SOME PROPERTIES AND THE DISTRIBUTIVITY OF THE (P, Q)-SUPERLATTICE

K. Serafimidis and M. Konstantinidou

Abstract

We consider a hyperstructure of the form $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$, where $(L, \vee, \wedge)$ is a lattice and the hyperoperations $\overset{P}{\vee}, \overset{Q}{\wedge}$ are defined as follows: $a \overset{P}{\vee} b = a \vee b \vee P$, $a \overset{Q}{\wedge} b = a \wedge b \wedge Q$. If the sets $P, Q \subseteq L$ satisfy appropriate conditions, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a superlattice. We explore some properties of $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ with special attention to various types of $\overset{P}{\vee}$ and $\overset{Q}{\wedge}$ distributivity.

AMS Subject Classification: 06B99, 06D99.
Key words: Hyperstructure, Superlattice, Distributivity

1 Introduction

A classical algebraic operation maps two elements from a reference set L to a third element from L. A *hyperoperation*, on the other hand, maps two elements of L to a *subset* of L. Using hyperoperations, one can generalize the classical algebraic structures (e.g. group, ring, lattice) to *hyperstructures*. For example, a hyperlattice [6] is a structure $(L, \curlyvee, \wedge)$ (where $\curlyvee$ is a hyperoperation and $\wedge$ is a classical operation) which generalizes a classical lattice. A *superlattice* [8] is a structure $(L, \curlyvee, \curlywedge)$ (where $\curlyvee$, $\curlywedge$ are hyperoperations) which generalizes both the classical lattice and the hyperlattice. In this paper we establish some of the properties of the (P, Q)-superlattice, and examine various types of $\overset{P}{\vee}$ and $\overset{Q}{\wedge}$ distributivity.

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2001,* 150-164

2 Definition of the (P,Q)-Superlattice

Let us first give the definition of a general superlattice. In fact, we will give two equivalent definitions. In what follows $\mathbf{P}(L)$ will denote the power set of a reference set L.

Definition 1 A superlattice is a partially ordered set $(L, \leq)$ with two hyperoperations $\curlyvee, \curlywedge$, where $\curlyvee : L \times L \to P(L)$, $\curlywedge : L \times L \to P(L)$, and the following properties are satisfied for all $a, b, c \in S$.

S1 $a \in (a \curlyvee a) \cap (a \curlywedge a)$,

S2 $a \curlyvee b = b \curlyvee a, a \curlywedge b = b \curlywedge a,$

S3 $(a \curlyvee b) \curlyvee c = a \curlyvee (b \curlyvee c), (a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$,

S4 $a \in [(a \curlyvee b) \curlywedge a] \cap [(a \curlywedge b) \curlyvee a]$,

S5a $a \leq b \Rightarrow (b \in a \curlyvee b \text{ and } a \in a \curlywedge b)$,

S5b $(b \in a \curlyvee b \text{ or } a \in a \curlywedge b) \Rightarrow a \leq b.$

The following definition is equivalent to Definition 1, as has been shown in [8]

Definition 2 A superlattice is a partially ordered set $(L, \leq)$ with two hyperoperations $\curlyvee, \curlywedge$, where $\curlyvee : L \times L \to P(L)$, $\curlywedge : L \times L \to P(L)$, and the following properties are satisfied for all $a, b, c \in S$.

S1 $a \in (a \curlyvee a) \cap (a \curlywedge a)$,

S2 $a \curlyvee b = b \curlyvee a, a \curlywedge b = b \curlywedge a,$

S3 $(a \curlyvee b) \curlyvee c = a \curlyvee (b \curlyvee c), (a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$,

S4 $a \in [(a \curlyvee b) \curlywedge a] \cap [(a \curlywedge b) \curlyvee a]$,

S6 $b \in a \curlyvee b \Leftrightarrow a \in a \curlywedge b,$

S7 $a, b \in a \curlyvee b \Longrightarrow a = b,$

S8 $b \in a \curlyvee b \text{ et } c \in b \curlyvee c \Longrightarrow c \in a \curlyvee c.$

Next let us define the (P,Q)-superlattice, which has been introduced in [7]. A (P,Q)-superlattice is a special kind of superlattice, which can be considered as a generalization of either the P-hyperlattice [4, 5] or the Q-d-hyperlattice[1]. (P,Q)-superlattices are constructed on a lattice $(L, \vee, \wedge)$ in a manner analogous to the construction of P-hypergroups [3, 9, 11] and P-hyperrings [10].

[1] A Q-d-hyperlattice can be defined analogously to a P-hyperlattice but utilizes an operation $\vee$ and a hyperoperation $\curlywedge = \overset{Q}{\wedge}$.

In what follows, $(L, \vee, \wedge)$ will always denote a lattice (with $L \neq \emptyset$) and $\leq$ will denote the order of $(L, \vee, \wedge)$. $\mathbf{J}(L)$ will denote the set of ideals of L, and $\mathbf{F}(L)$ will denote the set of filters of L. If L possesses a minimum (respectively maximum) element of L, this will be denoted by 0 (respectively 1).

Let us select two sets $P, Q \in \mathbf{P}(L)$ and define the following hyperoperations.

Definition 3 For all $a, b \in L$ we define $a \overset{P}{\vee} b \doteq a \vee b \vee P = \{a \vee b \vee p : p \in P\}$.

Definition 4 For all $a, b \in L$ we define $a \overset{Q}{\wedge} b \doteq a \wedge b \wedge Q = \{a \wedge b \wedge q : q \in Q\}$.

Remark. In [4] we have shown that if $P = L$, then $a \overset{L}{\vee} b = \{x \in L : a \vee b \leq x\}$. It can be shown similarly that, if $Q = L$, then $a \overset{L}{\wedge} b = \{y \in L : y \leq a \wedge b\}$.
Remark. It is easy to see that for $P, Q, P_1, Q_1 \in \mathbf{P}(L)$ such that $P \subseteq P_1$ and $Q \subseteq Q_1$ we have for all $a, b \in L$ that $a \overset{P}{\vee} b \subseteq a \overset{P_1}{\vee} b$ and $a \overset{Q}{\wedge} b \subseteq a \overset{Q_1}{\wedge} b$.

An $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ structure (with arbitrary choice of P, Q) is not necessary a superlattice. Consider the following example.

Example 5 Consider the lattice L of Figure 1.

Figure 1

(i) If we take $P = \{0, a\}$, $Q = \{b, 1\}$, then we see that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ satisfies the properties of Definition 1, i.e. it is a superlattice.
(ii) If we take $P = \{c, d\}$, $Q = \{c, d\}$, then we see that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ does not satisfy the properties of Definition 1, i.e. it is not a superlattice. For example $a \notin a \overset{P}{\vee} a = a \vee a \vee \{c, d\} = \{a \vee c, a \vee d\} = \{b, 1\}$. Similarly, $a \notin a \overset{Q}{\wedge} a = a \wedge a \wedge \{c, d\} = \{a \wedge c, a \wedge d\} = \{0\}$.

The necessary and sufficient conditions on P, Q for $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ to be a superlattice are easily stated in terms of the following two collections of sets.

Definition 6 $A(L) \doteq \{A \in P(L) : \forall x \in L \quad \exists a \in A \text{ such that } a \leq x\}$.

Definition 7 $B(L) \doteq \{B \in P(L) : \forall y \in L \quad \exists b \in B \text{ such that } y \leq b\}$.

It is clear that $L \in A_L \cap B_L$. Also if L has a 0 and a 1, then $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L) \Leftrightarrow (0, 1) \in P \times Q$. Furthermore, the following proposition yields the necessary and sufficient condition for $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ to be a superlattice.

Proposition 8 $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a superlattice $\Leftrightarrow (P, Q) \in A(L) \times B(L)$.

Proof. The proof appears in [7]. □

3 Properties of the (P, Q)-superlattice

In what follows we will assume that $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ (unless explicitly stated otherwise). Hence $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ will be a superlattice.

Definition 9 A superlattice $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ will be called proper iff there exist pairs (a, b), $(c, d) \in L \times L$, such that

$$card(a \overset{P}{\vee} b) \geq 2 \ \ and \ \ card(c \overset{Q}{\wedge} d) \geq 2 \ .$$

Proposition 10 *(i) If L possesses a maximum element 1 and we set $P \in \mathbf{A}(L)$, $Q = \{1\}$, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a P-hyperlattice.*

(ii) If L possesses a minimum element 0 and we set $P = \{0\}$, $Q \in \mathbf{B}(L)$, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a P-d-hyperlattice.

(iii) If L possesses a minimum element 0 and a maximum element 1, and we set $P = \{0\}$ and $Q = \{1\}$, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is the lattice $(L, \vee, \wedge)$.

Proof. (i) If $Q = \{1\}$, then for all $a, b \in L$ we have $a \overset{Q}{\wedge} b = a \wedge b \wedge 1 = a \wedge b$. Hence the $\overset{Q}{\wedge}$ is an operation, which yields the required result.

(ii) This is proved similarly to (i).

(iii) This is proved by combining (i) and (ii). □

Proposition 11 *For all $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ and all $a, b \in L$ we have: (i) $a \vee b = \min(a \overset{P}{\vee} b)$, (ii) $a \wedge b = \max(a \overset{Q}{\wedge} b)$.*

Proof. (i) Since $P \in \mathbf{A}(L)$ there will exist a $p \in P$ such that $p \leq a \vee b$. Hence $a \vee b = a \vee b \vee p \in a \overset{P}{\vee} b$. Clearly, for all $x \in a \overset{P}{\vee} b$ we have $a \vee b \leq x$, so we have proved $a \vee b = \min(a \overset{P}{\vee} b)$.

(ii) This is proved dually to (i). □

Remark. It follows that for all $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ we have that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a *strictly strong* superlattice [8].

Remark. A (P,Q)-superlattice, by its construction, preserves $\leq$, the original order of L. From **S5a** of Definition 1 follows that the $\leq$ order can be expressed in terms of the $\overset{P}{\vee}, \overset{Q}{\wedge}$ hyperoperations as follows:

$$a \leq b \Rightarrow (b \in a \overset{P}{\vee} b \text{ and } a \in a \overset{Q}{\wedge} b),$$

$$(b \in a \overset{P}{\vee} b \text{ or } a \in a \overset{Q}{\wedge} b) \Rightarrow a \leq b.$$

Proposition 12 *(i) If L has minimum element 0 and maximum element 1, then*

$$(card(P) \geq 2 \textit{ and } card(Q) \geq 2) \Leftrightarrow \left((L, \overset{P}{\vee}, \overset{Q}{\wedge}) \textit{ is a proper superlattice} \right).$$

(ii) If L does not have either minimum or maximum element, then for all $(P,Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ we have that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a proper superlattice.

Proof. (i) We have $0 \overset{P}{\vee} 0 = 0 \vee 0 \vee P = P$; hence $card(0 \overset{P}{\vee} 0) = card(P) \geq 2$. Similarly, $1 \overset{Q}{\wedge} 1 = 1 \wedge 1 \wedge Q = Q$; hence $card(1 \overset{Q}{\wedge} 1) = card(Q) \geq 2$. Hence $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a proper superlattice.

(ii) Assume that for some $(P,Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ the corresponding $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is *not* a proper superlattice. This means that for every $a \in L$ we will have $a \overset{P}{\vee} a = a$, or $a \overset{Q}{\wedge} a = a$, or both. But $a = a \overset{P}{\vee} a = a \vee P$ and so we conclude that for every $p \in P$ and for every $a \in L$ we have $p \leq a$. In particular, for any two $p, p_1 \in P \subseteq L$ we will have $p = p \vee p_1 = p_1$. It follows that $P = \{p\}$ and that (since $P \in \mathbf{A}(L)$) p is the minimum element of L. But this is in contradiction to the assumption. Dually, if $a \overset{Q}{\wedge} a$ we conclude that L has a maximum element, which again leads to a contradiction. $\square$

A special subset of $\mathbf{A}(L) \times \mathbf{B}(L)$ is $\mathbf{J}(L) \times \mathbf{F}(L)$, i.e. the Cartesian product of ideals and filters of L. In part (i) of the following proposition we do *not* initially assume $(P,Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$. However, (i) states that $\mathbf{J}(L) \times \mathbf{F}(L) \subseteq \mathbf{A}(L) \times \mathbf{B}(L)$, hence $(P,Q) \in \mathbf{J}(L) \times \mathbf{F}(L)$ implies that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a superlattice. Parts (ii) and (iii) of the proposition use stronger assumptions to reach more specialized conclusions.

Proposition 13 *(i)* $(P,Q) \in \mathbf{J}(L) \times \mathbf{F}(L) \Rightarrow (P,Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$.
(ii) $(P \in \mathbf{A}(L) \cap \mathbf{F}(L)) \Leftrightarrow P = L$.
(iii) $(Q \in \mathbf{B}(L) \cap \mathbf{J}(L)) \Leftrightarrow Q = L$.

Proof. (i) In [4] we have shown that $P \in \mathbf{J}(L)$ implies that $P \in \mathbf{A}(L)$. To prove that $Q \in \mathbf{B}(L)$ we proceed dually. Namely, if $Q \in \mathbf{F}(L)$, then $Q \vee L \subseteq Q$ [1]. Now take any $a \in L$ and any $q \in Q$. Then $(a,q) \in L \times Q$ and exists a $q_1 \in Q$ such that $a \vee q = q_1$, i.e. $a \leq q_1$. Hence $Q \in \mathbf{B}(L)$.

(ii) Now assume that $P \in \mathbf{A}(L) \cap \mathbf{F}(L)$. Since $P \in \mathbf{A}(L)$ for every $a \in L$ there exists $p \in P$ such that $p \leq a$, i.e. $a \vee p = a \in P \vee L$; but, since $P \in \mathbf{F}(L)$ we also have

$P \vee L \subseteq P$. Hence $a \in L \Rightarrow a \in P$ and so $L \subseteq P$. Since obviously $P \subseteq L$, we have $P = L$. The converse is obvious.

(iii) This is proved dually to (ii). $\square$

The converse of (i) in the above proposition does not hold, as we can see in the next example.

Example 14 If L is the lattice of Figure 2, and $P = Q = \{0, 1\}$, then $(P, Q) \in A(L) \times B(L)$, but $(P, Q) \notin \mathbf{J}(L) \times \mathbf{F}(L)$.

Figure 2

Proposition 15 *For all $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ we have:*

(i) $Q \subseteq P \vee Q$, $P \subseteq P \wedge Q$.

(ii) If for every $(p, q) \in P \times Q$ we have $p \leq q$, then $Q = P \vee Q$, $P = P \wedge Q$.

(iii) If L is distributive and P, Q are intervals, then $Q = P \vee Q$, $P = P \wedge Q$.

Proof. (i) Take any $q \in Q$. Since $P \in \mathbf{A}(L)$, it follows there will exist $p \in P$ such that $p \leq q$. Hence $q = p \vee q \in P \vee Q$. Hence $Q \subseteq P \vee Q$. It is proved dually that $P \subseteq P \wedge Q$.

(ii) This is easy to prove.

(iii) Now assume that $P = [a, b]$, $Q = [c, d]$. Then there will exist $q \in [c, d]$ such that $b \leq q \leq d$ and $p \in [a, b]$ such that $a \leq p \leq c$. Hence, since L is distributive, we will have: $P \vee Q = [a, b] \vee [c, d] = [a \vee c, b \vee d] = [c, d] = Q$ [2]. Dually we can prove that $P \wedge Q = P$. $\square$

We now introduce a relation $\preceq$ between elements of $\mathbf{P}(L)$; $\preceq$ is an order relation (see for instance [2]).

Definition 16 Take any $A, B \in P(L)$; we write $A \preceq B$ iff

$$\textit{(i)}\ \forall a \in A \quad \exists b_1 \in B : a \leq b_1, \quad \textit{(ii)}\ \forall b \in B \quad \exists a_1 \in A : a_1 \leq b.$$

Proposition 17 *If $a, b \in L$ and $a \leq b$, then for all $c \in L$ we have: (i) $a \overset{P}{\vee} c \preceq b \overset{P}{\vee} c$, (ii) $a \overset{Q}{\wedge} c \preceq b \overset{Q}{\wedge} c$.*

Proof. (i) If $x \in a \overset{P}{\vee} c$ then exists some $p_1 \in P$ such that $x = a \vee c \vee p_1 \leq b \vee c \vee p_1 = y \in b \overset{P}{\vee} c$. Similarly, if $z \in b \overset{P}{\vee} c$ then exists some $p_2 \in P$ such that $z = b \vee c \vee p_2 \geq b \vee c \vee p_2 = w \in a \overset{P}{\vee} c$. Hence $a \overset{P}{\vee} c \preceq b \overset{P}{\vee} c$.

(ii) It is proved dually. □

Remark. The relationship $\preceq$ defined above, generally is not an order relationship on $\mathbf{P}(L)$. It is an order relationship on $\mathbf{I}(L)$, the class of intervals of L[2].

Remark. If $a, b \in L$ are such that $a \leq b$ and $c \in L$, then it is possible to have $(a \overset{P}{\vee} c) \cap (b \overset{P}{\vee} c) = \emptyset$ and $(a \overset{Q}{\wedge} c) \cap (b \overset{Q}{\wedge} c) = \emptyset$. An example of this follows.

Example 18 Consider the lattice L of Figure 3 and take $P = \{0, a\} \in A_L$ and $Q = \{b, 1\} \in B_L$.

Figure 3

We observe that $a \leq 1$ and $a \overset{P}{\vee} a = a \vee P = a$ and $1 \overset{P}{\vee} a = 1 \vee P = 1$, i.e. $(a \overset{P}{\vee} a) \cap (1 \overset{P}{\vee} a) = \emptyset$. Similarly, $0 \leq b$ and $b \overset{Q}{\wedge} b = b \wedge Q = b$ and $0 \overset{Q}{\wedge} b = 0 \wedge Q = 0$, i.e. $(b \overset{Q}{\wedge} b) \cap (0 \overset{Q}{\wedge} b) = \emptyset$.

Clearly for $a, b \in L$ the form of $a \overset{P}{\vee} b$, $a \overset{Q}{\wedge} b$ depends on the form of P, Q. The following propositions examine some cases of this connection.

Proposition 19 *If $(L, \leq)$ is a distributive lattice, then for all $a, b \in L$ we have*

(i) P is an interval $\Rightarrow a \overset{P}{\vee} b$ is an interval.

(ii) Q is an interval $\Rightarrow a \overset{Q}{\wedge} b$ is an interval.

Proof. (i) Assume $P = [x, y]$, then (using [2]) we have $a \overset{P}{\vee} b = a \vee b \vee [x, y] = [a \vee b \vee x, a \vee b \vee y]$. But, since $\exists p \in P = [x, y]$ such that $p \leq a \vee b$, we will have $a \overset{P}{\vee} b = [a \vee b, a \vee b \vee y]$.

(ii) Assume $Q = [z, w]$, then (using [2]) we have $a \overset{Q}{\wedge} b = a \wedge b \wedge [z, w] = [a \wedge b \wedge z, a \wedge b \wedge w]$ and, similarly to (i), we get $a \overset{Q}{\wedge} b = [a \wedge b \wedge z, a \wedge b]$. □

Proposition 20 *If* $(L, \leq)$ *is a distributive lattice and* $a, b \in L$ *are such that* $a \leq b$, *then we have*

(i) P *is an interval* $\Rightarrow (a \overset{P}{\vee} c) \vee (b \overset{P}{\vee} c) = b \overset{P}{\vee} c$.

(ii) Q *is an interval* $\Rightarrow \ (a \overset{Q}{\wedge} c) \vee (b \overset{Q}{\wedge} c) = a \overset{Q}{\wedge} c$.

Proof. (i) Assume $P = [x, y]$, then $a \overset{P}{\vee} c = [a \vee c \vee x, a \vee c \vee y]$ and $b \overset{P}{\vee} c = [b \vee c \vee x, b \vee c \vee y]$. Since L is distributive, we will have

$$(a \overset{P}{\vee} c) \vee (b \overset{P}{\vee} c) = [a \vee c \vee x, a \vee c \vee y] \vee [b \vee c \vee x, b \vee c \vee y] =$$

$$[a \vee b \vee c \vee x, a \vee b \vee c \vee y] = [b \vee c \vee x, b \vee c \vee y] = b \vee c \vee [x, y] = b \overset{P}{\vee} c.$$

(ii) It is proved dually. □

Proposition 21 *If* $(L, \leq)$ *is a distributive lattice, then:*

(i) $(P$ *is a sublattice*$) \Rightarrow \left(\forall a, b \in L \quad a \overset{P}{\vee} b \text{ is a sublattice}\right)$.

(ii) $(P$ *is a sublattice*$) \Rightarrow \left(\forall a, b \in L \quad a \overset{Q}{\wedge} b \text{ is a sublattice}\right)$.

Proof. (i) Assume that P is a sublattice of L. Take any $a, b \in L$. For any $x_1, x_2 \in a \overset{P}{\vee} b$ there exist $p_1, p_2 \in P$ such that $x_1 = a \vee b \vee p_1$ and $x_2 = a \vee b \vee p_2$. Furthermore, $p_1 \vee p_2 = p_3 \in P$, $p_1 \wedge p_2 = p4 \in P$. Hence $x_1 \vee x_2 = a \vee b \vee p_3 \in a \overset{P}{\vee} b$ and $x_1 \wedge x_2 = (a \vee b \vee p_1) \wedge (a \vee b \vee p_2) = (a \vee b) \vee (p_1 \wedge p_2) = (a \vee b) \vee p_4 \in a \overset{P}{\vee} b$.

(ii) Is proved dually to (i). □

Proposition 22 *(i) If* $(L, \leq)$ *is a distributive lattice and has a minimum element 0, then we have:*

$$0 \overset{P}{\vee} 0 \text{ is a sublattice} \Leftrightarrow P \text{ is a sublattice.}$$

(ii) If $(L, \leq)$ *is a distributive lattice and has a maximum element 1, then we have:*

$$1 \overset{Q}{\wedge} 1 \text{ is a sublattice} \Leftrightarrow Q \text{ is a sublattice.}$$

Proof. (i) This is obvious, since $0 \overset{P}{\vee} 0 = 0 \vee P = P$.

(ii) It is proved dually to (i). □

Corollary 23 *(i) If* L *has a minimum element 0, then we have:*

$$\left(\forall a, b \in L \quad a \overset{P}{\vee} b \text{ is a sublattice}\right) \Leftrightarrow (P \text{ is a sublattice}).$$

(ii) If L *has a maximum element 1, then we have:*

$$\left(\forall a, b \in L \quad a \overset{Q}{\wedge} b \text{ is a sublattice}\right) \Leftrightarrow (Q \text{ is a sublattice}).$$

Proof. Follows immediately from Propositions 21 and 22. □

Proposition 24 *For all $a, b \in L$ we have*

(i) P is sublattice $\Rightarrow \left((a \overset{P}{\vee} b) \overset{P}{\vee} (a \overset{P}{\vee} b) = a \overset{P}{\vee} b \subseteq (a \overset{P}{\vee} b) \overset{Q}{\wedge} (a \overset{P}{\vee} b) \right)$.

(ii) Q is sublattice $\Rightarrow \left((a \overset{Q}{\wedge} b) \overset{Q}{\wedge} (a \overset{Q}{\wedge} b) = a \overset{Q}{\wedge} b \subseteq (a \overset{Q}{\wedge} b) \overset{P}{\vee} (a \overset{Q}{\wedge} b) \right)$.

Proof. (i) Take any $x_1, x_2 \in a \overset{P}{\vee} b$; then there exist $p_1, p_2 \in P$ such that $x_1 = a \vee b \vee p_1$ and $x_2 = a \vee b \vee p_2$. We have $x_1 \overset{P}{\vee} x_2 = \{(a \vee b \vee p_1) \vee (a \vee b \vee p_2) \vee p : p \in P\}$ $= \{a \vee b \vee (p_1 \vee p_2 \vee p) : p \in P\}$; and since P is a sublattice we have $x_1 \overset{P}{\vee} x_2 =$ $\{a \vee b \vee (p_1 \vee p_2 \vee p) : p \in P\} \subseteq a \overset{P}{\vee} b$, which finally implies

$$(a \overset{P}{\vee} b) \overset{P}{\vee} (a \overset{P}{\vee} b) \subseteq a \overset{P}{\vee} b. \tag{1}$$

Now take any $x \in a \overset{P}{\vee} b$. Then there exist $p, p_1 \in P$ such that $x = a \vee b \vee p$ and $p_1 \leq a \vee b \vee p$. Hence $x = a \vee b \vee p = (a \vee b \vee p) \vee (a \vee b \vee p) \vee p_1 \in (a \vee b \vee p) \vee (a \vee b \vee p) \vee P$ $= (a \vee b \vee p) \overset{P}{\vee} (a \vee b \vee p) \subseteq (a \overset{P}{\vee} b) \overset{P}{\vee} (a \overset{P}{\vee} b)$ which implies

$$a \overset{P}{\vee} b \subseteq (a \overset{P}{\vee} b) \overset{P}{\vee} (a \overset{P}{\vee} b). \tag{2}$$

From (1) and (2) we have that $(a \overset{P}{\vee} b) \overset{P}{\vee} (a \overset{P}{\vee} b) = a \overset{P}{\vee} b$.

Furthermore, if $x \in a \overset{P}{\vee} b$, then there exist $p \in P$ and $q \in Q$ such that $x = a \vee b \vee p \leq q$. Hence $a \vee b \vee p = (a \vee b \vee p) \wedge (a \vee b \vee p) \wedge q \in (a \vee b \vee p) \wedge (a \vee b \vee p) \wedge Q$ $= (a \vee b \vee p) \overset{Q}{\wedge} (a \vee b \vee p) \subseteq (a \overset{P}{\vee} b) \overset{Q}{\wedge} (a \overset{P}{\vee} b)$.

(ii) This is proved similarly to (ii). □

The inclusion in the above proposition can be proper, as can be seen in the following example

Example 25 Consider the distributive lattice of Figure 4. Here we have $(d \overset{P}{\vee} e) \overset{Q}{\wedge} (d \overset{P}{\vee} e)$ $= \{e, b, k, l, 1\} \supset \{k, l, 1\} = d \overset{P}{\vee} e$.$\mathbb{Z}$.

Example 26

Figure 4

Proposition 27 $(L, \vee, \wedge)$ *is distributive iff*

$$\text{For all } a, x, y \in L \text{ we have: } \left.\begin{array}{l} a \overset{P}{\vee} x = a \overset{P}{\vee} y \\ a \overset{Q}{\wedge} x = a \overset{Q}{\wedge} y \end{array}\right\} \Rightarrow x = y$$

Proof. (i) Let L be distributive. Pick any a, x, y such that $a \overset{P}{\vee} x = a \overset{P}{\vee} y$ and $a \overset{Q}{\wedge} x = a \overset{Q}{\wedge} y$. Then there exist $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ such that

$$a \vee x = a \vee y \vee p_1, \qquad a \vee y = a \vee x \vee p_2, \tag{3}$$

$$a \wedge x = a \wedge y \wedge q_1, \qquad a \wedge y = a \wedge x \wedge q_2. \tag{4}$$

From (3) we obtain $a \vee y \leq a \vee x$ and $a \vee x \leq a \vee y$, i.e. $a \vee x = a \vee y$; similarly, from (4) we obtain $a \wedge x = a \wedge y$. From these and distributivity we obtain $x = y$.

(ii) On the other hand, assume that for all $a, x, y \in L$ the implication

$$(a \overset{P}{\vee} x = a \overset{P}{\vee} y \text{ and } a \overset{Q}{\wedge} x = a \overset{Q}{\wedge} y) \Rightarrow x = y \tag{5}$$

is true. Now choose any $a, x, y \in L$ such that $a \vee x = a \vee y$ and $a \wedge x = a \wedge y$; then we will also have $a \overset{P}{\vee} x = a \overset{P}{\vee} y$ and $a \overset{Q}{\wedge} x = a \overset{Q}{\wedge} y$ and so from (5) follows that $x = y$. In short, for all $a, x, y \in L$ we have:

$$(a \vee x = a \vee y \text{ and } a \wedge x = a \wedge y) \Rightarrow x = y; \tag{6}$$

but, as is well known (6) is a necessary and sufficient condition for distributivity of L. $\square$

4 Distributivity

In this section we examine the distributivity of the $\overset{P}{\vee}$, $\overset{Q}{\wedge}$ hyperoperations. Since the outcome of each of these hyperoperations is generally a set, several forms of distributivity can be introduced. Let us first introduce the following definitions of distributivity for a general superlattice.

Definition 28 A superlattice $(L, \curlyvee, \curlywedge]$ is called weakly $\curlywedge$-distributive (also denoted by w-$\curlywedge$-d) iff

$$(a \curlywedge (b \curlyvee c)) \cap ((a \curlywedge b) \curlyvee (a \curlywedge c)) \neq \emptyset;$$

it is called weakly $\curlyvee$-distributive (also denoted by w-$\curlyvee$-d) iff

$$(a \curlyvee (b \curlywedge c)) \cap ((a \curlyvee b) \curlywedge (a \curlyvee c)) \neq \emptyset.$$

Definition 29 A superlattice $(L, \curlyvee, \curlywedge]$ is called feebly $\curlywedge$-distributive (also denoted by f-$\curlywedge$-d) iff

$$(a \curlywedge (b \curlyvee c)) \subseteq ((a \curlywedge b) \curlyvee (a \curlywedge c));$$

it is called feebly $\curlyvee$-distributive (also denoted by f-$\curlyvee$-d) iff

$$(a \curlyvee (b \curlywedge c)) \subseteq ((a \curlyvee b) \curlywedge (a \curlyvee c)).$$

Definition 30 A superlattice $(L, \curlyvee, \curlywedge]$ is called $\curlywedge$-distributive (also denoted by $\curlywedge$-d) iff

$$(a \curlywedge (b \curlyvee c)) = ((a \curlywedge b) \curlyvee (a \curlywedge c));$$

it is called $\curlyvee$-distributive (also denoted by $\curlyvee$-d) iff

$$(a \curlyvee (b \curlywedge c)) = ((a \curlyvee b) \curlywedge (a \curlyvee c)).$$

In the following, consider a distributive lattice $(L, \vee, \wedge)$ and choose sets $P, Q \subseteq \mathbf{P}(L)$ such that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a superlattice. The distributivity of $(L, \vee, \wedge)$ is connected to the various forms of distributivity of $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$, as will be seen by the following propositions.

Proposition 31 *For all* $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$, $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *is both w-*$\overset{P}{\vee}$*-d and w-*$\overset{Q}{\wedge}$*-d.*

Proof. Take any $a, b, c \in L$, then there exist $p \in P$ satisfying $p \leq b$, and $q \in Q$ satisfying $a \leq q$. Hence we will have that $a \wedge (b \vee c) = a \wedge (b \vee c \vee p) = a \wedge (b \vee c \vee p)$ $\wedge q \in a \wedge (b \vee c \vee p) \wedge Q \ = a \overset{Q}{\wedge} (b \vee c \vee p) \subseteq a \overset{Q}{\wedge} (b \vee c \vee P) = a \overset{Q}{\wedge} (b \overset{P}{\vee} c)$. On the other hand, $(a \wedge b) \vee (a \wedge c) \in (a \wedge b) \overset{P}{\vee} (a \wedge c) \subseteq (a \overset{Q}{\wedge} b) \overset{P}{\vee} (a \overset{Q}{\wedge} c)$. Since L is distributive, we will have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. In short, we have shown that

$$a \wedge (b \vee c) \in (a \overset{Q}{\wedge} (b \overset{P}{\vee} c)) \cap ((a \overset{Q}{\wedge} b) \overset{P}{\vee} (a \overset{Q}{\wedge} c))$$

and so we have shown that the (P, Q)-superlattice is w-$\overset{Q}{\wedge}$-d. It can be shown dually that it is also w-$\overset{P}{\vee}$-d. $\square$

Proposition 32 *For all* $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ *we have:*

(i) if $L \wedge P \wedge Q \subseteq P$, *then* $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *is f-*$\overset{Q}{\wedge}$*-d;*

(ii) if L *has a minimum element 0 and* $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *is f-*$\overset{Q}{\wedge}$*-d, then* $L \wedge P \wedge Q \subseteq P$*;*

(iii) if $L \vee P \vee Q \subseteq Q$, *then* $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *is f-*$\overset{P}{\vee}$*-d;*

(iv) if L *has a maximum element 1 and* $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *is f-*$\overset{P}{\vee}$*-d, then* $L \vee P \vee Q \subseteq Q$.

Proof. (i) If $a, b, c \in L$ and $x \in a \overset{Q}{\wedge} (b \overset{P}{\vee} c)$, then there exists a pair $(p, q) \in P \times Q$ such that

$$x = a \wedge (b \vee c \vee p) \wedge q = (a \wedge b \wedge q) \vee (a \wedge c \wedge q) \vee (a \wedge p \wedge q) \in$$

$$(a \wedge b \wedge q) \vee (a \wedge c \wedge q) \vee P = (a \wedge b \wedge q) \overset{P}{\vee} (a \wedge c \wedge q) \subseteq (a \overset{Q}{\wedge} b) \overset{P}{\vee} (a \overset{Q}{\wedge} c).$$

(ii) Now assume that L has a minimum element 0 and $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is f-$\overset{Q}{\wedge}$-d. Then, for every $x \in L$ we have

$$x \overset{Q}{\wedge} (0 \overset{P}{\vee} 0) = \cup_{p \in P} x \overset{Q}{\wedge} p = x \wedge P \wedge Q.$$

On the other hand,

$$(x \overset{Q}{\wedge} 0) \overset{P}{\vee} (x \overset{Q}{\wedge} 0) = (x \wedge 0 \wedge Q) \overset{P}{\vee} (x \wedge 0 \wedge Q) = 0 \overset{P}{\vee} 0 = P. \tag{7}$$

From (7) and the fact that $x \wedge P \wedge Q = x \overset{Q}{\wedge} (0 \overset{P}{\vee} 0) \subseteq (x \overset{Q}{\wedge} 0) \overset{P}{\vee} (x \overset{Q}{\wedge} 0)$, it follows that $x \wedge P \wedge Q \subseteq P$; since this is true for any $x \in L$, we conclude that $L \wedge P \wedge Q \subseteq P$.

(iii) and (iv) are proved dually. □

Corollary 33 *Let $P \in \mathbf{J}(L)$. Then we have the following.*

(i) If for all $(p, q) \in P \times Q$ we have $p \leq q$, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is f-$\overset{Q}{\wedge}$-d.

(ii) If P, Q are intervals, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is f-$\overset{Q}{\wedge}$-d.

Similarly, let $Q \in \mathbf{F}(L)$. Then we have the following.

(iii) If for all $(p, q) \in P \times Q$ we have $p \leq q$, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is f-$\overset{P}{\vee}$-d.

(iv) If P, Q are intervals, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is f-$\overset{P}{\vee}$-d.

Proof. (i) Since for every pair $(p, q) \in P \times Q$ we have $p \leq q$, it follows that $P \wedge Q = P$. Since P is an ideal of L, it follows that $L \wedge P \wedge Q = L \wedge P \subseteq P$. Now we can apply Proposition 32 to obtain the desired conclusion.

(ii) Since P, Q are intervals, then from Proposition 15 we have $P \wedge Q = P$ and we can apply part (i) of this proposition.

(iii) and (iv) are proved dually. □

Corollary 34 *The $(L, \overset{L}{\vee}, \overset{L}{\wedge})$ is f-$\overset{Q}{\wedge}$-d and f-$\overset{P}{\vee}$-d.*

Proof. Taking $P = L$, $Q = L$ and applying parts (i) and (iii) of Proposition 32, we immediately obtain the desired conclusion. □

The next proposition concerns $\overset{Q}{\wedge}$-distributivity.

Proposition 35 *(i) If the lattice L does not have a minimum element, there exist no (P, Q) pair such that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a proper $\overset{Q}{\wedge}$-distributive superlattice.*

(ii) If the lattice L has a minimum element 0, then for every (P, Q) pair such that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a $\overset{Q}{\wedge}$-distributive superlattice, we have $P = \{0\}$ (i.e. $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a Q-d-hyperlattice).

Proof. (i) Assume that there is a pair $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(\mathbf{L})$, such that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a proper superlattice which is also $\overset{Q}{\wedge}$-distributive. I.e. for all $a, b, c \in L$ we have

$$a \overset{Q}{\wedge} (b \overset{P}{\vee} c) = (a \overset{Q}{\wedge} b) \overset{P}{\vee} (a \overset{Q}{\wedge} c). \tag{8}$$

But we also have

$$a \overset{Q}{\wedge} (b \overset{P}{\vee} c) = a \wedge (b \vee c \vee P) \wedge Q \tag{9}$$

and

$$(a \overset{Q}{\wedge} b) \overset{P}{\vee} (a \overset{Q}{\wedge} c) = (a \wedge b \wedge Q) \vee (a \wedge c \wedge Q) \vee P. \tag{10}$$

From (8-10) follows that

$$a \wedge (b \vee c \vee P) \wedge Q = (a \wedge b \wedge Q) \vee (a \wedge c \wedge Q) \vee P \Rightarrow$$

$$a \vee (a \wedge (b \vee c \vee P) \wedge Q) = a \vee (a \wedge b \wedge Q) \vee (a \wedge c \wedge Q) \vee P. \tag{11}$$

Now

$$a \vee (a \wedge (b \vee c \vee P) \wedge Q) = \cup_{x \in a \wedge (b \vee c \vee P) \wedge Q} a \vee x = \{a\}. \tag{12}$$

Similarly,

$$a \vee (a \wedge b \wedge Q) \vee (a \wedge c \wedge Q) \vee P = (\cup_{x \in (a \wedge b \wedge Q) \vee (a \wedge c \wedge Q)} a \vee x) \vee P = a \vee P. \tag{13}$$

And from (11), (12) and (13) it follows that $a = a \vee P$. Using $a = a \vee P$ and referring to the proof of part (ii), Proposition 12, we conclude that $P = \{p\}$, which means that p will be the minimum element of L; but this is contrary to the hypothesis and we have reached a contradiction.

(ii) Now assume that L has a minimum element 0 and take any (P, Q) pair such that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a proper superlattice is a $\overset{Q}{\wedge}$-distributive superlattice. Duplicating the argument of part (i), we conclude that $P = \{0\}$. □

Remark. If L has a maximum element 1, and $Q = \{1\}$ and $card(P) \geq 2$, then $a \overset{Q}{\wedge} b = a \wedge b$ and $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a proper P-hyperlattice. But in [4] we have shown that a proper P-hyperlattice cannot be $\wedge$-d. If, on the other hand, L has a minimum element 0, and $P = \{0\}$ and $card(Q) \geq 2$, then we have a proper Q-d-hyperlattice, which may also be $\wedge$-d, as will be seen from the following proposition.

Proposition 36 *L is distributive iff $(L, \vee, \overset{L}{\wedge})$ is a $\overset{L}{\wedge}$-distributive L-d-hyperlattice.*

Proof. The proof is analogous to Corollary 2.3 of [4]. □

Proposition 37 *(i) If the lattice L does not have a maximum element, there exist no (P, Q) pair such that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a proper $\overset{P}{\vee}$-distributive superlattice.*

(ii) If the lattice L has a maximum element 1, then for every (P, Q) pair such that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a $\overset{P}{\vee}$-distributive superlattice, we have $Q = \{1\}$ (i.e. $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a P-hyperlattice.)

Proof. The proof of this proposition is dual to that of Proposition 35. □

Remark. If L has a minimum element 0, and $P = \{0\}$ and $card(Q) \geq 2$, then $a \overset{P}{\vee} b = a \vee b$ and $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a proper Q-d-hyperlattice. A proper Q-d-hyperlattice cannot be $\vee$-d (the proof is dual to the proof concerning P-hyperlattices [4]). If L has a maximum element 1, and $Q = \{1\}$ and $card(P) \geq 2$, then the (P, Q)-superlattice is a proper P-hyperlattice, which may be $\overset{P}{\vee}$-d.

References

[1] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 1990.

[2] Ath. Kehagias and M. Konstantinidou, *Lattice-ordered join space: an applications-oriented example*, to appear in *Italian Journal of Pure and Applied Mathematics*.

[3] L. Koguetsof and Th. Vougiouklis, *Constructions d'hyperanneaux à partir d'anneaux*, Acta Univ. Carolin. Math. Phys., vol. 28(1987).9–13.

[4] M. Konstantinidou, *On P-hyperlattices and their distributivity*, Rend. Circ. Mat. Palermo, vol. 42, (1994) 391–403.

[5] M. Konstantinidou, *A representation theorem for P-hyperlattices*, Riv. Mat. Pura Appl., vol. 18, (1996) 63–69.

[6] M. Konstantinidou and J. Mittas, *An introduction to the theory of hyperlattices*, Math. Balkanica, vol. 7, (1997)187–193.

[7] M. Konstantinidou and K. Serafimidis, *Sur les P-supertreillis*, in ,,New frontiers in hyperstructures" (Molise, 1995), Ser. New Front. Adv. Math. Ist. Ric. Base, Hadronic Press, Palm Harbor, FL, (1996) 139–148.

[8] J. Mittas and M. Konstantinidou, *Sur une nouvelle génération de la notion de treillis. Les supertreillis et certaines de leurs propriétés générales*, Ann. Sci. Univ. Blaise Pascal, Ser. Math., vol.25, (1989) 61-83.

[9] S. Spartalis, *Quotients of P-H_V-rings*, in ,,New frontiers in hyperstructures" (Molise, 1995), Ser. New Front. Adv. Math. Ist. Ric. Base, Hadronic Press, Palm Harbor, FL, (1996) 167–176.

[10] Th. Vougiouklis, *Isomorphisms on P-hypergroups and cyclicity*, Ars Combin., vol. 29, Ser. A, (1990) 241–245.

[11] Th. Vougiouklis and L. Koguetsof, *P-hypergroupes*, Acta Univ. Carolin. Math. Phys., vol.28, (1987) 15–20.

Authors' address:

K. Serafimidis and M. Konstantinidou
Division of Mathematics
Department of Mathematics, Physical and Computational Sciences
Aristotle University
GR 54006 Thessaloniki, GREECE