GEOMETRY OF BERWALD–CARTAN SPACES

Mihai Anastasiei

Abstract

The geometry of regular Hamiltonians as smooth functions on the cotangent bundle is mainly due to R. Miron and it is now systematically described in the monograph [4]. A manifold endowed with a regular Hamiltonian which is 2–homogeneous in momenta was called a Cartan space. An interesting particular class of Cartan spaces is given by the so–called Berwald–Cartan spaces. In this paper some new properties of the Berwald–Cartan spaces are proved.

AMS Subject Classification: 53C60.
Key words: cotangent bundle, homogeneous Hamiltonians.

Introduction

Analytical Mechanics and some theories in Physics brought into discussion regular Lagrangians and their geometry, [5]. A regular Lagrangian which is 2-homogeneous in velocities is nothing but the square of a fundamental Finsler function and its geometry is Finsler geometry. This geometry was developed since 1918 by P. Finsler, E. Cartan, L. Berwald and many others, see [2] and the most recent graduate text [1]. But in Mechanics and Physics there exists also regular Hamiltonians whose geometry is also useful. This geometry is mainly due to R. Miron ,[3], and it is now systematically presented in the monograph [4]. A manifold endowed with a regular Hamiltonian which is 2-homogeneous in momenta was called a Cartan space. The notion of Cartan space was introduced by R. Miron in [3]. A particular and interesting class of Cartan spaces is given by the so–called Berwald–Cartan spaces, shortly BC–spaces. The geometry of the BC–spaces can be found in [4], Chs. 6-7. Our purpose is to prove some new properties of these spaces. A Cartan space is a pair $(M, K)$ for $M$ a smooth manifold and $K$ a regular Hamiltonian which is 2-homogeneous in momenta. A BC space is defined as a Cartan space whose Chern–Rund connection coefficients of the canonical metrical connection do not depend on momenta, that is, $H_{jk}^i(x, p) = H_{jk}^i(x)$. For a Cartan space the pair $(T^*_x M, K(x, p))$ for any fixed $x \in M$ is a Minkowski space. We prove (Theorem 3.2) that for BC spaces the Minkowski spaces $(T^*_x M, K(x, p))$
are all linearly isometric to each other. Noticing that the functions $H^i_{jk}(x)$ defines a symmetric linear connection $\nabla$ on $M$ we prove (Theorem 3.3) that $\nabla$ is metrizable, that is, there exists a Riemannian metric on $M$ whose Levi-Civita connection is $\nabla$. These proofs are presented in Section 3. Some preliminaries from the geometry of cotangent bundle are given in Section 1, and Section 2 contains necessary facts from the geometry of Cartan spaces.

1 Preliminaries

Let $M$ be an $n$-dimensional $C^\infty$ manifold and $\tau^* : T^* M \rightarrow M$ its cotangent bundle. If $(x^i)$ are local coordinates on $M$, then $(x^i, p_i)$ will be taken as local coordinates on $T^* M$ with the momenta $(p_i)$ provided by $p = p_i dx^i$ where $p \in T^*_x M$, $x = (x^i)$ and $(dx^i)$ is the natural basis of $T^*_x M$. The indices $i, j, k...$ will run from 1 to $n$ and the Einstein convention on summation will be used. A change of coordinates $(x^i, p_i) \rightarrow (\tilde{x}^i, \tilde{p}_i)$ on $T^* M$ has the form

$$\begin{align*}
\tilde{x}^i &= \tilde{x}^i(x^1, ..., x^n), \\
\tilde{p}_i &= \frac{\partial x^i}{\partial \tilde{x}^j} (\tilde{x}) p_j, 
\end{align*}$$

(1.1)

where $\left( \frac{\partial x^i}{\partial \tilde{x}^j} \right)$ is the inverse of the Jacobian matrix $\left( \frac{\partial \tilde{x}^i}{\partial x^j} \right)$. Let $\left( \partial_i := \frac{\partial}{\partial x^i}, \partial^j := \frac{\partial}{\partial p_j} \right)$ be the natural basis in $T_{(x,p)}^* M$. The change of coordinates (1.1) produces

$$\begin{align*}
\partial_i &= (\partial_i \tilde{x}^j) \partial_j + (\partial_i \tilde{p}_j) \partial^j, \\
\partial^j &= (\partial_j \tilde{x}^i) \partial_i.
\end{align*}$$

(1.2)

The natural cobasis $(dx^i, dp_i)$ from $T_{(x,p)}^* M$ transforms as follows.

$$d\tilde{x}^i = (\partial_i \tilde{x}^j) dx^j, \quad d\tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} dp_j + \frac{\partial^2 x^j}{\partial \tilde{x}^i \partial \tilde{x}^j} p_j dx^k.
$$

(1.3)

The kernel $V_{(x,p)}$ of the differential $d\tau^* : T_{(x,p)}^* M \rightarrow T_x M$ is called the vertical subspace of $T_{(x,p)}^* M$ and the mapping $(x, p) \rightarrow V_{(x,p)}$ is a regular distribution on $T^* M$ called the vertical distribution. This is integrable with the leaves $T^*_x M$, $x \in M$ and is locally spanned by $(\partial^j)$. The vector field $C^* = p_j \partial^j$ is called the Liouville vector field and $\omega = p_i dx^i$ is called the Liouville 1-form on $T^* M$. Then $d\omega$ is the canonical symplectic structure on $T^* M$. For an easier handling of the geometrical objects on $T^* M$ it is usual to consider a supplementary distribution to the vertical distribution, $(x, p) \rightarrow N_{(x,p)}$, called the horizontal distribution and to report all geometrical objects on $T^* M$ to the decomposition

$$T_{(x,p)}^* M = N_{(x,p)} \oplus V_{(x,p)}.
$$

(1.4)
The pieces produced by the decomposition (1.4) are called $d$–geometrical objects ($d$ is for distinguished) since their local components behave like geometrical objects on $M$, although they depend on $x = (x^i)$ and momenta $p = (p_i)$.

The horizontal distribution is taken as being locally spanned by the local vector fields

$$\delta_i := \partial_i + N_{ij}(x, p)\partial^j,$$

and for a change of coordinates (1.1), the condition

$$\delta_i = (\partial_i x^j)\delta_j$$

for $\delta_j := \partial_j + \tilde{N}_{jk}(\bar{x}, \bar{p})\partial^k$,

is equivalent with

$$\tilde{N}_{ij}(\bar{x}, \bar{p}) = \frac{\partial x^s}{\partial x^i} \frac{\partial x^r}{\partial x^j} N_{sr}(x, p) + \frac{\partial^2 x^r}{\partial x^i \partial x^j} p_r.$$

The horizontal distribution is called also a nonlinear connection on $T^* M$ and the functions $(N_{ij})$ are called the local coefficients of this nonlinear connection. It is important to note that any regular hamiltonian on $T^* M$ determines a nonlinear connection whose local coefficients verify $N_{ij} = N_{ji}$.

The basis $(\delta_i, \partial^i)$ is adapted to the decomposition (1.4). The dual of it is $(dx^i, \delta p_i)$, for $\delta p_i = dp_i - N_{ji} dx^j$ and then $\delta \tilde{p}_i = \frac{\partial x^i}{\partial x^l} \delta \tilde{p}_j$.

## 2 Cartan spaces

A Cartan structure on $M$ is a function $K : T^* M \rightarrow [0, \infty)$ with the following properties:

1. $K$ is $C^\infty$ on $T^* M \setminus 0 = \{(x, 0), \ x \in M\}$.
2. $K(x, \lambda p) = \lambda K(x, p)$ for all $\lambda > 0$,
3. The $n \times n$ matrix $(g^{ij})$, where $g^{ij}(x, p) = \frac{1}{2} \partial^i \partial^j K^2(x, p)$, is positive–definite at all points of $T^* M \setminus 0$.

We notice that in fact $K(x, p) > 0$, whenever $p \neq 0$.

**Definition 2.1.** The pair $(M, K)$ is called a Cartan space.

**Example.** Let $(\gamma_{ij}(x))$ be the matrix of the local coefficients of a Riemannian metric on $M$ and $(\gamma^{ij}(x))$ its inverse. Then $K(x, p) = \sqrt{\gamma^{ij}(x)p_i p_j}$ gives a Cartan structure. Thus any Riemannian manifold can be regarded as a Cartan space. More examples can be found in Ch. 6 of [4].
We put $p^i = \frac{1}{2} \partial^i K^2$ and $C^{ij^k} = -\frac{1}{4} \partial^i \partial^j \partial^k K^2$. The properties of $K$ imply

\begin{align}
p^i &= g^{ij} p_j, \quad p_i = g_{ij} p^j,
K^2 &= g^{ij} p_i p_j = p_i p^i, 
C^{ij^k} p_k &= C^{ijk} p_k = 0.
\end{align}

One considers the formal Christoffel symbols

\begin{equation}
(2.2) \quad \gamma^i_{jk}(x, p) := \frac{1}{2} g^{is} (\partial_k g_{js} + \partial_j g_{sk} - \partial_s g_{jk})
\end{equation}

and the contractions $\gamma^i_{jk}(x, p) := \gamma^i_{jk}(x, p) p_i$, $\gamma^j_{io} := \gamma^i_{jk}(x, p) \partial^k g_{ij}$. Then the functions

\begin{equation}
(2.3) \quad N_{ij}(x, p) = \gamma^o_{ij}(x, p) - \frac{1}{2} \gamma^o_{ho}(x, p) \partial^h g_{ij}(x, p),
\end{equation}

verify (1.7). In other words, these functions define a nonlinear connection on $T^* M$. This nonlinear connection was discovered by R. Miron, [3]. Thus a decomposition (1.4) holds. From now on we shall use only the nonlinear connection given by (2.3).

A linear connection $D$ on $T^* M$ is said to be an $N$–linear connection if

1° $D$ preserves by parallelism the distributions $N$ and $V$,

2° $D \theta = 0$, for $\theta = \delta p_i \wedge dx^i$.

One proves that an $N$-linear connection can be represented in the adapted basis $(\delta_i, \partial^i)$ in the form

\begin{align}
(2.4) \quad &D_{\delta_i} \delta_j = H^i_{jk} \delta_j, \quad D_{\delta_j} \partial^i = -H^i_{kj} \partial^k, \\
&\quad D_{\partial^i} \delta_j = V^i_{kj} \delta_k, \quad D_{\partial^k} \partial^i = -V^i_{k} \delta^k,
\end{align}

where $V^i_{kj}$ is a $d$–tensor field and $H^i_{jk}(x, p)$ behave like the coefficients of a linear connection on $M$. The functions $H^i_{jk}$ and $V^i_{kj}$ define operators of $h$–covariant and $v$–covariant derivatives in the algebra of $d$-tensor fields, denoted by $\partial^k|_i$ and $\partial^k|_j$, respectively. For $g^{ij}$ these are given by

\begin{align}
(2.5) \quad &g^{ij}|_k = \delta_k g^{ij} + g^{sj} H^i_{sk} + g^{sk} H^j_{hk}, \\
&g^{ij}|^k = \partial^k g^{ij} + g^{sj} V^k_{sj} + g^{sk} V^j_{sk}.
\end{align}

An $N$-linear connection given in the adapted basis $(\delta_i, \partial^i)$ as $D \Gamma(N) = (H^i_{jk}, V^i_{kj})$ is called metrical if

\begin{equation}
(2.6) \quad g^{ij}|_k = 0, \quad g^{ij}|^k = 0.
\end{equation}

One verifies that the $N$-linear connection $C \Gamma(N) = (H^i_{jk}, C^i_{jk})$ with

\begin{align}
(2.7) \quad &H^i_{jk} = \frac{1}{2} g^{js} (\partial_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}), \\
&\quad C^i_{jk} = -\frac{1}{2} g_{js} (\partial^i g^{sk} + \partial^j g^{sj} - \partial^s g^{ik}) = g_{ks} C^{sj}.
\end{align}
is metrical and its h-torsion $T_{jk} := H_{jk}^i - H_{kj}^i = 0$, v-torsion $S_{jk}^i := C_{ij}^k - C_{ik}^j = 0$ and the deflection tensor $\Delta_{ij} = N_{ij} - p_k H_{ij}^k = 0$. Moreover, it is unique with these properties. This is called the canonical metrical connection of the Cartan space $(M, K)$. It has also the following properties:

\[(2.8)\]
\[
\begin{align*}
K_{ij} &= 0, & K^i_j &= \frac{\partial^i}{\partial x^j}, & K^2_{ij} &= 0, & K^2_{ij} &= 2p^i, \\
p_{ij} &= 0, & p^i_j &= \delta^i_j, & p^i_j &= 0, & p^i_j &= g^{ij}.
\end{align*}
\]

Besides $CT(N)$ one may consider on $T^* M$ three other important $N$-linear connection which are partially or not at all metrical: Chern–Rund connection $\Gamma(N) = (H_{jk}^i, 0)$, the Hashiguchi connection $H\Gamma(N) = (\partial^i N_{jk}, C^i_j)$ and the Berwald connection $B\Gamma(N) = (\partial^i N_{jk}, 0)$.

## 3 Berwald–Cartan spaces

Let $C^m = (M, K)$ be a Cartan space with the canonical metrical connection $CT(N) = (H_{jk}^i, C^i_j)$ given by (2.7).

**Definition 3.1.** The Cartan space $C^m$ is called a Berwald–Cartan space, shortly a BC space, if the connection coefficients $H_{jk}^i$ do not depend on momenta, that is, $H_{jk}^i(x, p) = H_{jk}^i(x)$.

In [4], by direct methods or using the duality between Finsler and Cartan spaces given by the Legendre map, one proves

**Theorem 3.1.** The following assertions are equivalent:

1° The Cartan space $C^m$ is a BC space,

2° The coefficients $B_{jk}^i = \partial^i N_{jk}$ of the Berwald connection are functions of position only, that is $B_{jk}^i(x, p) = B_{jk}^i(x)$,

3° The curvature $P_{jk}^{ih} := \partial^h B_{jk}^i$ of the Berwald connection vanishes.

4° $C^{ij} = 0$.

For the Cartan space $C^m = (M, K)$, the function $K_x := K(x, \cdot) : T^*_x M \rightarrow \mathbb{R}$ is a Minkowski norm for every $x \in M$. Thus we have the Minkowski spaces $(T^*_x M, K_x)$, $x \in M$. For BC spaces, the following theorem holds.

**Theorem 3.2.** Let $(M, K)$ be a BC space. Whenever $M$ is connected the Minkowski spaces $(T^*_x M, K_x)$ are all linearly isometric to each other.

**Proof.** Let $\omega = \omega_i dx^i$ an 1-form and $v = v^j \partial_j$ a vector field on $M$. Using the connection coefficients $H_{jk}^i(x)$ we may define a covariant derivative of $\omega$ in the direction of $v$ as follows: $\nabla_v \omega = v^k (\partial_k \omega_i - H_{ik}^j \omega_j) dx^i$. 

We restrict $\omega$ to a curve $c : t \to x(t), t \in \mathbb{R}$, on $M$, define the covariant derivative of $\omega$ along $c$ by
\[
\nabla c_t = \frac{d}{dt} \left[ \frac{d\omega_i^j}{dt} - H_{ik}^j \frac{dx_k}{dt} \right] dx^i \quad \text{and we say that $\omega$ is parallel along $c$ if $\nabla c_t = 0$.} \]
Let us estimate $\frac{dK^2(x(t),\omega(t))}{dt}$. We write the equality $K^2(x,p) = g^{ij}(x,p)p_ip_j$ for $(x(t),\omega(t))$ and we obtain that along the curve $c:\frac{dK^2}{dt} = \frac{dg^{ij}}{dt} \omega^j + 2g^{ij} \omega_i \frac{dx_i}{dt}$. But $\frac{d}{dt}(g^{ij}) = (\delta_k g^{ij}) \frac{dx_k}{dt} + (\partial^k g^{ij}) \frac{dx_i}{dt}$ and using $g^{ij} = 0$ as well as the last equation (2.1) one gets:
\[
\frac{dK^2}{dt} = 2g^{ij} \omega_i \left( \frac{dx_j}{dt} - H_{jk}^s \omega_s \frac{dx^k}{dt} \right).
\]
From here we read

**Lemma 3.1.** If the 1-form $\omega$ is parallel along the curve $c : t \to x(t)$, then the function $K(t) := K(x(t),\omega(t))$ is constant along the curve $c$.

Let $x,y$ be points of $M$ joined by a curve $c : [0,1] \to M$ such that $c(0) = x$, $c(1) = y$. Let $\alpha \in T_x^*M$. We consider the unique solution $\omega = (\omega_i)$ of the system of linear ordinary differential equations $\frac{d\omega_i}{dt} - H_{ik}^j \omega_j \frac{dx_k}{dt} = 0$ with the initial condition $\omega(0) = \alpha$ and we associate to $\alpha$ the element $\alpha' = \omega(1)$ of $T_y^*M$. The mapping $T_x^*M \to T_y^*M$ given by $\alpha \to \alpha'$ is a linear isomorphism. By Lemma 3.1, $K(x(t),\omega(t))$ has the same values at $t = 0$. Hence $K_x(\alpha) = K_y(\alpha')$. This means that the Minkowski spaces $(T_x^*M,K_x)$ and $(T_y^*M,K_y)$ are linearly isometric for every $x,y \in M$, $\Box$

Another interesting property of $BC$ spaces is as follows.

The connection coefficients $H_{jk}^s(x,p) = H_{jk}^s(x)$ define a symmetric linear connection $\nabla$ on $M$ and it happens that this is metrizable, that is, there exists on $M$ a Riemannian metric $h$ such that $\nabla$ is the Levi–Civita connection associated to it. This $h$ is not unique.

We prove this fact by adapting an idea of Z.I. Szabó [6]. The duality with Finsler spaces is not used.

**Theorem 3.3.** Let $C^\alpha = (M,K)$ be a $BC$ space with $M$ connected and $\nabla$ the symmetric linear connection on $M$ of local coefficients $H_{jk}^s(x,p) = H_{jk}^s(x)$. Then there exists a Riemannian metric $h$ on $M$ such that $\nabla$ is the Levi–Civita connection of it.

**Proof.** Let be the Minkowski space $(T_x^*M,K_{x_0})$ for a fixed $x_0 \in M$. Then $S_{x_0} = \{ \omega \mid K_{x_0}(\omega) = 1 \}$ is a compact subset of $T_{x_0}^*M$. Let $G$ be the group of all linear isomorphisms of $T_{x_0}^*M$ that preserve $S_{x_0}$. This $G$ is a compact Lie group. It contains as a subgroup the holonomy group $H_{x_0}$ defined by $(H_{jk}^i(x))$ according to Lemma 3.1. In general, $H_{x_0}$ is not compact.
Let $\langle \cdot, \cdot \rangle$ be any inner product in $T^*_x M$. Define a new inner product on $T^*_x M$ by

$$
(3.1) \quad h_{x_0}(\varphi, \omega) = \frac{1}{\text{vol}(G)} \int_G \langle a\varphi, a\omega \rangle \mu_G, \quad \varphi, \omega \in T^*_x M,
$$

for $a \in G$, where $\mu_G$ denotes the bi-invariant Haar measure on $G$. It results $h_{x_0}(b\varphi, b\omega) = h_{x_0}(\varphi, \omega)$ for every $b \in G$ (from the properties of $\mu_G$), that is $h_{x_0}$ is $G$-invariant. In particular, $h_{x_0}$ is $H_{x_0}$-invariant.

Let now any $x \in M$ and a curve $c : t \to c(t)$ joining $x$ with $x_0$, $c(0) = x$, $c(1) = x_0$. Denote by $P_c : T^*_x M \to T^*_x M$ the parallel transport of covectors defined by $H^i_{jk}(x)$.

For every $\varphi \in T^*_x M$, $P_c(\varphi) = \omega(1) \in T^*_x M$, where $\omega = (\omega_i)$ is the unique solution of the system of linear differential equations

$$
(3.2) \quad \frac{d\omega_i}{dt} - H^i_{jk}\omega_j \frac{dx^k}{dt} = 0, \quad \text{with} \quad \omega(0) = \varphi.
$$

In the proof of Theorem 3.2 we have seen that $P_c$ is a linear isometry of Minkowski spaces. We define an inner product on $T^*_x M$ by

$$
(3.3) \quad h_x(\varphi, \psi) = h_{x_0}(P_c\varphi, P_c\psi), \quad \varphi, \psi \in T^*_x M.
$$

Lemma 3.2. $h_x$ does not depend on the curve $c$.

Indeed, if $\tilde{c}$ is another curve joining $x$ and $x_0$, denote by $c_-$ the reverse of $c$ and consider the loop $\tilde{c} \circ c_-$. Then $P_{c_-} \in H_{x_0}$ and from the $H_{x_0}$-invariance of $h_{x_0}$, that is, $h_{x_0}(P_{c_-} \varphi, P_{c_-} \psi) = h_{x_0}(\varphi, \psi)$ we get $h_{x_0}(P_c \varphi, P_c \psi) = h_{x_0}(P_c \varphi, P_c \psi)$ as we claimed.

The mapping $x \rightarrow h_x : T^*_x M \times T^*_x M \rightarrow R$ is smooth since $P_c$ smoothly depends on $x$, according to a general result regarding the dependence of solution of system of differential equations by initial data. Thus we have constructed a Riemannian metric $h$ in the cotangent bundle of $M$.

The connection coefficients $(H^i_{jk}(x))$ define a linear connection $\nabla$ in the cotangent bundle as follows:

$$
\nabla : \mathcal{X}(M) \times \Gamma(T^* M) \rightarrow \Gamma(T^* M), \quad (X, \omega) \rightarrow \nabla_X \omega = X^i \left( \frac{\partial \omega_i}{\partial x^k} - H^i_{jk}\omega_j \right) dx^i
$$

and the operator $\nabla_X$, $X \in \mathcal{X}(M)$, extends to the tensorial algebra of the cotangent bundle. For instance, if we regard $h$ as a section in the vector bundle $L^2(T^* M, \mathbb{R})$, then we have

$$
(3.4) \quad (\nabla_X h)(\varphi, \psi) = X(h(\varphi, \psi)) - h(\nabla_X \varphi, \psi) - h(\varphi, \nabla_X \psi).
$$

Lemma 3.3. $\nabla_X h = 0$, $X \in \mathcal{X}(M)$.
Proof. We choose a basis \((\varphi_i(x))\) in \(T_x^*M\). It suffices to show that 
\([\nabla_X h](\varphi_i(x), \varphi_j(x)) = 0\). Let be the vector \(X = \frac{dc}{dt}\) tangent to a curve \(c\) starting from \(x \in M\) at \(t = 0\). We parallel translate \(\varphi_i(x)\) along \(c\) and we obtain a field of basis \(\varphi_i(t)\) along \(c\). The general formula
\[
\frac{\nabla h}{dt}(\varphi, w) = \frac{dh(\varphi, \psi)}{dt} - h \left( \frac{\nabla \varphi}{dt}, \psi \right) - h \left( \varphi, \frac{\nabla \psi}{dt} \right),
\]
gives
\[
\frac{\nabla h}{dt}(\varphi_i(x), \varphi_j(x)) = \left. \frac{dh(\varphi_i, \varphi_j)}{dt} \right|_{t=0}
\]
because of \(\frac{\nabla \varphi_i}{dt} = 0\).

Now we show that \(h(\varphi_i(t), \varphi_j(t))\) does not depend on \(t\).
Indeed, \(h_{\varphi(t)}(\varphi_i(t), \varphi_j(t)) = h_{\varphi(0)}(P_{\varphi_i}, P_{\varphi_j})\), where \(P\) is the parallel translation from \(T_{\varphi(t)}M\) to \(T_{\varphi(0)}M\). This \(P\) may be thought as the composition of a parallel translation \(P_2\) from \(T_{\varphi(t)}M\) to \(T_{\varphi(0)}M\) and of a parallel translation \(P_1\) from \(T_{\varphi(0)}M\) to \(T_{\varphi(0)}M\). We have \(h_{\varphi(t)}(\varphi_i(t), \varphi_j(t)) = h_{\varphi(0)}((P_2 \circ P_1)\varphi_i, (P_2 \circ P_1)\varphi_j) = h_{\varphi(0)}(P_1\varphi_i, P_2\varphi_j) = h_x(\varphi_i(x), \varphi_j(x))\). Hence \(h_{\varphi(t)}(\varphi_i(t), \varphi_j(t))\) does not depend on \(t\), as we claimed.

This fact ends the proof of Lemma 3.3.

To end the proof of Theorem, we take the covariant part of \(h\) as a section in the vector bundle \(\mathcal{L}_X(TM, \mathbb{R})\) and so we get a Riemannian metric on \(M\), denoted with the same letter \(h\). The operator \(\nabla_X\) acts also on vector fields on \(M\) by the rule
\[
\nabla_X Y = X^k \left( \frac{\partial Y^i}{\partial x^k} + H^i_{jk} Y^j \right) \text{ for } Y = Y^i \frac{\partial}{\partial x^i} \text{ and } (X,Y) \rightarrow \nabla_X Y \text{ gives a linear connection on } M \text{ such that } \nabla_X h = 0. \text{ As } \nabla \text{ has no torsion, it coincides with the Levi–Civita connection of } h, \Box
\]

Remark. An alternative way to prove Lemma 3.3 is to prove first that \(\frac{\nabla h}{dt}(\varphi, \psi) = \lim_{t \to 0} \frac{h(P_{\varphi}, P_{\psi}) - h(\varphi, \psi)}{t}\), where \(P_t\) is the parallel translation from \(c(0)\) to \(c(t)\).

Acknowledgements. The author is grateful to Professor Radu Miron for stimulating discussions during the preparation of this work.

References


Author’s address:

Mihai Anastasiei
*Faculty of Mathematics,*
*University “Al.I.Cuza” Iaşi,*
*6600, Iaşi, Romania*