Non-holonomic Economical Systems

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Abstract

We analyze a global nonholonomic economic system defined in [8], inspired from Gibbs modelization in thermodynamics. Four types of "simple" economic models are derived, via integral manifolds of the initial distribution.

We translate some plausible economic hypothesis into a mathematical formalism and prove local and global results about the evolution of the modelled economical systems.

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1 Introduction

In Theoretical physics, modelization has a long history, which often is identified with (a large part of) the history of great mathematical ideas. In contrast, social sciences are only at the begining of this process: a huge "data base" of observations and experiment records are still waiting for phenomenological interpretation and theories building. It is not by chance that one tries to adopt and (eventually adapt) the successful models from Physics, via a translation of notions and results.

The mathematical core of Economy already contains keywords with physical resonance, as equilibrium, entropy, stability, dynamics, and so on. The general hope is twofold: first, to pick a lucky existing matematical theory (well fitted to an economic reality) and to use it as a tool box, for interpreting the known economic data and for making economic previsions.

Secondly, we may hope that via some epistemological feed-back, ideas from Economy will impose the construction of new mathematical theories (or fundamental original new ideas discovered inside old mathematical theories), in a beneficent symbiose for both disciplines.

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Our paper contribute to the first attempt: we adopt a non-holonomic economic model from [8], analogous to one used in thermodynamics (based on a Gibbs-Pfaff equation). Via a translation into economic terms, we use geometrical information about the integral manifolds of a 1-form, in order to describe the evolution of some global or local economic systems.

2 Non-holonomic economic systems

In this section we consider a model of an economic system, similar to a thermodynamic system, defined in [8].

Denote on $\mathbf{R}^5$ the global coordinates (G, I, E, P, Q), with the following signification: G = potential of growth; I = internal stability ; E = entropy; P= price; Q = production quantity .

Consider the distribution D expressed as a kernel of an 1-form

$$\omega = dG - IdE + PdQ.$$

The equation $\omega = 0$ is the analogue of the equation Gibbs- Pfaff from thermodynamics, where the same variables have the following meanings: internal energy, temperature, entropy, pression and, respectively, volume ([8]).

The distribution D may be expressed contravariantly (in the vector field language) by

$$D = sp\{\partial_I, I\partial_G + \partial_E, \partial_P, \partial_Q - P\partial_G\}.$$

A complementary distribution (1-dimensional and even orthogonal to D with respect to the Euclidian metric on $\mathbf{R}^5$) is

$$D' = sp\{\partial_G - I\partial_E + P\partial_Q\}.$$

One knows ([2], p. 384, Obs.1.5.3.) D doesn't admit integral manifolds of dimension greater than 2 (thus it admits only integral curves and integral surfaces).

An integral surface of D is called *simple economic system* ([8], p.146).

Remark 1. Due to a theorem from [1], p. 453 (reformulated in [8], p. 146), there are four families of integral surfaces, which we describe now:

Type I. Let $\alpha : \mathbf{R}^5 \to \mathbf{R}$, $\alpha = \alpha(E, Q)$. The integral surface admits the global parametrization on $\mathbf{R}^2$, expressed in the coordinates (G, I, E, P, Q) through

$$\tilde{\alpha}(E, Q) = (\alpha, \partial_E\alpha, E, -\partial_Q\alpha, Q).$$

We remark that it is a "graphic-like" parametrization, so it has rank 2 and defines a 2-dimensional submanifold in $\mathbf{R}^5$. For simplicity, we will reprezent only the projection on the 3-plane from $\mathbf{R}^5$

$$E = 0 \quad , \quad Q = 0.$$

We denote

$$\overline{\alpha}(E,Q) = (\alpha, \partial_E \alpha, -\partial_Q \alpha).$$

Type II. Let $\beta : \mathbf{R}^5 \to \mathbf{R}$, $\beta = \beta(I,Q)$. The integral surface admits the global parametrization on $\mathbf{R}^2$, expressed in the coordinates (G,I,E,P,Q) through

$$\tilde{\beta}(I,Q) = (\beta - I\partial_I \beta, I, -\partial_I \beta, -\partial_Q \beta, Q).$$

We remark it is a "graphic-like" parametrization, so it has rank two, defining a 2-dimensional submanifold in $\mathbf{R}^5$. For simplicity, we represent only the projection on the 3-plane of $\mathbf{R}^5$, given by

$$I = 0 \quad , \quad Q = 0.$$

We denote

$$\overline{\beta}(I,Q) = (\beta - I\partial_I \beta, -\partial_I \beta, -\partial_Q \beta).$$

Type III. Let $\gamma : \mathbf{R}^5 \to \mathbf{R}$, $\gamma = \gamma(E,P)$. The integral surface admits the global parametrization on $\mathbf{R}^2$, expressed in coordinates (G,I,E,P,Q) by

$$\tilde{\gamma}(E,P) = (\gamma - P\partial_P \gamma, \partial_E \gamma, E, P, \partial_P \gamma).$$

We remark it is a "graphic-like" parametrization, so it has rank 2, defining a 2-dimensional submanifold in $\mathbf{R}^5$. For simplicity, we shall represent only the projection on the 3-plane of $\mathbf{R}^5$, given by

$$E = 0 \quad , \quad P = 0.$$

We denote

$$\overline{\gamma}(E,P) = (\gamma - P\partial_P \gamma, \partial_E \gamma, \partial_P \gamma).$$

Type IV. Let $\delta : \mathbf{R}^5 \to \mathbf{R}$, $\delta = \delta(I,P)$. The surface admits the global parametrization on $\mathbf{R}^2$, expressed in coordonates (G,I,E,P,Q) as

$$\tilde{\delta}(I,P) = (\delta - I\partial_I \delta - P\partial_P \delta, I, -\partial_I \delta, P, \partial_P \delta).$$

We remark this is also a "graphic-like" parametrization, of rank 2, defining a 2-dimensional submanifold in $\mathbf{R}^5$. For simplicity, we reprezent only the projection on the 3-plane of $\mathbf{R}^5$, given by

$$I = 0 \quad , \quad P = 0.$$

We denote

$$\overline{\delta}(I, P) = (\delta - I\partial_I\delta - P\partial_P\delta, -\partial_I\delta, \partial_P\delta).$$

3 Interaction between simple economic systems

(i) Let consider the simple economic systems, expressed by integral surfaces of D of type I, respectively II, which correspond to the functions α, $\beta : \mathbf{R}^2 \to \mathbf{R}$ respectively, defined by

$$\alpha(E, Q) = E^2 + Q^2 \quad , \quad \beta(I, Q) = I^2 + Q^2.$$

We compute

$$\tilde{\alpha}(E, Q) = (E^2 + Q^2, 2E, E, -2Q, Q),$$

$$\overline{\alpha}(E, Q) = (E^2 + Q^2, 2E, -2Q),$$

$$\tilde{\beta}(I, Q) = (-I^2 + Q^2, I, -2I, -2Q, Q),$$

$$\overline{\beta}(I, Q) = (-I^2 + Q^2, -2I, -2Q).$$

Generally, the interaction of such economic systems may be expressed also by the intersection of the corresponding integral surfaces ; thus, through a common point of these surfaces, may pass an integral surface, an integral curve or a 0-dimensional integral manifold of D (i.e. an "integral" point).

In previous example, the intersection of the two integral surfaces includes the integral curve $c : \mathbf{R} \to \mathbf{R}^5$, $c(Q) = (Q^2, 0, 0, -2Q, Q)$.

Let now consider the objective function $f : \mathbf{R}^5 \to \mathbf{R}$, $f(G, I, E, P, Q) = Q$. We remark f models the production quantity of the economic system generated by D and also that it has no (unconstrained) extremum point. (At the first view, the negative values of f seem impossible to attain in practice ; it's possible to "explain" this as import, etc.)

Consider that f is restricted to a first type simple economic system ; therefore 0 is a minimum point for f, constrained by D.

Similarly, the function f restricted to the second type simple economic system has in 0 a minimum point, constrained by D. Moreover, the restriction of the function f along the curve c has 0 as minimum point, constrained by D.

(ii) In the previous construction, the dependence of E and I on the "generating" functions α and β, respectively was not essential. So, it is possible to generalize the conclusion for the case of $\alpha(E,Q) = a^2(E) + Q^2$ and $\beta(I,Q) = b^2(I) + Q^2$, with a and b differentiable functions.

4 Interaction between global economic systems

(i) A more complex economic model supposes the existence of two or more economic systems, similar to that described by the nonholonomic manifold $(\mathbf{R}^5, \omega)$. In [8] one constructs the following model of a double economic system in which two nonholonomic manifolds interact: $(\mathbf{R}^5, \omega_1)$ and $(\mathbf{R}^5, \omega_2)$, with the global coordinates $(G_1, I_1, E_1, P_1, Q_1)$ and , respectively, $(G_2, I_2, E_2, P_2, Q_2)$. We denoted

$$\omega_1 = dG_1 - I_1 dE_1 + P_1 dQ_1,$$

$$\omega_2 = dG_2 - I_2 dE_2 + P_2 dQ_2.$$

We consider on $\mathbf{R}^5 \times \mathbf{R}^5$ the product coordinates

$$(G_1, I_1, E_1, P_1, Q_1, G_2, I_2, E_2, P_2, Q_2)$$

and the 1-forms on $\mathbf{R}^{10}$ induced by ω_1 and ω_2 (denoted in the same manner). The system

$$\omega_1 = 0 \quad , \quad \omega_2 = 0$$

generates a distribution D of constant (=8) rank, noninvolutive (thus not completely integrable) on $\mathbf{R}^{10}$. The integral manifolds of D are product of integral manifolds for the distributions on every factor, and having dimensions only 0, 1, 2, 3 or 4.

In [8] are considered the "total growth" function, "total entropy" function and "total production quantity" function, defined by $G = G_1 + G_2$, $E = E_1 + E_2$, $Q = Q_1 + Q_2$ respectively. The results proved there refer to the behaviour of the critical points of the function G, constrained by D, for points with $E =$ constant and $Q =$ constant (and similar results, where the task of E, G and Q are permutated).

The economic systems modelled by $(\mathbf{R}^5, \omega_1)$ and $(\mathbf{R}^5, \omega_2)$ are independent from the geometric point of view, as factors in the product manifold. Their only interaction manifests through the global functions G, E and Q on $\mathbf{R}^{10}$.

Next, we propose a few alternative models for the interaction of two economic systems.

(ii) **Constant coordinate model.** Let suppose that the potential of growth is the same (with small variations, this assumption is equivalent to the condition $G =$ constant from the result quoted in (i) and proved in [8]). We have $G_2 = G_1$ and it is possible to eliminate a coordinate from $\mathbf{R}^{10}$.

We consider on $\mathbf{R}^9$ the coordinates

$$(G_1, I_1, E_1, P_1, Q_1, I_2, E_2, P_2, Q_2)$$

and the distributions D_1, D_2 and D, generated by the kernel of 1-forms

$$\omega_1 = dG_1 - I_1 dE_1 + P_1 dQ_1,$$

$$\omega_2 = dG_1 - I_2 dE_2 + P_2 dQ_2,$$

$$\omega = dG_1 - I_1 dE_1 + P_1 dQ_1 - I_2 dE_2 + P_2 dQ_2.$$

Each of them has rank 8 and thus codimension 1, in $\mathbf{R}^9$. For every such distribution we can resume the study from Remark 1, with the four types of integral surfaces families.

We denote these models by $(\mathbf{R}^9, \omega_1 = 0)$, $(\mathbf{R}^9, \omega_2 = 0)$, respectively $(\mathbf{R}^9, \omega = 0)$.

The interaction between the initial economic models is possible to appear also as part of following "mixed" models: $(\mathbf{R}^9, \omega_1 = 0, \omega_2 = 0)$, $(\mathbf{R}^9, \omega_2 = 0, \omega = 0)$, $(\mathbf{R}^9, \omega_1 = 0, \omega = 0)$ (which contains the distributions of rank 7) in $\mathbf{R}^9$.

The interaction of the initial economic models is more sophisticated within the framework of these six "mixed" models than in the case of the product of the models described in (i).

(iii) Similar constructions may be obtained if we suppose another variable between I, E, P and Q to be constant.

Also, a very rich family of models appears when we fix (constant) two, three or four variables between G, I, E, P and Q; thus we obtain economic systems in $\mathbf{R}^8$, in $\mathbf{R}^7$ and in $\mathbf{R}^6$, respectively.

Case studies: (i) The function $f : \mathbf{R}^5 \to \mathbf{R}$, $f(G, I, E, P, Q) = PQ$ has no extremum points. We define $\alpha((E, Q)) = (Q - 1)^2 +$ a function of E.

Then f has a maximum for $Q = \frac{1}{2}$.

Comment: *in the simple economic system of type I, we consider a generating function α (with quadratic variation in Q (production)) and a minimum for Q = 1; (the E-variation is not important). The maximum profit is attained for the production quantity value Q = 1/2 .*

(ii) An integral curve of D admits the parametrization

$$c(t) = (\int_0^1 \{I(s)E'(s) - P(s)Q'(s)\}ds + x_0, I(t), E(t), P(t), Q(t)),$$

with I, E, P, Q differentiable arbitrary functions of t, x_0 a real constant. We suppose $x_0 = 0$.

Particular case: $c(t) = (-1/2t^2, 0, 0, t, 1-t)$; then $f(c(t)) = t - t^2$ has a maximum in $1/2$.

Comment : *consider an " economic factor" described by c, with linear variation in price and production quantity, in inverse proportionality. It follows the growth potential has quadratic variation (with minus), and the maximum profit is attaind for* $t = 1/2$.

(iii) Alternatively, an integral curve of D admits also the parametrization

$$c(t) = (G(t), I(t), E(t), \{I(t)E'(t) - G'(t)\}/Q'(t), Q(t)),$$

where the fourth component is the price.

In particular, if $G(t) = t$; $I(t) = sint$; $E(t) = cost$; $Q(t) = t$, we obtain $P(t) = -sin2t$; $f(c(t)) = -tsin2t$, with an infinity of extremum points.

Comment: *initially (at* $t = 0$*) the profit decreases; next, it fluctuates cvasiperiodically, with increasing amplitude as in* Fig. 1.

Fig.1: the variation of the profit vs. time

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