THE SPECTRUM OF THE LAPLACE OPERATOR FOR THE MANIFOLD $SO\left(2p+2q+1\right)/SO\left(2p\right)\times SO\left(2q+1\right)$

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Abstract

The aim of the present paper is to determine the spectrum of the Laplace operator acting on the functions on the symmetric manifold $SO\left(2p+2q+1\right)/SO\left(2p\right)\times SO\left(2q+1\right)$. We also study the existance of isospectral manifolds modeling of the given one.

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1 Introduction

Let (M,g) be a compact Riemannian manifold of dimension n. Let $D^{o}\left(M\right)$ be the algebra of differential functions on M. The Laplace operator Δ 0n $D^{o}\left(M\right)$ gives a spectrum $Sp\left(M\right)$.

One of the problems of the spectrum theory for Δ is to determine $Sp\left(M\right)$, when (M,g) is given Riemannian manifold. Another basic problem is to find under which conditions $Sp\left(M\right)$, determines the geometry on M. Another problem is to determine another Riemannian manifold (N,h) such that $Sp\left(M\right)=Sp\left(N\right)$ but (M,g) is not isometric onto (N,h).

This paper contains four sections. Each of them is analyzed as follows. The first section is the introduction. The second section contains basic elements about the spectrum and the Riemannian geometry on a compact Riemannian manifold. The spectrum of the Laplace operator Δ on the functions on the Grasmann manifold $M=SO\left(2p+2q+1\right)/SO\left(2p\right)\times SO\left(2q+1\right)$ is computed in the third section.

In the fourth section, using the results of the third section and the space forms associated to the manifold $SO\left(2p+2q+1\right)/SO\left(2p\right)\times SO\left(2q+1\right)$, we obtain that

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there are not isospectral manifolds modeling on the given Grasmann manifold which are not isometric.

2 The spectrum of a Riemannian manifold.

Let (M, g) be a compact Riemannian manifold. We denote by $D^o(M)$ the algebra of differentiable functions on M. Let $\Delta = \delta d$ be the Laplace operator acting on $D^o(M)$, that is

$$\Delta = \delta d : D^o(M) \longrightarrow D^o(M),$$

$$\Delta : f \longrightarrow \Delta f = \delta d(f).$$

If we have

$$\Delta f = \lambda f \qquad \lambda \in I\!R,$$

then λ is called eigenvalue of Δ associated to the eigenfunction.

The set of eigenvalues, denoted by $Sp(M, g)$, is called spectrum of Δ on $D^o(M)$, which has the form

$$Sp(M) = Sp(M, g) = \left\{ 0 < \underbrace{\lambda_1 = \lambda_1 = ... = \lambda_1}_{m_1} < \lambda_2 ... < ... < \infty \right\}.$$

The $Sp(M, g)$ is discrete and its eigenvalue has finite multiplicity.

There are many problems which are related to the spectrum of the Riemannian Geometry on M.

We refer some of these problems.

Problem 1 *Let (M, g) be a compact Riemannian manifold. Compute the $Sp(M, g)$.*

Problem 2 *Does $Sp(M, g)$ determine the geometry on (M, g)?*

Problem 3 *Are there two compact Riemannian manifolds (M, g) and (N, h) such that $Sp(M, g) = Sp(N, h)$ but not isometric?*

In the present paper we study the problems I and III.

Let $M = G/H$ be a compact homogeneous manifold where G is a compact semi-simple Lie group and H a closed subgroup of G. Let t, v be the Lie algebras of G and H respectively. We assume that the metric g on M comes from the Killing-Cartan form on t.

It is known that t can be written

$$t = v + s,$$

where v is the eigenspace with eigenvalue 1 and s the eigenspace with eigenvalue -1 and the same time that s can be identified with the tagent space of M at its origin.

Let b be a Cartan subalgebra of t. We denote by Λ^t the system of positive roots of t_C with respect to b_C. Let α be the maximal abelian subspace of s. We denote by Λ_α^+ the system of positive roots belonging in α_C, that is

$$\Lambda_\alpha^+ = \left\{\lambda/\alpha : \lambda \in \Lambda^+, \lambda/\alpha \neq 0\right\}.$$

We denote by κ the subset of v defined by

$$v = \{x \in v \ / \ [x, \alpha] = 0\}$$

and by b_1 the Cartan subalgebra of κ.

The following relation is valid

$$b = b_1 + \alpha.$$

Finally, we construct the set B_α as follows

$$B_\alpha = \left\{\mu \in \alpha^* \ / \ < \mu, \lambda > / < \lambda, \lambda > \succeq 0, \forall \lambda \in \Lambda_\alpha^+\right\}.$$

Now, we state the following theorem

Theorem 4 *Let M=G/H be a compact homogeneous symmetric space, where G a semi-simple compact Lie group and H is a closed subgroup of G. The Riemannian metric g on M is obtained from the Killing-Cartan form on the Lie algebra t of G. The spectrum $Sp(M, g)$ is described by the relation*

$$Sp(M, g) = \left\{\|\mu + \delta\alpha\|_{\kappa - c}^2 - \|\delta_\alpha\|^2 \ / \ \mu \in B_\alpha\right\}, \quad (1)$$

where δ_α is the restriction of δ where δ is the half of the sum of the positive roots, on α. The multiplicity of this eigenvalue is given by

$$d\mu^2 = \prod_{\lambda \in \Lambda^+} \frac{< \mu + \delta_\alpha, \lambda >}{< \delta_\alpha, \lambda >}. \quad (2)$$

3 The spectrum of the Grasmann Manifold.

We consider the Grasmann manifold

$$M = \frac{SO(2p + 2q + 1)}{SO(2p) \times SO(2q + 1)}. \quad (3)$$

Let $so(2p + 2q + 1)$, $so(2p)$ and $so(2q + 1)$ be the Lie algebras of $SO(2p + 2q + 1)$, $SO(2p)$ and $SO(2q + 1)$ respectively . It is known that the following relation holds

$$so(2p + 2q + 1) = so(2p) + so(2q + 1) + m, \quad (4)$$

where m is the tangent space of M at its origin O.

In order to determine the $Sp\,(M,g)$, where M is given by (3), we distinguish the following case

$$(p,q)\begin{cases} (p>q)\begin{cases}(p-q>1)\,,\\ (p-q=1)\,,\end{cases}\\ (p=q)\,,\\ (p<q)\,.\end{cases}$$

We have the above table because the set Λ_α is different in each case.

Therefore we have the following theorem:

Theorem 5 *The spectrum of the Laplace operator Δ on the functions on $SO\,(2p+2q+1)\ /\ SO\,(2p)\times SO\,(2q+1)$ with $p>q$ and $p-q=1$ is given by the relation*

$$\begin{aligned} Sp\,(M,g) &= \left\{ \frac{1}{8q+2}\left[\sum_{i=1}^{2q}\left(\sum_{j=1}^{2q} m_j + \frac{m_{q+p}}{2}\right)^2 + \left(\frac{m_{2q+1}}{2}\right)^2\right] + \right. \\ & \left. 2\frac{1}{8q+2}\left[\sum_{j=1}^{2q}\left(\sum_{j=1}^{2q} m_j + \frac{m_{2q+1}}{2}\right)\left(\frac{4q-2i+3}{2}\right) + \frac{m_{2q+1}}{4}\right]\right\}. \end{aligned} \tag{5}$$

The multiplicity of this eigenvalue is given by

$$\begin{aligned} d\mu^2 &= \prod_{1\preceq i\preceq 2q}\left[\frac{\sum_{j=i}^{2q} m_j + \frac{m_{2q+1}}{2} + \frac{4q-2i+3}{2}}{\left(\frac{4q-2i+3}{2}\right)}\right](m_{2q+1}) \prod_{1\preceq i<j\preceq 2q}\left[\frac{\sum_{k=i}^{j-1} m_k + j - i}{j-i}\right] \\ & \prod_{1\preceq i\preceq 2q}\left[\frac{\sum_{k=i}^{2q} m_k + 2q - i + 1}{2q-i+1}\right] \\ & \prod_{1\preceq i<j\preceq 2q}\left[\frac{\sum_{k=i}^{2q} m_k + \sum_{\lambda=j}^{2q} m_\lambda + m_{2q+1} + 4q - i - j + 3}{4q-i-j+3}\right] \qquad (6) \\ & \prod_{1\preceq i\preceq 2q}\left[\frac{\sum_{k=i}^{2q} m_k + m_{2q+1} + 2q - i + 2}{2q-i+2}\right]. \qquad (7) \end{aligned}$$

If $p>q$ and $q-p>1$, then we have

$$Sp\,(M,g) \;=\; \Bigg\{ \frac{1}{4\,(p+q)-2} \left[\sum_{i=p-q}^{p+q-1} \left(\sum_{j=i}^{p+q-1} m_j + \frac{m_{p+q}}{2} \right)^2 + \left(\frac{m_{p+q}}{2} \right)^2 \right] + 2$$

$$\frac{1}{4\,(p+q)-2} \left[\sum_{i=p-q}^{p+q-1} \left(\sum_{j=i}^{p+q-1} m_j + \frac{m_{p+q}}{2} \right) \left(\frac{2\,(p+q)-2i+1}{2} \right) + \left(\frac{m_{p+q}}{4} \right) \right] \Bigg\} \quad (8)$$

and the multiplicity of this eigenvalue is given by

$$d\mu^2 = \prod_{p-q \preceq i \preceq p+q-1} \left[\frac{\sum\limits_{j=i}^{p+q-1} m_j + \frac{m_{p+q}}{2} + \frac{2(p+q)-2i+1}{2}}{\left(\frac{m_{p+q}+1}{2} \right)} \right] \left(\frac{m_{p+q}+1}{2} \right)$$

$$\prod_{\substack{1 \preceq i \preceq p+q-1 \\ p-q \preceq i \preceq p+q-1}} \left[\frac{\sum\limits_{k=j}^{p+q-1} m_k + \frac{m_{p+q}}{2} + \frac{2(p+q)-2j+1}{2}}{\frac{2(p+q)-2j+1}{2}} \right] (m_{p+q}+1)^{p-q-1}$$

$$\prod_{p-q \preceq i < j \preceq p+q-1} \left[\frac{\sum\limits_{k=i}^{j-1} m_k + j - 1}{j-i} \right] \prod_{p-q \preceq i \preceq p+q-1} \left[\frac{\sum\limits_{k=i}^{p+q-1} m_k + p + q - i}{p+q-i} \right]$$

$$\prod_{\substack{1 \preceq i \preceq p-q-1 \\ p-q \preceq i \preceq p+q-1}} \left[\frac{\sum\limits_{k=j}^{p+q-1} m_k + \frac{m_{p+q}}{2} + \frac{2(p+q)-2j+1}{2}}{\frac{2(p+q)-2j+1}{2}} \right] (m_{p+q}+1)^{p-q-1}$$

$$\prod_{p-q \preceq i < j \preceq p+q-1} \left[\frac{\sum\limits_{k=i}^{p+q-1} m_k + \sum\limits_{\lambda=j}^{p+q-1} m_\lambda + m_{p+q} + (p+q) - i - j + 1}{2\,(p+q) - i - j + 1} \right]$$

$$\prod_{p+q \preceq i \preceq p+q-1} \left[\frac{\sum\limits_{k=i}^{p+q-1} m_k + m_{p+q} + p + q - i + 1}{p+q-i+1} \right]. \quad (9)$$

If $p = q$, then the spectrum $Sp\,(M,g)$ has the form

$$
\begin{aligned}
Sp\,(M,g) \quad = \quad & \left\{ \frac{1}{8p-2} \left[\sum_{i=1}^{2p-1} \left(\sum_{j=i}^{2p-1} m_j + \frac{m_{p+q}}{2} \right)^2 \right] + 2 \right. \\
& \left. \frac{1}{8p-2} \left[\sum_{i=1}^{2p-1} \left(\sum_{j=i}^{2p-1} m_j + \frac{m_{p+q}}{2} \right) \left(\frac{4p-2i+1}{2} \right) \right] \right\}. \qquad (10)
\end{aligned}
$$

The eigenvalue has the multiplicity

$$
\begin{aligned}
d\mu^2 \quad = \quad & \prod_{1 \preceq i \preceq 2p-1} \left[\frac{\sum\limits_{j=i}^{2p-1} m_j + \frac{m_{2p}}{2} + \frac{4p-2i+1}{2}}{\left(\frac{4p-2i+1}{2} \right)} \right] \prod_{1 \preceq i < j \preceq 2p-1} \left[\frac{\sum\limits_{k=i}^{j-1} m_k + j - i}{j-i} \right] \\
& \prod_{1 \preceq i \preceq 2p-1} \left[\frac{\sum\limits_{k=i}^{2p-1} m_k + \frac{4p-2i+1}{2}}{\frac{4p-2i+1}{2}} \right] \\
& \prod_{1 \preceq i < j \preceq 2p-1} \left[\frac{\sum\limits_{k=i}^{2p-1} m_k + \sum\limits_{\lambda=j}^{2p-1} m_\lambda + m_{2p} + 4p - i - j + 1}{4p - i - j + 1} \right] \\
& \prod_{1 \preceq i \preceq 2p-1} \left[\frac{\sum\limits_{k=i}^{2p-1} m_k + \frac{m_{2p}}{2} + \frac{4p-2i+1}{2}}{\frac{4p-2i+1}{2}} \right]. \qquad (11)
\end{aligned}
$$

Finally if $p < q$, then we obtain

$$
\begin{aligned}
Sp\,(M,g) \quad = \quad & \left\{ \frac{1}{4\,(p+q)-2} \left[\sum_{i=1}^{2p-1} \left(\sum_{j=i}^{p+q} m_j \right)^2 \right] + 2 \right. \\
& \left. \frac{1}{4\,(p+q)-2} \left[\sum_{i=1}^{2p-1} \left(\sum_{j=i}^{p+q} m_j \right) \left(\frac{2\,(p+q) - 2i + 1}{2} \right) \right] \right\}. \qquad (12)
\end{aligned}
$$

The multiplicity of the eigenvalue is given by

$$d\mu^2 = \prod_{1\preceq i\preceq 2p-1}\left[\frac{\sum_{j=i}^{p+q} m_j+\frac{2(p+q)-2i+1}{2}}{\frac{2(p+q)-2i+1}{2}}\right]\prod_{1\preceq i<j\preceq 2p-1}\left[\frac{\sum_{k=i}^{j-1} m_k+j-i}{j-i}\right]$$

$$\prod_{\substack{1\preceq i\preceq 2p-1\\ 2p\preceq j\preceq p+q}}\left[\frac{\sum_{k=i}^{p+q} m_k+\frac{2(p+q)-2i+1}{2}}{\frac{2(p+q)-2i+1}{2}}\right]$$

$$\prod_{1\preceq i<j\preceq 2p-1}\left[\frac{\sum_{k=i}^{p+q} m_k+\sum_{\lambda=j}^{p+q} m_\lambda+2\,(p+q)-i-j+1}{2\,(p+q)-i-j+1}\right]$$

$$\prod_{\substack{1\preceq i\preceq 2p-1\\ 2p\preceq j\preceq p+q}}\left[\frac{\sum_{k=i}^{p+q} m_k+\frac{2(p+q)-2i+1}{2}}{\frac{2(p+q)-2i+1}{2}}\right]. \tag{13}$$

Proof. We shall prove the first part of this theorem because the other theorems are proved with the same method. Firstly we construct the weights of the Lie algebra $so\,(2p+2q+1)$, which are given by

$$w_i = e_1+e_2+...+e_i \quad\Longrightarrow\quad w_{i/m}=0 \;\; 1\preceq i\preceq p-q-1, \tag{14}$$

$$w_i = (e_1+...+e_{p-q-1})+(e_{p-q}+...+e_i)\,, \tag{15}$$

which implies

$$w_{i/m}=\sum_{j=p-q}^{i} e_j \quad p-q\preceq i\preceq p+q-1 \tag{16}$$

and

$$w_{p+q}=\frac{1}{2}\left(\sum_{i=1}^{p+q} e_i\right)\Longrightarrow w_{p+q/m}=\frac{1}{2}\left(\sum_{j=p-q}^{p+q} e_j\right) \quad i=p+q. \tag{17}$$

The total weight is given by

$$\mu = m_1 w_{1/m}+...+m_{p+q}w_{p+q/m},\;\; m_1,...,m_{p+q}\in Z^+. \tag{18}$$

The formula (17) by means (13), (15) and (16) takes the form

$$\mu = \sum_{i=p-q}^{p+q-1} \left(\sum_{j=1}^{p+q-1} m_j + m_{p+q/2} \right) e_i + m_{p+q/2} e_{p+q}. \tag{19}$$

We construct the half of the sum of positive roots δ whose restriction on the tangent space has the expression

$$\delta/m = \sum_{i=p-q}^{p+q} \frac{2\,(p+q)+1-i}{2} e_i. \tag{20}$$

The formulas (1) and (2) by means (18), (19) and the form of positive roots of this manifold (3) implies the formulas (5) and (6) respectively. □

4 The problem of isospectrality

Let (M, g) be a compact Riemannian manifold of dimension n. The problem of determination another compact Riemannian manifold which is isospectral with (M, g) but not isometric is a basic in the theory of isospectrality. Another problem is to prove that we can not determine other manifolds from (M, g) with the property of isospectrality.

The classification of Riemannian manifolds which are covered be a given compact simply connected Riemannian is an open problem. This is well known as space form problem.

Now, we consider the Grassmann manifold

$$M = SO\,(2p+2q+1)\,/SO\,(2p) \times SO\,(2q+1) \tag{21}$$

and we study the above problems for this manifold.

The dimension of the manifold M is $2p\,(2q+1)$ which is even. On the other hand $2p+2q+1$ is an odd number.

According the classification of space form for the Grassmann manifold (20) and take and the above remarks we conclude that the manifold which is space form of (20) is the manifold

$$N = SO\,(2p+2q+1)\,/SO\,(2p) \times SO\,(2q+1)\,/\Gamma_2,$$

where Γ is the group isomorphic onto Z_2 which has the form $\{1, w\}$, where w is the change of orientation, isometry of $SO\,(2p+2q+1)\,/SO\,(2p) \times SO\,(2q+1)$.

Therefore using as a model

$$M = SO\,(2p+2q+1)\,/SO\,(2p) \times SO\,(2q+1)\,, \tag{22}$$

this Grassmann manifold and the theory of space forms, we conclude that we can not determine two space forms of (21) which are isospectral manifolds.

References

[1] M. Berger, J. Gaudachon and E. Mazet, *Le spectre d' une variete Riemanienne*, Lecture Notes in Math. 194, Spring-Verlag, Berlin-New York, 1971.

[2] R. Cahn, and J. Wels, *Zeta functions and their asymptotic expansions for compact symmetric spaces of rank 1*, Com. Math. Helv. 51 (1976) 11-21.

[3] Gr. Tsagas, *Generating functions for space forms*, Proceedings of the Intern. Conf. Of Complex Analysis and Applications, Varna (Bulgaria).

[4] Gr. Tsagas, *The spectrum of the Laplacian for a spherical space form*, Bulletin de la Soc. des Sc. et de letters des Lod , Vol. XXXIX, 1, No. 55, 1-10.

[5] Gr. Tsagas and G. Dimou, *The spectrum of the Laplace operator for a generalized Dold manifold*, Jour. Inst. Math. And Comp. Sci., Vol. 4, No. 3 (1991) 355-360.

[6] Gr. Tsagas and G. Dimou, *The spectrum of the Laplace operator for a generalized Dold manifold*, Radov. Matematicki, Vol. 8 (1992) 115-125.

[7] Gr. Tsagas, *Determination of the spectrum for special manifolds*, Proceedings of the International Conference of Differential Geometry in Bucharest 31-7-2000/2-8-2000.

[8] J. Wolf, *Spaces of constant curvatures*, Publish or Perish Inc., Berv. 1977.

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