SOME PROPERTIES AND THE DISTRIBUTIVITY OF THE 
(P, Q)-SUPERLATTICE

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Abstract

We consider a hyperstructure of the form \((L, P, Q, \vee, \wedge)\), where \((L, \vee, \wedge)\) is a lattice and the hyperoperations \(P, Q\) are defined as follows: 
\[ a P b = a \vee b \vee P, \]
\[ a Q b = a \wedge b \wedge Q. \]
If the sets \(P, Q \subseteq L\) satisfy appropriate conditions, then \((L, P, Q, \vee, \wedge)\) is a superlattice. We explore some properties of \((L, \vee, \wedge)\) with special attention to various types of \(P\) and \(Q\) distributivity.

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1 Introduction

A classical algebraic operation maps two elements from a reference set \(L\) to a third element from \(L\). A hyperoperation, on the other hand, maps two elements of \(L\) to a subset of \(L\). Using hyperoperations, one can generalize the classical algebraic structures (e.g. group, ring, lattice) to hyperstructures. For example, a hyperlattice [6] is a structure \((L, \gamma, \wedge)\) (where \(\gamma\) is a hyperoperation and \(\wedge\) is a classical operation) which generalizes a classical lattice. A superlattice [8] is a structure \((L, \gamma, \wedge)\) (where \(\gamma, \wedge\) are hyperoperations) which generalizes both the classical lattice and the hyperlattice. In this paper we establish some of the properties of the \((P, Q)\)-superlattice, and examine various types of \(P\) and \(Q\) distributivity.
2 Definition of the (P,Q)-Superlattice

Let us first give the definition of a general superlattice. In fact, we will give two equivalent definitions. In what follows $P(L)$ will denote the power set of a reference set $L$.

**Definition 1** A superlattice is a partially ordered set $(L, \leq)$ with two hyperoperations $\gamma, \lambda$, where $\gamma : L \times L \to P(L)$, $\lambda : L \times L \to P(L)$, and the following properties are satisfied for all $a, b, c \in S$.

$S1 \ a \in (a \gamma a) \cap (a \lambda a),$

$S2 \ a \gamma b = b \gamma a, a \lambda b = b \lambda a,$

$S3 \ (a \gamma b) \gamma c = a \gamma (b \gamma c), (a \lambda b) \lambda c = a \lambda (b \lambda c),$

$S4 \ a \in [(a \gamma b) \lambda a] \cap [(a \lambda b) \gamma a],$

$S5a \ a \leq b \Rightarrow (b \in a \gamma b \text{ and } a \in a \lambda b),$

$S5b \ (b \in a \gamma b \text{ or } a \in a \lambda b) \Rightarrow a \leq b.$

The following definition is equivalent to Definition 1, as has been shown in [8].

**Definition 2** A superlattice is a partially ordered set $(L, \leq)$ with two hyperoperations $\gamma, \lambda$, where $\gamma : L \times L \to P(L)$, $\lambda : L \times L \to P(L)$, and the following properties are satisfied for all $a, b, c \in S$.

$S1 \ a \in (a \gamma a) \cap (a \lambda a),$

$S2 \ a \gamma b = b \gamma a, a \lambda b = b \lambda a,$

$S3 \ (a \gamma b) \gamma c = a \gamma (b \gamma c), (a \lambda b) \lambda c = a \lambda (b \lambda c),$

$S4 \ a \in [(a \gamma b) \lambda a] \cap [(a \lambda b) \gamma a],$

$S6 \ b \in a \gamma b \Rightarrow a \in a \lambda b,$

$S7 \ a, b \in a \gamma b \Rightarrow a = b,$

$S8 \ b \in a \gamma b \text{ et } c \in b \gamma c \Rightarrow c \in a \gamma c.$

Next let us define the $(P,Q)$-superlattice, which has been introduced in [7]. A $(P,Q)$-superlattice is a special kind of superlattice, which can be considered as a generalization of either the $P$-hyperlattice [4, 5] or the $Q$-d-hyperlattice\(^1\). $(P,Q)$-superlattices are constructed on a lattice $(L, \lor, \land)$ in a manner analogous to the construction of $P$-hypergroups [3, 9, 11] and $P$-hyperrings [10].

\(^1\)A $Q$-d-hyperlattice can be defined analogously to a $P$-hyperlattice but utilizes an operation $\lor$ and a hyperoperation $\lambda = Q$. 

In what follows, \((L, \vee, \wedge)\) will always denote a lattice (with \(L \neq \emptyset\)) and \(\leq\) will denote the order of \((L, \vee, \wedge)\). \(\mathbf{J}(L)\) will denote the set of ideals of \(L\), and \(\mathbf{F}(L)\) will denote the set of filters of \(L\). If \(L\) possesses a minimum (respectively maximum) element of \(L\), this will be denoted by \(0\) (respectively \(1\)).

Let us select two sets \(P, Q \in \mathbf{P}(L)\) and define the following hyperoperations.

**Definition 3** For all \(a, b \in L\) we define \(a \uplus b \triangleq a \vee b \vee P = \{a \vee b \vee p : p \in P\}\).

**Definition 4** For all \(a, b \in L\) we define \(a \ominus b \triangleq a \wedge b \wedge Q = \{a \wedge b \wedge q : q \in Q\}\).

**Remark.** In [4] we have shown that if \(P = L\), then \(a \uplus b = \{x \in L : a \vee b \leq x\}\). It can be shown similarly that, if \(Q = L\), then \(a \ominus b = \{y \in L : y \leq a \wedge b\}\).

**Remark.** It is easy to see that for \(P, Q, P_1, Q_1 \in \mathbf{P}(L)\) such that \(P \subseteq P_1\) and \(Q \subseteq Q_1\), we have for all \(a, b \in L\) that \(a \uplus b \subseteq a \uplus P_1\) and \(a \ominus b \subseteq a \ominus Q_1\).

An \((L, \vee, \wedge)\) structure (with arbitrary choice of \(P, Q\)) is not necessary a superlattice. Consider the following example.

**Example 5** Consider the lattice \(L\) of Figure 1.

(i) If we take \(P = \{0, a\}, Q = \{b, 1\}\), then we see that \((L, \uplus, \ominus)\) satisfies the properties of Definition 1, i.e. it is a superlattice.

(ii) If we take \(P = \{c, d\}, Q = \{c, d\}\), then we see that \((L, \uplus, \ominus)\) does not satisfy the properties of Definition 1, i.e. it is not a superlattice. For example \(a \notin a \uplus a = a \vee a \vee \{c, d\} = \{a \vee c, a \vee d\} = \{b, 1\}\). Similarly, \(a \notin a \ominus a = a \wedge a \wedge \{c, d\} = \{a \wedge c, a \wedge d\} = \{0\}\).

The necessary and sufficient conditions on \(P, Q\) for \((L, \vee, \wedge)\) to be a superlattice are easily stated in terms of the following two collections of sets.

**Definition 6** \(A(L) \triangleq \{A \in \mathbf{P}(L) : \forall x \in L \ \exists a \in A \text{ such that } a \leq x\}\).

**Definition 7** \(B(L) = \{B \in \mathbf{P}(L) : \forall y \in L \ \exists b \in B \text{ such that } y \leq b\}\).
It is clear that $L \subseteq A_L \cap B_L$. Also if $L$ has a 0 and a 1, then $(P, Q) \in A(L) \times B(L) \Leftrightarrow (0, 1) \in P \times Q$. Furthermore, the following proposition yields the necessary and sufficient condition for $(L, \vee, \wedge)$ to be a superlattice.

**Proposition 8** $(L, \vee, \wedge)$ is a superlattice if and only if $(P, Q) \in A(L) \times B(L)$.

*Proof.* The proof appears in [7]. □

### 3 Properties of the $(P, Q)$-superlattice

In what follows we will assume that $(P, Q) \subseteq A(L)$ (unless explicitly stated otherwise). Hence $(L, \vee, \wedge)$ will be a superlattice.

**Definition 9** A superlattice $(L, \vee, \wedge)$ will be called proper if there exist pairs $(a, b)$, $(c, d) \in L \times L$, such that

\[ \text{card}(a^P b) \geq 2 \quad \text{and} \quad \text{card}(c^Q d) \geq 2 . \]

**Proposition 10** (i) If $L$ possesses a maximum element 1 and we set $P = A(L)$, $Q = \{1\}$, then $(L, \vee, \wedge)$ is a $P$-hyperlattice.

(ii) If $L$ possesses a minimum element 0 and we set $P = \emptyset$, $Q = B(L)$, then $(L, \vee, \wedge)$ is a $P$-hyperlattice.

(iii) If $L$ possesses a minimum element 0 and a maximum element 1, and we set $P = \emptyset$ and $Q = \{1\}$, then $(L, \vee, \wedge)$ is the lattice $(L, \_ \_ \_ \wedge \_ \_ \_)$.

*Proof.* (i) If $Q = \{1\}$, then for all $a, b \in L$ we have $a \wedge b \subseteq a \wedge b \subseteq a \wedge b$.

(ii) This is proved similarly to (i).

(iii) This is proved by combining (i) and (ii). □

**Proposition 11** For all $(P, Q) \subseteq A(L)$ and all $a, b \in L$ we have: (i) $a \vee b = \min(a^P b)$, (ii) $a \wedge b = \max(a^Q b)$.

*Proof.* (i) Since $P \subseteq A(L)$ there will exist a $p \in P$ such that $p \leq a \vee b$. Hence $a \vee b = a \vee b \vee p \in a \vee b$. Clearly, for all $x \in a \vee b$ we have $a \vee b \leq x$, so we have proved $a \vee b = \min(a^P b)$.

(ii) This is proved dually to (i). □

**Remark.** It follows that for all $(P, Q) \subseteq A(L)$ we have that $(L, \vee, \wedge)$ is a strictly strong superlattice [8].
Remark. A \((P, Q)\)-superlattice, by its construction, preserves \(\leq\), the original order of \(L\). From S5a of Definition 1 follows that the \(\leq\) order can be expressed in terms of the \(\lor, \land\) hyperoperations as follows:
\[
a \leq b \Rightarrow (b \in a \lor b \text{ and } a \in a \land b),
\]
\[
(b \in a \lor b \text{ or } a \in a \land b) \Rightarrow a \leq b.
\]

**Proposition 12** (i) If \(L\) has minimum element \(0\) and maximum element \(1\), then
\[
\text{(card}(P) \geq 2 \text{ and card}(Q) \geq 2) \iff (L, \lor, \land) \text{ is a proper superlattice}.
\]

(ii) If \(L\) does not have either minimum or maximum element, then for all \((P, Q) \in A(L) \times B(L)\) we have that \((L, P, Q, \lor, \land)\) is a proper superlattice.

**Proof.** (i) We have \(P \lor 0 = 0 \lor 0 \lor P = P\); hence \(\text{card}(0 \lor 0) = \text{card}(P) \geq 2\).
Similarly, \(1 \land 1 = 1 \land 1 \land Q = Q\); hence \(\text{card}(1 \land 1) = \text{card}(Q) \geq 2\). Hence \((L, \lor, \land)\) is a proper superlattice.

(ii) Assume that for some \((P, Q) \in A(L) \times B(L)\) the corresponding \((L, P, Q, \lor, \land)\) is not a proper superlattice. This means that for every \(a \in L\) we will have \(a \lor a = a\), or \(a \land a = a\), or both. But \(a = a \lor a = a \lor P\) and so we conclude that for every \(p \in P\) and for every \(a \in L\) we have \(p \leq a\). In particular, for any two \(p, p_1 \in P \subseteq L\) we will have \(p = p \lor p_1 = p_1\). It follows that \(P = \{p\}\) and that (since \(P \in A(L)\)) \(p\) is the minimum element of \(L\). But this is in contradiction to the assumption. Dually, if \(a \land a\) we conclude that \(L\) has a maximum element, which again leads to a contradiction.

A special subset of \(A(L) \times B(L)\) is \(J(L) \times F(L)\), i.e. the Cartesian product of ideals and filters of \(L\). In part (i) of the following proposition we do not initially assume \((P, Q) \in A(L) \times B(L)\). However, (i) states that \(J(L) \times F(L) \subseteq A(L) \times B(L)\), hence \((P, Q) \in J(L) \times F(L)\) implies that \((L, \lor, \land)\) is a superlattice. Parts (ii) and (iii) of the proposition use stronger assumptions to reach more specialized conclusions.

**Proposition 13** (i) \((P, Q) \in J(L) \times F(L)\Rightarrow (P, Q) \in A(L) \times B(L)\).

(ii) \((P \in A(L) \cap F(L)) \Rightarrow P = L\).

(iii) \((Q \in B(L) \cap J(L)) \Rightarrow Q = L\).

**Proof.** (i) In [4] we have shown that \(P \in J(L)\) implies that \(P \in A(L)\). To prove that \(Q \in B(L)\) we proceed dually. Namely, if \(Q \in F(L)\), then \(Q \lor L \subseteq Q\) [1]. Now take any \(a \in L\) and any \(q \in Q\). Then \((a, q) \in L \times Q\) and exists a \(q_1 \in Q\) such that \(a \lor q = q_1\), i.e. \(a \leq q_1\). Hence \(Q \in B(L)\).

(ii) Now assume that \(P \in A(L) \cap F(L)\). Since \(P \in A(L)\) for every \(a \in L\) there exists \(p \in P\) such that \(p \leq a\), i.e. \(a \lor p = a \lor P\lor L\); but, since \(P \in F(L)\) we also have
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Since obviously \( P \subseteq L \), we have \( P = L \). The converse is obvious. (iii) This is proved dually to (ii).

The converse of (i) in the above proposition does not hold, as we can see in the next example.

Example 14 If \( L \) is the lattice of Figure 2, and \( P = Q = f_0, \ldots, g \), then \((P, Q) \not\in J(L) \times F(L)\).

Proposition 15 For all \((P, Q) \in A(L) \times B(L)\) we have:

(i) \( Q \subseteq P \lor Q, P \subseteq P \land Q \).

(ii) If for every \((p, q) \in P \times Q\) we have \( p \leq q \), then \( Q = P \lor Q, P = P \land Q \).

(iii) If \( L \) is distributive and \( P, Q \) are intervals, then \( Q = P \lor Q, P = P \land Q \).

Proof. (i) Take any \( q \in Q \). Since \( P \in A(L) \), it follows there will exist \( p \in P \) such that \( p \leq q \). Hence \( q = p \lor q \in P \lor Q \). Hence \( Q \subseteq P \lor Q \). It is proved dually that \( P \subseteq P \land Q \).

(ii) This is easy to prove.

(iii) Now assume that \( P = [a, b], Q = [c, d] \). Then there will exist \( q \in [c, d] \) such that \( b \leq q \leq d \) and \( p \in [a, b] \) such that \( a \leq p \leq c \). Hence, since \( L \) is distributive, we will have: \( P \lor Q = [a, b] \lor [c, d] = [a \lor c, b \lor d] = [c, d] = Q \) [2]. Dually we can prove that \( P \land Q = P \).

We now introduce a relation \( \preceq \) between elements of \( P(L) \); \( \preceq \) is an order relation (see for instance [2]).

Definition 16 Take any \( A, B \in P(L) \); we write \( A \preceq B \) iff

\[
\begin{align*}
(i) &\quad \forall a \in A \; \exists b_1 \in B : a \leq b_1, \\
(ii) &\quad \forall b \in B \; \exists a_1 \in A : a_1 \leq b.
\end{align*}
\]

Proposition 17 If \( a, b \in L \) and \( a \leq b \), then for all \( c \in L \) we have:

(i) \( a \lor c \leq b \lor c \),

(ii) \( a \land c \leq b \land c \).
Proof. (i) If $x \in a^P \lor c$ then exists some $p_1 \in P$ such that $x = a \lor c \lor p_1 \leq b \lor c \lor p_1 = y \in b^P \lor c$. Similarly, if $z \in b^P \lor c$ then exists some $p_2 \in P$ such that $z = b \lor c \lor p_2 \geq b \lor c \lor p_2 = w \in a^P \lor c$. Hence $a^P \lor c \leq b^P \lor c$.

(ii) It is proved dually. $\Box$

Remark. The relationship $\triangleleft$ defined above, generally is not an order relationship on $P(L)$. It is an order relationship on $I(L)$, the class of intervals of $L$.2.

Remark. If $a, b \in L$ are such that $a \leq b$ and $c \in L$, then it is possible to have $(a \lor c) \cap (b \lor c) = \emptyset$ and $(a \land c) \cap (b \land c) = \emptyset$. An example of this follows.

Example 18 Consider the lattice $L$ of Figure 3 and take $P = \{0, a\} \in A_L$ and $Q = \{b, 1\} \in B_L$.

Figure 3

We observe that $a \leq 1$ and $a^P \lor a = a \lor P = a$ and $1^P \lor a = 1 \lor P = 1$, i.e. $(a^P \lor a) \cap (1^P \lor a) = \emptyset$. Similarly, $0 \leq b$ and $b \land Q = b$ and $0 \land Q = 0$, i.e. $(b \land Q) \cap (0 \land b) = \emptyset$.

Clearly for $a, b \in L$ the form of $a^P \lor b$, $a \land b$ depends on the form of $P, Q$. The following propositions examine some cases of this connection.

Proposition 19 If $(L, \triangleleft)$ is a distributive lattice, then for all $a, b \in L$ we have

(i) $P$ is an interval $\Rightarrow a^P \lor b$ is an interval.

(ii) $Q$ is an interval $\Rightarrow a^Q \land b$ is an interval.

Proof. (i) Assume $P = [x, y]$, then (using $[2]$) we have $a^P \lor b = a \lor b \lor [x, y] = [a \lor b \lor x, a \lor b \lor y]$. But, since $\exists p \in P = [x, y]$ such that $p \leq a \lor b$, we will have $a^P \lor b = [a \lor b, a \lor b \lor y]$.

(ii) Assume $Q = [z, w]$, then (using $[2]$) we have $a^Q \land b = a \land b \land [z, w] = [a \land b \land z, a \land b \land w]$ and, similarly to (i), we get $a^Q \land b = [a \land b \land z, a \land b]$. $\Box$
Proposition 20 If \((L, \leq)\) is a distributive lattice and \(a, b \in L\) are such that \(a \leq b\), then we have

(i) \(P\) is an interval \(\Rightarrow (a \lor c) \lor (b \lor c) = b \lor c\).

(ii) \(Q\) is an interval \(\Rightarrow (a \land c) \land (b \land c) = a \land c\).

Proof. (i) Assume \(P = [x, y]\), then \(a \lor c = [a \lor c \lor x, a \lor c \lor y]\) and \(b \lor c = [b \lor c \lor x, b \lor c \lor y]\). Since \(L\) is distributive, we will have

\((a \lor c) \lor (b \lor c) = [a \lor c \lor x, a \lor c \lor y] \lor [b \lor c \lor x, b \lor c \lor y] = [a \lor b \lor c \lor x, a \lor b \lor c \lor y] = [b \lor c \lor x, b \lor c \lor y] = b \lor c\).

(ii) It is proved dually. \(\Box\)

Proposition 21 If \((L, \leq)\) is a distributive lattice, then:

(i) \((P\) is a sublattice \(\Rightarrow (\forall a, b \in L\) \(a \lor b\) is a sublattice).

(ii) \((P\) is a sublattice \(\Rightarrow (\forall a, b \in L\) \(a \land b\) is a sublattice).

Proof. (i) Assume that \(P\) is a sublattice of \(L\). Take any \(a, b \in L\). For any \(x_1, x_2 \in a \lor b\) there exist \(p_1, p_2 \in P\) such that \(x_1 = a \lor b \lor p_1\) and \(x_2 = a \lor b \lor p_2\).

Furthermore, \(p_1 \lor p_2 = p_3 \in P\), \(p_1 \land p_2 = p_4 \in P\). Hence \(x_1 \lor x_2 = a \lor b \lor p_3 \in a \lor b\) and \(x_1 \land x_2 = (a \lor b \lor p_1) \land (a \lor b \lor p_2) = (a \lor b) \lor (p_1 \land p_2) = (a \lor b) \lor p_4 \in a \lor b\).

(ii) Is proved dually to (i). \(\Box\)

Proposition 22 (i) If \((L, \leq)\) is a distributive lattice and has a minimum element 0, then we have:

\(0 \lor 0\) is a sublattice \(\Leftrightarrow P\) is a sublattice.

(ii) If \((L, \leq)\) is a distributive lattice and has a maximum element 1, then we have:

\(1 \land 1\) is a sublattice \(\Leftrightarrow Q\) is a sublattice.

Proof. (i) This is obvious, since \(0 \lor 0 = 0 \lor P = P\).

(ii) It is proved dually to (i). \(\Box\)

Corollary 23 (i) If \(L\) has a minimum element 0, then we have:

\(\left(\forall a, b \in L\ a \lor b\text{ is a sublattice}\right) \Leftrightarrow (P\text{ is a sublattice})\).

(ii) If \(L\) has a maximum element 1, then we have:

\(\left(\forall a, b \in L\ a \land b\text{ is a sublattice}\right) \Leftrightarrow (Q\text{ is a sublattice})\).
Proof. Follows immediately from Propositions 21 and 22. □

**Proposition 24** For all \( a, b \in L \) we have

(i) \( P \) is sublattice \( \Rightarrow \) \((a \lor b) \lor (a \land b) = a \lor b \subseteq (a \lor b) \land (a \land b)\).

(ii) \( Q \) is sublattice \( \Rightarrow \) \((a \land b) \land (a \lor b) = a \land b \subseteq (a \land b) \lor (a \lor b)\).

Proof. (i) Take any \( x_1, x_2 \in a \lor b \); then there exist \( p_1, p_2 \in P \) such that \( x_1 = a \lor b \lor p_1 \)
and \( x_2 = a \lor b \lor p_2 \). We have \( x_1 \lor x_2 = \{(a \lor b \lor p_1) \lor (a \lor b \lor p_2) \lor p \ : \ p \in P\} \)
\( = \{a \lor b \lor (p_1 \lor p_2 \lor p) : p \in P\} \); and since \( P \) is a sublattice we have \( x_1 \lor x_2 = \{a \lor b \lor (p_1 \lor p_2 \lor p) : p \in P\} \subseteq a \lor b \), which finally implies
\[
(a \lor b) \lor (a \lor b) \subseteq a \lor b. \tag{1}
\]
Now take any \( x \in a \lor b \). Then there exist \( p, p_1 \in P \) such that \( x = a \lor b \lor p \) and \( p_1 \leq a \lor b \lor p \). Hence \( x = a \lor b \lor p = (a \lor b \lor p) \lor (a \lor b \lor p) \lor p_1 \in (a \lor b \lor p) \lor (a \lor b \lor p) \lor P = (a \lor b \lor p) \lor (a \lor b \lor p) \subseteq (a \lor b) \lor (a \lor b) \) which implies
\[
a \lor b \subseteq (a \lor b) \lor (a \lor b). \quad \tag{2}
\]
From (1) and (2) we have that \((a \lor b) \lor (a \lor b) = a \lor b\).

Furthermore, if \( x \in a \lor b \), then there exist \( p \in P \) and \( q \in Q \) such that \( x = a \lor b \lor p \leq q \). Hence \( a \lor b \lor p = (a \lor b \lor p) \land (a \lor b \lor p) \land q \in (a \lor b \lor p) \land (a \lor b \lor p) \land Q = (a \lor b \lor p) \land (a \lor b \lor p) \subseteq (a \lor b) \land (a \lor b) \).

(ii) This is proved similarly to (ii). □

The inclusion in the above proposition can be proper, as can be seen in the following example

**Example 25** Consider the distributive lattice of Figure 4. Here we have \((d \lor e) \land (d \lor e)\)
\( = \{e, b, k, l, 1\} \supset \{k, l, 1\} = d \lor e \in \mathbb{Z}\).

**Example 26**
Proposition 27 \((L, \lor, \land)\) is distributive iff
\[
\begin{align*}
\forall a, x, y \in L : \quad & a \lor x = a \lor y \\
& a \land x = a \land y
\end{align*}
\Rightarrow x = y
\]

Proof. (i) Let \(L\) be distributive. Pick any \(a, x, y\) such that \(a \lor x = a \lor y\) and \(a \land x = a \land y\). Then there exist \(p_1, p_2 \in P\) and \(q_1, q_2 \in Q\) such that
\[
\begin{align*}
a \lor x &= a \lor y \lor p_1, & a \lor y &= a \lor x \lor p_2, \\
a \land x &= a \land y \land q_1, & a \land y &= a \land x \land q_2.
\end{align*}
\]
From (3) we obtain \(a \lor y \leq a \lor x\) and \(a \lor x \leq a \lor y\), i.e. \(a \lor x = a \lor y\); similarly, from (4) we obtain \(a \land x = a \land y\). From these and distributivity we obtain \(x = y\).

(ii) On the other hand, assume that for all \(a, x, y \in L\) the implication
\[
(a \lor x = a \lor y \text{ and } a \land x = a \land y) \Rightarrow x = y
\]
is true. Now choose any \(a, x, y \in L\) such that \(a \lor x = a \lor y\) and \(a \land x = a \land y\); then we will also have \(a \lor x = a \lor y\) and \(a \land x = a \land y\) and so from (5) follows that \(x = y\).

In short, for all \(a, x, y \in L\) we have:
\[
(a \lor x = a \lor y \text{ and } a \land x = a \land y) \Rightarrow x = y;
\]
but, as is well known (6) is a necessary and sufficient condition for distributivity of \(L\).

\[
\square
\]

4 Distributivity

In this section we examine the distributivity of the \(\lor, \land\) hyperoperations. Since the outcome of each of these hyperoperations is generally a set, several forms of distributivity can be introduced. Let us first introduce the following definitions of distributivity for a general superlattice.

Definition 28 A superlattice \((L, \gamma, \lambda)\) is called weakly \(\lambda\)-distributive (also denoted by w-\(\lambda\)-d) iff
\[
(a \land (b \lor c)) \cap ((a \land b) \lor (a \land c)) \neq \emptyset;
\]
it is called weakly \(\gamma\)-distributive (also denoted by w-\(\gamma\)-d) iff
\[
(a \lor (b \land c)) \cap ((a \lor b) \land (a \lor c)) \neq \emptyset.
\]
Definition 29 A superlattice $(L, \lor, \land)$ is called feebly $\land$-distributive (also denoted by f-$\land$-d) iff
\[
(a \land (b \lor c)) \subseteq ((a \land b) \lor (a \land c));
\]
it is called feebly $\lor$-distributive (also denoted by f-$\lor$-d) iff
\[
(a \lor (b \land c)) \subseteq ((a \lor b) \land (a \lor c)).
\]

Definition 30 A superlattice $(L, \lor, \land)$ is called $\land$-distributive (also denoted by $\land$-d) iff
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(a \land (b \lor c)) = ((a \land b) \lor (a \land c));
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it is called $\lor$-distributive (also denoted by $\lor$-d) iff
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(a \lor (b \land c)) = ((a \lor b) \land (a \lor c)).
\]

In the following, consider a distributive lattice $(L, \lor, \land)$ and choose sets $P, Q \subseteq P(L)$ such that $(L, P, Q, \land)$ is a superlattice. The distributivity of $(L, \lor, \land)$ is connected to the various forms of distributivity of $(L, P, Q, \land)$, as will be seen by the following propositions.

Proposition 31 For all $(P, Q) \in A(L) \times B(L)$, $(L, P, Q, \land)$ is both $w-P$-d and $w-Q$-d.

Proof. Take any $a, b, c \in L$, then there exist $p \in P$ satisfying $p \leq b$, and $q \in Q$ satisfying $a \leq q$. Hence we will have that $a \land (b \lor c) = a \land (b \lor c \lor p) = a \land (b \lor c \lor Q) \land Q = (a \land (b \lor c \lor P)) = a \land (b \lor c \lor Q)$. On the other hand, $(a \land b) \lor (a \land c) \subseteq (a \land b) \lor (a \land c) \subseteq (a \land b) \lor (a \land c)$. Since $L$ is distributive, we will have $a \land (b \lor c) = (a \land b) \lor (a \land c)$. In short, we have shown that
\[
a \land (b \lor c) \subseteq (a \land (b \lor c)) \cap ((a \land b) \lor (a \land c))
\]
and so we have shown that the $(P, Q)$-superlattice is $w-Q$-d. It can be shown dually that it is also $w-P$-d. $\square$

Proposition 32 For all $(P, Q) \in A(L) \times B(L)$ we have:

(i) if $L \land P \land Q \subseteq P$, then $(L, P, Q, \land)$ is f-$Q$-d;

(ii) if $L$ has a minimum element $\theta$ and $(L, P, Q, \land)$ is f-$\land$-d, then $L \land P \land Q \subseteq P$;

(iii) if $L \lor P \lor Q \subseteq Q$, then $(L, P, Q, \land)$ is f-$P$-d;

(iv) if $L$ has a maximum element $1$ and $(L, P, Q, \land)$ is f-$\lor$-d, then $L \lor P \lor Q \subseteq Q$.

Proof. (i) If $a, b, c \in L$ and $x \in a \land (b \lor c)$, then there exists a pair $(p, q) \in P \times Q$ such that
\[
x = a \land (b \lor c \lor p) \land q = (a \land b \land q) \lor (a \land c \land q) \lor (a \land p \land q) \subseteq
\]
(a \land b \land q) \lor (a \land c \land q) \lor P = (a \land b \land q) \lor (a \land c \land q) \subseteq (a \lor b) \lor (a \lor c).

(ii) Now assume that $L$ has a minimum element 0 and $(L, \lor, \land)$ is f-$\land$-d. Then, for every $x \in L$ we have

$$x \land (0 \lor P) = \lor_{p \in P} x \land p = x \land P \land Q.$$ 

On the other hand,

$$(x \land 0) \lor (x \land 0) = (x \land 0 \land Q) \lor (x \land 0 \lor Q) = 0 \lor 0 = P.$$ (7)

From (7) and the fact that $x \land P \land Q = x \land (0 \lor P) \subseteq (x \land 0) \lor (x \land 0)$, it follows that $x \land P \land Q \subseteq P$; since this is true for any $x \in L$, we conclude that $L \land P \land Q \subseteq P$.

(iii) and (iv) are proved dually. $\square$

**Corollary 33** Let $P \in J(L)$. Then we have the following.

(i) If for all $(p, q) \in P \times Q$ we have $p \leq q$, then $(L, \lor, \land)$ is f-$\land$-d.

(ii) If $P, Q$ are intervals, then $(L, \lor, \land)$ is f-$\land$-d. Similarly, let $Q \in F(L)$. Then we have the following.

(iii) If for all $(p, q) \in P \times Q$ we have $p \leq q$, then $(L, \lor, \land)$ is f-$\land$-d.

(iv) If $P, Q$ are intervals, then $(L, \lor, \land)$ is f-$\land$-d.

**Proof.** (i) Since for every pair $(p, q) \in P \times Q$ we have $p \leq q$, it follows that $P \land Q = P$. Since $P$ is an ideal of $L$, it follows that $L \land P \land Q = L \land P \subseteq P$. Now we can apply Proposition 32 to obtain the desired conclusion.

(ii) Since $P, Q$ are intervals, then from Proposition 15 we have $P \land Q = P$ and we can apply part (i) of this proposition.

(iii) and (iv) are proved dually. $\square$

**Corollary 34** The $(L, \lor, \land)$ is f-$\land$-d and f-$\land$-d.

**Proof.** Taking $P = L, Q = L$ and applying parts (i) and (iii) of Proposition 32, we immediately obtain the desired conclusion. $\square$

The next proposition concerns $\land$-distributivity.

**Proposition 35** (i) If the lattice $L$ does not have a minimum element, there exist no $(P, Q)$ pair such that $(L, \lor, \land)$ is a proper $\land$-distributive superlattice.

(ii) If the lattice $L$ has a minimum element 0, then for every $(P, Q)$ pair such that $(L, \lor, \land)$ is a $\land$-distributive superlattice, we have $P = \{0\}$ (i.e. $(L, \lor, \land)$ is a $Q$-$d$-hyperlattice).
Proof. (i) Assume that there is a pair \((P, Q) \in A(L) \times B(L)\), such that \((L, \lor, \land)^Q\) is a proper superlattice which is also \(\land\)-distributive. I.e. for all \(a, b, c \in L\) we have
\[
a \land (b \lor c) = (a \land b) \lor (a \land c).
\]
But we also have
\[
a \land (b \lor c) = a \land (b \lor c) \lor (a \land Q) \land P
\]
and
\[
(a \land b) \lor (a \land c) = (a \land b) \lor (a \land c) \lor (a \land Q) \lor P.
\]
From (8-10) follows that
\[
a \land (b \lor c \lor P) \lor (a \land Q) \lor P \Rightarrow a \lor (a \land b \land Q) \lor (a \land c \land Q) \lor P.
\]
Now
\[
a \lor (a \land (b \lor c \lor P) \lor (a \land Q) \lor P) = \bigcup_{x \in a \land (b \lor c \lor Q) \lor P} x = \{a\}.
\]
Similarly,
\[
a \lor (a \land (b \lor c \land Q) \lor (a \land Q) \lor P) = \bigcup_{x \in (a \land b \land Q) \lor (a \land c \land Q) \lor P} x = a \lor P.
\]
And from (11), (12) and (13) it follows that \(a = a \lor P\). Using \(a = a \lor P\) and referring to the proof of part (ii), Proposition 12, we conclude that \(P = \{p\}\), which means that \(p\) will be the minimum element of \(L\); but this is contrary to the hypothesis and we have reached a contradiction.

(ii) Now assume that \(L\) has a minimum element \(0\) and take any \((P, Q)\) pair such that \((L, \lor, \land)^Q\) is a proper superlattice which is also \(\land\)-distributive superlattice. Duplicating the argument of part (i), we conclude that \(P = \{0\}\). \(\Box\)

Remark. If \(L\) has a maximum element \(1\), and \(Q = \{1\}\) and \(\text{card}(P) \geq 2\), then \(a \land b = a \land b\) and \((L, \lor, \land)^Q\) is a proper \(P\)-hyperlattice. But in [4] we have shown that a proper \(P\)-hyperlattice cannot be \(\land\)-d. If, on the other hand, \(L\) has a minimum element \(0\), and \(P = \{0\}\) and \(\text{card}(Q) \geq 2\), then we have a proper \(Q\)-\(d\)-hyperlattice, which may also be \(\land\)-d, as will be seen from the following proposition.

Proposition 36 \(L\) is distributive iff \((L, \lor, \land)^L\) is a \(\land\)-distributive \(L\)-\(d\)-hyperlattice.

Proof. The proof is analogous to Corollary 2.3 of [4]. \(\Box\)

Proposition 37 (i) If the lattice \(L\) does not have a maximum element, there exist no \((P, Q)\) pair such that \((L, \lor, \land)^P\) is a proper \(\lor\)-distributive superlattice.

(ii) If the lattice \(L\) has a maximum element \(1\), then for every \((P, Q)\) pair such that \((L, \lor, \land)^P\) is a \(\lor\)-distributive superlattice, we have \(Q = \{1\}\) (i.e. \((L, \lor, \land)^P\) is a \(P\)-hyperlattice.)
Proof. The proof of this proposition is dual to that of Proposition 35. □

Remark. If \( L \) has a minimum element 0, and \( P = \{0\} \) and \( \text{card}(Q) \geq 2 \), then \( a \uparrow b = a \lor b \) and \( (L, \lor, \land) \) is a proper \( Q \)-d-hyperlattice. A proper \( Q \)-d-hyperlattice cannot be \( \lor \)-d (the proof is dual to the proof concerning \( P \)-hyperlattices [4]). If \( L \) has a maximum element 1, and \( Q = \{1\} \) and \( \text{card}(P) \geq 2 \), then the \( (P, Q) \)-superlattice is a proper \( P \)-hyperlattice, which may be \( \lor \)-d.

References


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