HOMOGENEOUS STRUCTURES ON THE LEVICHEV SPACETIMES IIa

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Abstract

The homogeneous Lorentzian structures and the associated reductive decompositions for the Levichev homogeneous spacetimes of type IIa, are determined.

AMS Subject Classification: 53B30, 53C30, 53C50.
Key words: homogeneous Lorentzian structures, reductive decomposition, Levichev homogeneous spacetimes.

1 Introduction

The classical É. Cartan's characterization of Riemannian symmetric spaces as the spaces of parallel curvature was extended by Ambrose and Singer, who gave in [1] a characterization for a connected, simply connected and complete Riemannian manifold to be homogeneous, in terms of a (1,2) tensor field $S$, called by Tricerri and Vanhecke in [5] a homogeneous Riemannian structure, which satisfies certain equations (see (1.1) below). In [2] it is defined a homogeneous pseudo-Riemannian structure on a pseudo-Riemannian manifold $(M,g)$ as a tensor field $S$ of type (1,2) such that $\nabla$ being the Levi-Civita connection and $R$ its curvature tensor, the connection $\tilde{\nabla} = \nabla - S$ satisfies the Ambrose-Singer equations

\begin{align}
\tilde{\nabla}g &= 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0.
\end{align}

In [2] it is proved that if the pseudo-Riemannian manifold $(M,g)$ is connected, simply connected and geodesically complete then it admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.

Let $(M,g)$ be a connected, simply connected, and geodesically complete pseudo-Riemannian manifold, and suppose that $S$ is a homogeneous pseudo-Riemannian
structure on \((M,g)\). We fix a point \(o \in M\) and put \(m = T_o(M)\). If \(\tilde{R}\) is the curvature tensor of the connection \(\nabla = \nabla - S\), we can consider the holonomy algebra \(\tilde{h}\) of \(\nabla\) as the Lie subalgebra of “skew-symmetric” endomorphisms of \((m,g_o)\) generated by the operators \(\tilde{R}_{ZW}\), where \(Z,W \in m\). Then, according to the Ambrose-Singer construction [1, 5], a Lie bracket is defined in the vector space direct sum \(\tilde{g} = \tilde{h} \oplus m\) by

\[
[U, V] = UV - VU, \quad U, V \in \tilde{h},
\]

\[
[U, Z] = U(Z), \quad U \in \tilde{h}, \ Z \in m,
\]

\[
[Z, W] = \tilde{R}_{ZW} + S_Z W - S_W Z, \quad Z, W \in m,
\]

and we say that \((\tilde{g}, \tilde{h})\) is the reductive pair associated to the homogeneous pseudo-Riemannian structure \(S\).

Tricerri and Vanhecke [5] have classified the homogeneous Riemannian structures into eight classes, which are defined by the invariant subspaces of certain space \(S_1 \oplus S_2 \oplus S_3\). In [3] a similar classification for the pseudo-Riemannian case is given.

On the other hand, Levichev defines in [4] several families of metrics on the G"odel group \(G\), obtaining several types of homogeneous Lorentz spaces. The ones of type \(IIa\) are connected, simply connected, and geodesically complete. In the present note we determine the homogeneous Lorentzian structures on these homogeneous space-times and their type in Tricerri-Vanhecke’s classification, and the associated reductive decompositions.

An extended version of this work will appear elsewhere.

In a forthcoming paper we shall study the groups of isometries corresponding to the above reductive decompositions.

2 Homogeneous structures

The G"odel group is the simply connected Lie group \(G\) whose Lie algebra \(\mathfrak{g}\) has four generators \(e_1, e_2, e_3, e_4\), with the only nonvanishing bracket \([e_4, e_1] = e_1\). \(G\) admits a realization as \(\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4)\}\) with multiplication \(z = x \cdot y\) obtained from the matrix expression

\[
x(\begin{array}{cccc}
e^{x_4} & 0 & 0 & x_1 \\
0 & 1 & 0 & x_2 \\
0 & 0 & 1 & x_3 \\
0 & 0 & 0 & 1
\end{array})
\]

Consider the subspaces \(L_1, L_2, L_3\) of \(\mathfrak{g}\) generated respectively by \(e_1, e_2, e_3\); and \(e_1, e_2, e_3\). Then the homogeneous Lorentzian group of type \(IIa\) is defined by the conditions: \(L_2, L_3\) are timelike, and \(L_1\) is spacelike (for more details see [4]). Then, for each couple of real numbers \(p, q\) with \(0 \leq p < 1, q > 0\), the left-invariant Lorentzian metric \(g_{p,q}\) on \(G\) obtained by left translations from the scalar product at the origin
with matrix given, with respect to the above basis of $g$, by

$$
\langle , \rangle_{p,q} = \begin{pmatrix}
p & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{pmatrix},
$$

is given by

$$
g_{p,q} = \begin{pmatrix}
e^{-2x_p} & e^{-2x_p}p & 0 & 0 \\
e^{-2x_p}p & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}.
$$

Then we have

**Theorem 1.** The homogeneous Lorentzian structures on $(G, g_{p,q})$ are given by

$$
\theta^1 \otimes \theta^1 \wedge \theta^4 + (1 - p^2) \theta^2 \otimes \theta^4 + \frac{p}{2} (\theta^2 \otimes \theta^1 \wedge \theta^4 + \theta^4 \otimes \theta^1 \wedge \theta^2).
$$

**Proof.** We use the conventions $S_{XYZ} = g(S_X Y, Z), R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$, and $R(X, Y, Z, W) = g(R(Z, W)Y, X)$. In order to determine the homogeneous Lorentzian structures on $(G, g_{p,q})$, one must solve the Ambrose-Singer equations (1.1). The first Ambrose-Singer equation amounts to $S_{XYZ} = - S_{XZY}$ for any homogeneous pseudo-Riemannian structure $S$. One can write the second Ambrose-Singer equation $e^r S^{r} = 0$ as

$$
R_{\nabla_X YZW} + R_{X \nabla_Y ZW} + R_{XY \nabla vZW} + R_{XYZ \nabla vW} = S_{UXR(Z, W)Y} - S_{UYR(Z, W)X} + S_{UZR(X, Y)W} - S_{UWR(X, Y)Z}.
$$

Solving, one computes the nonvanishing components of $S$ except for $S_{e_i, e_i, e_i}, i = 1, \ldots, 4$, for which we must use the third Ambrose-Singer equation $\nabla S = 0$. This can be written here as

$$
S_{\nabla_X YZW} + S_{Y \nabla_X ZW} + S_{YZ \nabla vW} = S_{S_X YZW} + S_{YS_X ZW} + S_{YZS_X W},
$$

for $X, Y, Z, W \in g$. Solving, we obtain the rest of nonzero components. Then, with the convention $v \wedge w = v \otimes w - w \otimes v$ for the exterior product, we obtain the expression in the statement.

As for Tricerri-Vanhecke’s decomposition $S_1 \oplus S_2 \oplus S_3$ mentioned in the Introduction, in the present case we deduce

**Corollary 1.** The homogeneous Lorentzian structures on $(G, g_{p,q})$ belong to

$$
S_1 \oplus S_2 \oplus S_3 - \{(S_1 \oplus S_2) \cup (S_1 \oplus S_3) \cup (S_2 \oplus S_3)\}.
$$

In particular none of the associated reductive homogeneous spaces is either Lorentzian symmetric, or naturally reductive or cotorsionless.
3 Reductive decompositions

Consider the Ambrose-Singer connection $\tilde{\nabla} = \nabla - S$. According to Ambrose-Singer’s Theorem on holonomy, the algebra of holonomy of a connection is generated by the curvature operators. In the present case, as calculation shows, the holonomy algebra $\mathfrak{h}$ has the only generator $V = \tilde{R}_{e_1 e_4}$. Putting $\mathfrak{m}$ for $\mathfrak{g}$, on account of the expressions (1.2), and taking $T = V + e_1$, we have

**Theorem 2.** The reductive pairs $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ associated to the reductive decompositions $\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \mathfrak{m}$ corresponding to the homogeneous Lorentzian structures on $(G, g_{p,q})$ given in Theorem 1, are given in terms of the basis $\{e_1, e_2, e_3, e_4, T\}$ by the (nonvanishing) Lie brackets $[T, e_4] = 2e_1 - T$ and

$$[e_1, e_2] = -\frac{2p^2 + p - 2}{2q} e_4, \quad [e_1, e_4] = T - \frac{2p^3 - 3p^2 - 2p + 4}{2(1 - p^2)} e_1 + \frac{2p^2 + p - 2}{2(1 - p^2)} e_2.$$

Acknowledgments

This paper has been partially supported by DGICYT (Spain) under grant no. PB89-0533. The second author (A.P.R.) is grateful to Professors G. Tsagas and V. Balan for their kind hospitality in Thessaloniki’s Meeting.

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