}_2 \oplus \mathcal{S}_3)\}.$$

*In particular none of the associated reductive homogeneous spaces is either Lorentzian symmetric, or naturally reductive or cotorsionless.*

# 3 Reductive decompositions

Consider the Ambrose-Singer connection $\widetilde{\nabla} = \nabla - S$. According to Ambrose-Singer's Theorem on holonomy, the algebra of holonomy of a connection is generated by the curvature operators. In the present case, as calculation shows, the holonomy algebra $\tilde{\mathfrak{h}}$ has the only generator $V = \widetilde{R}_{e_1 e_4}$. Putting $\mathfrak{m}$ for $\mathfrak{g}$, on account of the expressions (1.2), and taking $T = V + e_1$, we have

**Theorem 2.** *The reductive pairs* $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ *associated to the reductive decompositions* $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \mathfrak{m}$ *corresponding to the homogeneous Lorentzian structures on* $(G, g_{p,q})$ *given in* Theorem 1*, are given in terms of the basis* $\{e_1, e_2, e_3, e_4, T\}$ *by the* (*nonvanishing*) *Lie brackets* $[T, e_4] = 2e_1 - T$ *and*

$$[e_1, e_2] = -\frac{2p^2 + p - 2}{2q} e_4, \; [e_1, e_4] = T - \frac{2p^3 - 3p^2 - 2p + 4}{2(1 - p^2)} e_1 + \frac{2p^2 + p - 2}{2(1 - p^2)} e_2.$$

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