MONOTONE OPERATOR FROM TANGENT BUNDLE TO COTANGENT BUNDLE OF A SYMPLECTIC MANIFOLD

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To a field of cones in the tangent bundle of a symplectic manifold we associat a natural field of cones in the cotangent bundle and a natural positive monotone operator from tangent bundle to cotangent bundle. An important example is the cotangent bundle of a symplectic manifold because this is the phase space of classical mechanics.

Recall that a symplectic manifold is a couple (M, ω), where M is a smooth manifold and ω is a symplectic form i.e., a nondegenerate closed 2-form on M.

If (M, ω) is a symplectic manifold, each tangent space $(T_x M, \omega_x)$ is a symplectic vector space and the manifold M is necessarily of even dimension. If $2n$ is the dimension of the manifold M, the product $\omega^n = \omega \wedge \omega \wedge \ldots \wedge \omega$ (n-factors) never vanishes. The manifold M is orientable and any symplectic diffeomorphism preserves the volume.

Definition 1. A subset K of a real vector space V is a cone if $\forall v \in K$, $\lambda \in \mathbf{R}_+$ imply $\lambda v \in K$. If moreover $v \in K$ and $-v \in K$ imply $v = 0$, then K is called *pointed cone.*

Definition 2. A field of cones in a vector bundle (E, p, M) is a map $K : x \in M \longrightarrow K(x) \subset E_x = p^{-1}(x)$ such that the following two condition are satisfied:

i) for each $x \in M$ the set $K(x)$ is a convex pointed closed cone with interior points of E_x;

ii) the sets $\bigcup_{x \in M} IntK(x)$ and $\bigcup_{x \in M} (E_x - K(x))$ are open in E.

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A vector bundle (E, p, M) endowed with a field of cones is denoted by $[(E, p, M); K]$ ([9])

It is not difficult to see that the structures $[(E, p, M); K]$ are the objects of a category. A morphism from $[(E, p, M); K]$ to $[(E', p', M'); K']$ in this category is a morphisms $f : E \longrightarrow E'$ of vector bundles such that $f(K(x)) \subset K(\underline{f}(x))$, $\forall x \in M$. (The map $\underline{f} : M \longrightarrow M$ is defined such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ p\downarrow & & \downarrow p' \\ M & \xrightarrow{\underline{f}} & M \end{array}$$ commutes).

In particular if (TM, p, M) is the tangent bundle of smooth manifold M, the structures $[(TM, p, M); K]$ are the objects of a subcategory of the previous category.

Proposition 1. *Let (M, ω) be a symplectic manifold (paracompact connected without boundary) and K a field of cones on the tangent bundles (TM, p, M). There is a natural field of cones K^* on the cotangent vector bundle (T^*M, p^*, M) such that, the structures $[(TM, p, M); K]$ and $[(T^*M, p^*, M); K^*]$ are isomorphic.*

Proof. Because M is a symplectic manifold each tangent space (T_xM, ω_x), $x \in M$ is symplectic vector space.

For each $x \in M$ we can define the map

$$h_x : T_xM \longrightarrow T_x^*M,\ X_x \longrightarrow h_x(X_x) = i_{X_x}\omega_x = \omega_x(X_x).$$

Since ω_x is nondegenerate, this map is an isomorphism between the tangent space T_xM and cotangent space T_x^*M.

The map $h : TM \longrightarrow T^*M, h/_{T_xM} = h_x, x \in M$ is an isomorphism of tangent fiber bundle TM and cotangent fiber bundle T^*M.

Then, we define the map $K^* = h \circ K : M \longrightarrow T^*M$.

It is easy to see that K^* is a field of cones of the cotangent bundle (T^*M, p^*, M) and $h : TM \longrightarrow T^*M$ is an isomorphism of structures $[(TM, p, M); K]$ and $[(T^*M, p^*, M); K^*]$, i.e., an isomorphism of vector bundles (TM, p, M) and (T^*M, p^*, M) such that $h(K) = K^*$

Remark. Let M be a smooth manifold. If K is a field of cones on the tangent bundle (TM, p, M), then $x \in M \longrightarrow \{\alpha_x \in T_x^*M / \alpha_x(X_x) \geq 0, \forall X_x \in K(x), \}$ is a field of cones on the cotangent bundle (T^*M, p^*, M) [9].

The field of cones K^* from Proposition 1 was defined using the structure of the symplectic manifold. Using the field of cones K^* one can define a positive Dirac current [10].

Let M be a symplectic manifold and a K a field of cones on the tangent bundle (TM, p, M).

Let $\chi(M)$ be the set of vector fields of M (the set of sections of tangent bundle) and $\Omega^1 M$ be the set of 1-forms of M (the set of sections of cotangent bundle). On the sets $\chi(M)$, $\Omega^1 M$ we can introduce ordering relations.

If $X, Y \in \chi(M)$, then $X \leq Y \Leftrightarrow X_x - Y_x \in K(x)$, $\forall x \in M$. Consequently, $(\chi(M), \leq)$ is an ordered vector space directed on both sides. The set of positive vector fields $K_\chi = \{X \in \chi(M)/\ X \geq 0\}$ is a convex pointed cone of $\chi(M)$. For every $Z \in IntK_\chi$, the set $\chi(M)$ is Z- measurable , i.e. $\forall X \in K_\chi, \exists \lambda \in \mathbf{R}, \lambda \geq 0$ such that $-\lambda Z \leq X \leq \lambda Z$.

Then, for a fixed $Z \in IntK_\chi$, there is a norm on $\chi(M)$:

$\|X\|_Z = \min\{\lambda \in \mathbf{R}/\lambda \geq 0, -\lambda Z \leq X \leq \lambda Z\}\ \forall X \in \chi(M)$ and this norm is monotone, i.e.,

$$0 \leq X \leq Y \Rightarrow 0 \leq \|X\|_Z \leq \|Y\|_Z$$

If $\alpha, \beta \in \Omega^1 M$, then $\alpha \leq \beta \Leftrightarrow \alpha_x - \beta_x \in K^*(x), \forall x \in M$. In this way, the set $(\Omega^1 M, \leq)$ is an ordered vector space directed on both sides.

The set of positive covector fields $K^*_\Omega = \{\alpha \in \Omega^1 M/\ \alpha \geq 0\}$ is a convex pointed cone of $\Omega^1 M$.

For every $\gamma \in IntK^*_\Omega$, the set $\Omega^1 M$ is γ-measurable , i.e., $\forall \alpha \in K^*_\Omega, \exists \lambda \in \mathbf{R}$, $\lambda \geq 0$ such that $-\lambda\gamma \leq \alpha \leq \lambda\gamma$.

Then, for a fixed $\gamma \in IntK^*_\Omega$, there is a norm on $\Omega^1 M$:

$\|\alpha\|_\gamma = \min\{\lambda \in \mathbf{R}/\lambda \geq 0, -\lambda\gamma \leq \alpha \leq \lambda\gamma\}\ \forall \alpha \in \Omega^1 M$ and this norm is monotone, i.e., $0 \leq \alpha \leq \beta \Rightarrow 0 \leq \|\alpha\|_\gamma \leq \|\beta\|_\gamma$.

The structures $(\chi(M), K_\chi)$ and $(\Omega^1 M, K^*_\Omega)$ are Krein spaces if and only if the manifold M is compact ([5], [6]).

Remark. Let (M, ω) be a symplectic manifold. Using the natural isomorphism between $\Omega^k M$ and $\Omega^{n-k} M$, to any convex pointed cone of $\Omega^k M$, we associate a pointed cone of $\Omega^{n-k} M$.

Proposition 2. *If (M, ω) is a symplectic manifold and K a field of cones on the tangent bundle (TM, p, M), there is a positive monotone operator from (TM, p, M) to (T^*M, p^*, M).*

Proof. If K is a field of cones on the tangent bundle (TM, p, M), there is a natural field of cones K^*on the cotangent vector bundle (T^*M, p^*, M) (Proposition 1.)

Recall that an operator ([3]) is a rule transforming the sections of a fiber bundles (E, p, M) into sections of another fiber bundle (E', p', M'). A positive operator between the structures $[(TM, p, M); K]$ and $[(T^*M, p^*, M); K^*]$ transform the positive sections of the fiber bundles (TM, p, M) (positive vector fields) into positive sections of the fiber bundle (TM^*, p^*, M) (positive covector fields).

There is a one-to-one correspondence between vector fields and 1-forms of manifold M , given by the map

$$H : X \in \chi(M) \longrightarrow H(X) \in \Omega^1 M, H(X) = h \circ X = i_X \omega = \omega(X, .), \forall X \in \chi(M)$$

So, H is an operator from (TM, p, M) to (T^*M, p^*, M).

If $X_x \in K(x)$, then $\forall x \in M \Rightarrow H(X)(x) \in K^*(x)$, $\forall x \in M$, i.e. H is a positive operator.

If $X, Y \in \chi(M)$, then $X \leq Y \Leftrightarrow X_x - Y_x \in K(x)$, $\forall x \in M \Leftrightarrow i_{X_x}\omega_x - i_{Y_x}\omega_x \in K^*(x)$, $\forall x \in M$

$\Leftrightarrow H(X)(x) \leq H(Y)(x), \forall x \in M \Leftrightarrow H(X) \leq H(Y)$, i.e., H is monotone operator.

Moreover, H is a regular operator because every smoothly parametrized family of vector fields is transformed into a smoothly family of covector fields.

Let (M, ω) be a symplectic manifold and let J be an almost complex structure on a manifold M (a section of $End(TM)$ such as $J^2 = -Id$). The almost complex structure J on the manifold M is tamed to the symplectic form ω if $\omega(X, JX) > 0$, $\forall X \in T(M) - \{0\}$. If moreover ω is J-invariant, J is said to be calibrated. We know that any symplectic manifold have a lot of almost complex structures. The spaces of almost complex structures on a given symplectic manifold (M, ω) which are tamed (resp. calibrated) by ω is nonempty and contractible.

Let J be an almost complex structure on the manifold M, tamed by the symplectic form ω . We define the map $g(X, Y) = \omega(X, JY) - \omega(JX, Y)$, $\forall X, Y \in T(M)$.

Based on biliniarity of ω and linearity of J, the map g is bilinear map.

Also

$$g(X, X) = \omega(X, JX) - \omega(JX, X) = 2\omega(X, JX) > 0, \ \forall X \in T(M) - \{0\};$$

$$g(JX, JY) = \omega(JX, J^2Y) - \omega(J^2X, JY) = \omega(JX, -Y) - \omega(-X, JY) =$$

$$= -\omega(JX, Y) + \omega(X, JY) = g(X, Y), \ \forall X, Y \in T(M);$$

$$g(Y, X) = g(JY, JX) = \omega(JY, J^2X) - \omega(J^2Y, JX) = \omega(JY, -X)$$

$$-\omega(-Y, JX) = \omega(X, JY) - \omega(JX, Y) = g(X, Y), \ \forall X, Y \in T(M).$$

Then, g is a J-invariant Riemannian metric of TM.

Remark. Based on isomorphism $h : TM \longrightarrow T^*M$ between tangent fiber bundles TM and cotangent fiber bundle T^*M of a symplectic manifold M, if g is a Riemannian metric on TM we can construct a metric g^*on T^*M:

$$g^*(\alpha_x, \beta_x) = g(h_x^{-1}(\alpha_x), h_x^{-1}(\beta_x)), \forall x \in M, \alpha_x, \beta_x \in T_x^*M$$

Proposition 3. *Let (M, ω) be a symplectic manifold and a vector field $X \in \chi(M)$ such that $X_x \neq 0_x, \forall x \in M$. Then, there exists a field of cones on the tangent bundle (TM, p, M).*

Proof. Let g be a Riemannian metric on the TM.

For each $x \in M$, we define the set

$$K(x) = \{Y_x \in M / \frac{1}{2}(g(X_x, X_x)g(Y_x, Y_x))^{\frac{1}{2}} \leq g(X_x, Y_x) \leq$$

$$\leq (g(X_x, X_x)g(Y_x, Y_x))^{\frac{1}{2}}\} \forall Y_x \in K(x),$$

$$\lambda \in \mathbf{R}, \lambda \geq 0 \Longrightarrow \frac{1}{2}(g(X_x, X_x)g(\lambda Y_x, \lambda Y_x))^{\frac{1}{2}} \leq g(X_x, \lambda Y_x) \leq$$

$$\leq (g(X_x, X_x)g(\lambda Y_x, \lambda Y_x))^{\frac{1}{2}} \Rightarrow \lambda Y_x \in K(x),$$

then $K(x)$ is a cone. If $Y_x \in K(x)$ and $-Y_x \in K(x) \Longrightarrow Y_x = 0 \Rightarrow K(x)$ is pointed cone. Then $K(x)$ is a convex pointed closed with interior points cone of T_xM and the sets $\bigcup_{x \in M} IntK(x)$ and $\bigcup_{x \in M} (E_x - K(x))$ are open in TM.

Proposition 4. *Let (M, ω) be a symplectic manifold and smooth function $f : M \longrightarrow \mathbf{R}$ without singular points. There exists a field of cones on the tangent bundle (TM, p, M).*

Proof. Since the covector field *df* never wanish using a metric g^* on T^*M there is a field of cones on the tangent bundle (T^*M, p, M).Due of the isomorphism of structures $[(TM, p, M); K]$ and $[(T^*M, p^*, M); K^*]$, there exists a field of cones on the tangent bundle (T^*M, p, M).

Example. The cotangent bundle form a fundamental class of symplectic manifolds because they are the phase spaces of classical mechanics. Let N be a n - dimensional smooth manifold and let (T^*N, p, N) be the cotangent bundle of N. We can define a symplectic structure on the $2n$ -dimensional manifold T^*N .

Let be $q = (x, \theta) \in T^*N$ where $x = p(q) \in N$ and $\theta \in T_x^*N$.

Let $T_qp : T_q(T^*N) \longrightarrow T_xN$ be the tangent map of p.

We define a 1-form λ on manifold T^*N , $\lambda_q(X_q) = \theta(T_qp(X_q)), \forall X_q \in T_q(T^*N)$.

This 1-form is the Liouville form on T^*N. Then $\omega = -d\lambda$ is a symplectic form on T^*N.

The Liouville form is a section of nonzero covectors in cotangent fiber bundle of symplectic manifold T^*N, then there is a field of cones in tangent bundle of T^*N.

References

1. Audin, M.,Lafontaine, J., Holomorphic Curves in symplectic Geometry, ed. Basel; Boston; Berlin; Birkhauser,1994.

2. Guillemin, V., Lerman, E., Sternberg, S., Simpletic Fibration and Multiplicity Diagrams, Cambridge University Press, 1996.

3. Kolar ,I., Michor, P. W., Slovak I., Natural Operations in Differential Geometry; Springer -Verlag Berlin Heidelberg; 1993.

4. McDuff, D., Salamon, D., Introduction to symplectic Topology, Oxford Science Publications, 1994.

5. Dan I. Papuc, Partial ordering on differentiable manifolds, An. Univ. Timisoara, Seria matematica, vol. XXVI, f.1, 55-73, 1988.

6. Dan I. Papuc, On the geometry of a differentiable manifold with a partial ordering, An. Univ. Timisoara, Seria matematica, vol. XXVI, f.2, 39-48, 1988.

7. Dan I. Papuc, Field of cones and positive operators on a vector bundle, An. Univ. Timisoara, Seria matematica, vol. XXX, f.1, 39-58, 1992.

8. Dan I. Papuc, Field of cones on a tensor bundle, Proceedings of the 24 National Conference of Geometry and Topology , Timisoara, Romania, July 5-9 1994, vol.II, pp.237-242.

9. Dan I. Papuc, The Geometry of a Vector Bundle Endowen with a Cone Field, Proceedings of the third International Workshop on Differential Geometry and its applications,Sibiu, september 18-27 1997,volume 5, pp.314-321.

10. Denis Sullivan, Cycles for the dynamical study of foliated manifolds and complex manifolds, Invention Math., 36,(1976) 225-255.

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