HIGHER ORDER GEOMETRY AND INDUCED OBJECTS

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Abstract

The higher order bundles are considered using a vector pseudo-field on, in an inductive manner. The main ideas of our construction can be used as well in other cases.

A dual theory between lagrangians and hamiltonians (via Legendre transformations) is considered, in a similar way as R.Miron. A canonical way to induce a hamiltonian on an affine subbundle is given, too.

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A theory of higher order Finsler and Lagrange spaces was developped in [17, 10] using the bundles of accelerations. A dual theory of higher order Hamilton spaces was recentely studied in [12, 13]. The theory of Finsler and Lagrange submanifolds is studied in many papers (for example [1, 2, 3, 4, 5, 15, 19, 20, 21, 22, 28]). A theory of Hamilton submanifolds of order one was also studied by R. Miron in [18, 11] and the case of higher order is considered in [14]. In these approaches one define an induced hamiltonian on the submanifold, which is not intrinsic (the induction procedure is not uniquely defined by the hamiltonian and the submanifold). A canonical way to induce a hamiltonian (of order one) on a submanifold is given in [30, 31], solving a problem of R. Miron [18, 11] concerning the possibility to induce a hamiltonian on a submanifold *in an intrinsic way.*

The aim of this paper is to induce canonically a hamiltonian of higher order on a submanifold using the construction given in [32]. In the first section we give a recursive definition of the k-acceleration bundles (as affine bundles) and we revise some known constructions and results related to the geometry of the k-acceleration bundle and its dual [14]. A *canonical way* to induce a hamiltonian on an submanifold is given in the second section.

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1 An inductive construction of the higher acceleration bundles

Let M be a manifold of dimension m and $\tau M = (TM, \pi, M)$ its tangent bundle. Considering an atlas of M, we denote by (x^i) the coordinates on an arbitrary domain $U \subset M$ and by (x^i, y^j) the coordinates on the domain $\pi^{-1}(U) \subset TM$ $(i, j = \overline{1, m})$. On the intersection of two open domains of coordinates on TM, the coordinates change according the rule

$$x^{i\prime} = x^{i\prime}(x^i),\ y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i.$$

A surjective submersion $E \xrightarrow{\pi} M$ is usually called a *fibered manifold*. An *affine bundle* $E \xrightarrow{\pi} M$ is a fibered manifold which the change rules of the local coordinates on E have the form

$$\bar{x}^i = \bar{x}^i(x^j), \bar{y}^\alpha = g^\alpha_\beta(x^j) y^\beta + v^\alpha(x^j). \tag{1}$$

An *affine section* in the bundle E is a differentiable map $M \xrightarrow{s} E$ such that $\pi \circ s = id_M$ and its local components change according to the rule $\bar{s}^\alpha(\bar{x}^i) = g^\alpha_\beta(x^j)\bar{s}^\beta(x^j) + v^\alpha(x^j)$. The set of affine sections is denoted by $\Gamma(E)$ and it is an affine module over $\mathcal{F}(M)$, i.e. for every $f_1, \ldots, f_p \in \mathcal{F}(M)$ such that $f_1 + \cdots + f_p = 1$ and $s_1, \ldots, s_p \in \Gamma(E)$, then $f_1 s_1 + \cdots + f_p s_p \in \Gamma(E)$, where the affine combination is taken in every point of the base. Using a partition of unity on the base M it can be easily proved that every affine bundle allows an affine section.

A vector bundle $\bar{E} \xrightarrow{\bar{\pi}} M$ can be canonically associated with the affine bundle $E \xrightarrow{\pi} M$. More precisely, using local coordinates, the coordinates change on $\bar{E}$ following the rules $\bar{x}^i = \bar{x}^i(x^j)$, $\bar{z}^\alpha = g^\alpha_\beta(x^j) z^\beta$, when the coordinates on E change accoding the formulas (1).

Every vector bundle is an affine bundle, called a *central affine bundle*. In this case $v^\alpha(x^j) = 0$.

The tangent bundle TM is an affine bundle, but, for $k \geq 2$, the k-accelerations bundles $T^k(M)$ are affine bundles over $T^{k-1}(M)$. They can be defined inductively as follows.

Let us denote $M = T^0 M$, $\pi = \pi_1$ and $TM = T^1 M$ and consider element $\Gamma \in \mathcal{F}(TM) \otimes_{\mathcal{F}(M)} \mathcal{X}(M)$ defined locally by $\Gamma = y^i \dfrac{\partial}{\partial x^i}$, where $\dfrac{\partial}{\partial x^i}$ are local fields on M. Let us associate with every domain $\pi^{-1}(U) \subset TM$ the vector field $\Gamma^{(1)}_U = y^i \dfrac{\partial}{\partial x^i}$, , where $\dfrac{\partial}{\partial x^i}$ are local fields on TM. We call this association a *vector pseudofield* on TM.

Let us suppose that the vector pseudofields $\Gamma^{(r)}$ on $T^r M$ and the r-acceleration bundles $T^r M$ have been defined for $1 \leq r \leq k-1$ as affine bundles over $T^{r-1}M$. We define $T^k M$ by the following change rule of the local coordinates:

$$k y^{(k)i'} = k \frac{\partial x^{i'}}{\partial x^i} y^{(k)i} + \Gamma^{(k-1)}_U (y^{(k-1)i'})$$

and $\Gamma_U^{(k)} = \Gamma_U^{(k-1)} + ky^{(k)i}\dfrac{\partial}{\partial y^{)k-1)i}}$, where $\Gamma_U^{(k-1)}$ is considered as a (local) vector field on T^kM and U is the domain which corresponds to the coordinates (x^i). Notice that

$$\Gamma_U^{(k)} = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}$$

and on the intersection of two domains corresponding to U and U', we have:

$$\Gamma_{U'}^{(k)} = \Gamma_U^{(k)} - \Gamma_U^{(k)}(y^{(k)i'})\frac{\partial}{\partial y^{(k)i'}}$$

Indeed, according to the recursive definitions of $\Gamma_U^{(r)}$ and $\Gamma_{U'}^{(r)}$ we have $\Gamma_{U'}^{(k)}(x^{i'}) = \Gamma_U^{(k)}(x^{i'})$, $\Gamma_{U'}^{(k)}(y^{(r)i'}) = \Gamma_U^{(k)}(y^{(r)i'})$, for $r = \overline{1, r-1}$ and
$\Gamma_{U'}^{(k)}(y^{(k)i'}) = 0$, $(\forall)i' = \overline{1,m}$; thus $\Gamma_U^{(k)} = \Gamma_U^{(k)}(x^{i'})\dfrac{\partial}{\partial x^{i'}} + \Gamma_U^{(k)}(y^{(1)i'})\dfrac{\partial}{\partial y^{(k)i'}} + \cdots + \Gamma_U^{(k)}(y^{(k-1)i'})\dfrac{\partial}{\partial y^{(k-1)i'}} + \Gamma_U^{(k)}(y^{(k)i'})\dfrac{\partial}{\partial y^{(k)i'}} = \Gamma_{U'}^{(k)} + \Gamma_U^{(k)}(y^{(k)i'})\dfrac{\partial}{\partial y^{(k)i'}}$.

Proposition 1.1 *The fibered manifold $(T^kM, p_k, T^{k-1}M)$ is an affine bundle, for $k \geq 2$.*

Proof. Let us consider $(x^i, y^{(1)i}, \ldots, y^{(k)i})$, $(x^{i'}, y^{(1)i'}, \ldots, y^{(k)i'})$ and $(x^{i''}, y^{(1)i''}, \ldots, y^{(k)i''})$ as being coordinates on three domains of adapted coordinates on T^kM, corresponding to U, U' and U'' and having nonvoid intersection.

We have $ky^{(k)i'} = k\dfrac{\partial x^{i'}}{\partial x^i}y^{(k)i} + \Gamma_U^{(k)}(y^{(k-1)i'})$ and $ky^{(k)i''} = k\dfrac{\partial x^{i''}}{\partial x^{i'}}y^{(k)i'} + \Gamma_{U'}^{(k)}(y^{(k-1)i''})$. It suffices to prove that

$$ky^{(k)i''} = k\frac{\partial x^{i''}}{\partial x^i}y^{(k)i} + \Gamma_U^{(k-1)}(y^{(k-1)i''}).$$

Indeed, $ky^{(k)i''} = k\dfrac{\partial x^{i''}}{\partial x^{i'}}y^{(k)i'} + \Gamma_{U'}^{(k-1)}(y^{(k-1)i''}) = \dfrac{\partial x^{i''}}{\partial x^{i'}}(\dfrac{\partial x^{i'}}{\partial x^i}ky^{(k)i} + \Gamma_U^{(k-1)}(y^{(k-1)i'})) + \Gamma_U^{(k-1)}(y^{(k-1)i''}) - \Gamma_U^{(k-1)}(y^{(k)i'})\dfrac{\partial y^{(k-1)i''}}{\partial y^{(k-1)i'}} = k\dfrac{\partial x^{i'}}{\partial x^i}y^{(k)i} + \Gamma_U^{(k-1)}(y^{(k-1)i''})$, since $\dfrac{\partial x^{i''}}{\partial x^{i'}}\dfrac{\partial x^{i'}}{\partial x^i} = \dfrac{\partial x^{i''}}{\partial x^i}$ and $\dfrac{\partial y^{(k-1)i''}}{\partial y^{(k-1)i'}} = \dfrac{\partial x^{i''}}{\partial x^{i'}}$. □

Let $\ker \pi_{k*} = V_0T^kM$, be the vertical vector bundle of T^kM (viewed as a fibered manifold over $T^{k-1}M$) and $\Gamma(V_0T^kM)$ be the module of the vertical sections. The local coordinates on the fibers of V_0T^kM have the form (Y^i) and change according to the rules $Y^{i'} = \dfrac{\partial x^{i'}}{\partial x^i}Y^i$. If $S : T^kM \to V_0T^kM$ is a section, then it has the local form $S = S^i\dfrac{\partial}{\partial y^{(k)i}}$ and $T_S = \dfrac{\partial S^i}{\partial y^{(k)j}}\dfrac{\partial}{\partial y^{(k)i}} \otimes dy^{(k)j}$ defines an endomorphism on the fibers of V_0T^kM.

A *Liouville type* section is a vertical section $S \in \Gamma(V_0T^kM)$ which T_S is the identity on the fibers of V_0T^kM; it has the local form
$S^i(x^i, y^{(1)i}, \ldots, y^{(k)i}) = (y^{(k)i} + t^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}))\dfrac{\partial}{\partial y^{(k)i}}$.

Proposition 1.2 *There is an one to one correspondence between the Liouville type sections in* $\Gamma(V_0T^kM)$ *and the affine sections in* $T^kM \to T^{k-1}M$.

Proof. Let $S \in \Gamma(V_0T^kM)$ be a Liouville section. The change rules of the local functions S^i are $S^{i'} = \dfrac{\partial x^{i'}}{\partial x^i}S^i$. Taking into account the forms of $S^{i'}$ and S^i, it follows that $y^{(k)i'} + t^{i'}(x^{i'}, y^{(1)i'}, \ldots, y^{(k-1)i'}) = \dfrac{\partial x^{i'}}{\partial x^i}(y^{(k)i} + t^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}))$. Since $ky^{(k)i'} = k\dfrac{\partial x^{i'}}{\partial x^i}y^{(k)i} + \Gamma_U^{(k-1)}(y^{(k-1)i'})$, it follows that $-kt^{i'}(x^{i'}, y^{(1)i'}, \ldots, y^{(k-1)i'}) = -k\dfrac{\partial x^{i'}}{\partial x^i}t^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}) + \Gamma_U^{(k-1)}(y^{(k-1)i'})$, thus the local functions $(-t^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}))$ are the local components of a global section from $\Gamma(T^kM)$. Conversely, for a global section $s \in \Gamma(T^kM)$ having the local components $(s^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}))$, the local functions $(y^{(k)i} - s^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}))$ on T^kM are the local components of a Liouville type section. □

The composition $\pi^k = \pi_1 \circ \cdots \circ \pi_k : T^kM \to M$ define a fibered manifold on T^kM on the base M. It is not an affine bundle. The vector bundle $VT^kM = \ker \pi^k_*$ is the vertical bundle of T^kM. The Liouville section is the vertical section defined (locally) by $\overset{k}{\Gamma} = y^{(1)i}\dfrac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\dfrac{\partial}{\partial y^{(2)i}} + \cdots + ky^{(k)i}\dfrac{\partial}{\partial y^{(k)i}}$. Notice that it is a global section on VT^kM. A special endomorphism on T^kM is defined by

$$\begin{aligned} J\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial}{\partial y^{(1)i}}, J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, \ldots, J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = \frac{\partial}{\partial y^{(k)i}}, \\ J\left(\frac{\partial}{\partial y^{(k)i}}\right) &= 0, \end{aligned}$$

called the k-tangent structure.

A k-(semi)spray is a vector field $S \in \mathcal{X}(T^kM)$ with the property $J(S) = \overset{k}{\Gamma}$. A k-(semi)spray S has the local form $S = \Gamma^{(k)} + S^i(x^i, y^{(1)i}, \ldots, y^{(k)i})\dfrac{\partial}{\partial y^{(k)i}}$. The change rule of the local functions S^i shows that they define a (global) affine section on the affine bundle $T^{k+1}M$ over the base T^kM. Conversely, every affine section on the affine bundle $T^{k+1}M$ define, using their components, a k-(semi)spray.

Proposition 1.3 *There is an one to one correspondence between the Liouville type sections in* $\Gamma(V_0T^kM)$ *and the* $(k-1)$*-(semi)sprays on* M.

A nonlinear connection on the k-tangent bundle T^kM is a left splitting C of the inclusion $VT^kM \to T(T^kM)$ or, equivalently, a right splitting D of the projection

$T(T^kM) \to (\pi^k)^*TM$, which define the horizontal lift. Using local coordinates, one can consider the coefficients and the dual coefficients of a nonlinear connections. The coefficients of the nonlinear connection are defined by the condition

$$D\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}_i^j \frac{\partial}{\partial y^{(1)j}} - \cdots - \underset{(k)}{N}{}_i^j \frac{\partial}{\partial y^{(k)j}} \overset{not.}{=} \frac{\delta}{\delta x^i}.$$

An adapted base of the local vector fields on T^kM is

$$\mathcal{B}_1 = \left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}} = J\left(\frac{\delta}{\delta x^i}\right), \ldots, \frac{\delta}{\delta y^{(k)i}} = J^{k-1}\left(\frac{\delta}{\delta x^i}\right), \frac{\delta}{\delta y^{(k)i}} = \frac{\partial}{\partial y^{(k)i}} \right\}.$$

The expression of the dual base of $\mathcal{B}_1$:

$$\mathcal{B}_1^* = \left\{ \delta x^i = \partial x^i, \partial y^{(1)i}, \ldots, \partial y^{(r)i} \right\}$$

with respect to the canonical dual base $\mathcal{B}^* = \{dx^i, dy^{(1)i}, \ldots, dy^{(k)i}\}$, in the form:

$$\begin{aligned} \delta x^i &= dx^i \\ \delta y^{(1)i} &= dy^{(1)i} + \underset{(1)}{M}{}_j^i dx^j \\ & \ldots \\ \delta y^{(1)i} &= dy^{(k)i} + \underset{(1)}{M}{}_j^i dy^{(k-1)i} + \cdots + \\ & \quad + \underset{(k-1)}{M}{}_j^i dy^{(1)i} + \underset{(k)}{M}{}_j^i dx^i \end{aligned}$$

define the dual coefficients.

In [12]-[14] there is used the Liouville type vector field (called a k-Liouville d-vector field) on T^kM, given by a nonlinear connection on $T^{k-1}M$, which has the form

$$kS^{(k)i} = ky^{(k)i} + (k-1)\underset{(1)}{M}{}_j^i y^{(k-1)j} + \cdots + \underset{(k-1)}{M}{}_j^i y^{(1)j}.$$

We recall breafly the construction of the Legendre and Legendre* transformations using a Lagrangian and a Hamiltonian respectively.

A *lagrangian* of order k on M is a continous function $L : T^kM \to I\!R$, differentiable on $\widetilde{T^kM}$ (i.e. T^kM without the null section). The lagrangian is *regular* if the vertical Hessian $\left(\frac{\partial^2 L}{\partial y^\alpha y^\beta}\right)$ of L is non-degenerate. In this case the vertical hessian defines a (pseudo)metric structure on the fibers of the vertical bundle $V\widetilde{T^kM}$. In order to have more generality, we remove in that follows the continuity of L in the points situated in the image of the null section.

The vector bundle canonically associated with the affine bundle $(T^kM, p_k, T^{k-1}M)$ is the vector bundle q_{k-1}^*TM, where $q_{k-1} : T^{k-1}M \to M$ is $q_{k-1} = p_1 \circ p_2 \circ \cdots \circ p_{k-1}$. The fibered manifold (T^kM, q_k, M) is systematically used in [?, 10] in the study of the geometrical objects of order k on M, in particular the Lagrangians of order k on M.

The total space of the dual $q_{k-1}^*T^*M$ of the vector bundle q_{k-1}^*TM is also the total space of the fibered manifold $(T^{k-1}M \times_M T^*M, r_k, M)$ and is used in [12, **?**, 14] in the study of the dual geometrical objects of order k on M, in particular the Hamiltonians of order k on M. In the sequel we denote $q_{k-1}^*T^*M = T^{k*}M$ as an affine bundle over $T^{k-1}M$.

A *hamiltonian* of order k on M is a continuous function $H : T^{k*}M \to I\!R$, differentiable on $\widetilde{T^{k*}M}$ (i.e. $T^{k*}M$ without the null section). The hamiltonian is *regular* if the vertical Hessian $\left(\dfrac{\partial^2 H}{\partial p_i p_j}\right)$ of H is non-degenerate. In this case the vertical hessian defines a (pseudo)metric structure on the fibers of the vertical bundle $V\widetilde{T^{k*}M}$. In order to have more generality, as in the case of lagrangians, we remove in that follows the continuity of H in the points situated in the image of the null section.

If $L : T^kM \to I\!R$ is a lagrangian, then the *Legendre transformation* is the fibered manifold map $\mathcal{L} : \widetilde{T^kM} \to \widetilde{T^{k*}M}$ (both on the base $T^{k-1}M$) defined in local coordinates on the fibers by $(y^{(k)i}) \xrightarrow{\mathcal{L}} (p_i = \dfrac{\partial L}{\partial y^{(k)i}}(x^i, y^{(1)i}, \ldots, y^{(k)i}))$. It is easy to see that if L is a regular lagrangian, then $\mathcal{L}$ is a local diffeomorphism. Considering a regular lagrangian locally, we can suppose that $\mathcal{L}$ is a global diffeomorphism.

The Legendre transformation defines an $\mathcal{L}$-morphism of the vertical vector bundles $V\widetilde{T^kM} \to V\widetilde{T^{k*}M}$ (called the *vertical Legendre morphism*) and expressed in local coordinates on fibers by $(y^{(k)i}, Y^j) \to (\dfrac{\partial L}{\partial y^{(k)i}}, Y^j \dfrac{\partial^2 L}{\partial y^{(k)j} y^{(k)k}})$.

Theorem 1.1 *Let $s : T^{k-1}M \to T^kM$ be an affine section and $L : T^kM \to I\!R$ be a regular lagrangian.*

Then there is a hamiltonian $H : T^{k*}M \to I\!R$ defined by L and s such that the vertical Legendre morphism is an isometry and the vertical hessian of H does not depend on the section s.

Proof. Let $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}) \xrightarrow{s} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, s^j(x^i, y^{(1)i}, \ldots, y^{(k-1)i}))$ be the local form of the section s. According to Proposition 1.2, the section s defines a Liouville section $S : T^{k-1}M \to VT^{k-1}M$ given in local coordinates by $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}) \xrightarrow{S} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, y^{(k-1)i} - s^i(y^{(1)i}, \ldots, y^{(k-1)i}))$. Since L is non-degenerate it means that $\mathcal{L}$ is a diffeomorphism, thus consider $\mathcal{H} = \mathcal{L}^{-1} : T^{k*}M \to T^kM$ and denote by $\bar{S} = S \circ \mathcal{H}{:}T^{k*}M \to VT^{k-1}M$. Notice that $\mathcal{H}$ has the local form

$$(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \xrightarrow{\mathcal{H}} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i(x^i y^{(1)i}, \ldots, y^{(k-1)i}, p_i)),$$

where

$$\begin{aligned} H^j(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \frac{\partial L}{\partial y^{(k)i}}(x^i, y^{(1)i}, \ldots, y^{(k)i})) &= y^j \text{ and} \\ \frac{\partial L}{\partial y^{(k)j}}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)) &= p_j. \end{aligned}$$

Differentiating the first formula, we obtain:

$$\frac{\partial^2 L}{\partial y^{(k)u}\partial y^{(k)w}}(x^i, y^{(1)i}, \ldots, y^{(k)i}) \cdot \tag{2}$$
$$\frac{\partial H^w}{\partial p_v}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \frac{\partial L}{\partial y^{(k)i}}(x^i, y^{(1)i}, \ldots, y^{(k)i})) \quad = \quad \delta_{uv}.$$

Substituting $y^{(k)j} = H^j(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)$ we also have

$$\frac{\partial^2 L}{\partial y^{(k)u}\partial y^{(k)w}}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \ldots, y^{(k)i}, p_i)) \cdot \tag{3}$$
$$\frac{\partial H^w}{\partial p_v}(x^i, y^{(1)i}, \ldots, y^{(k)i}) \quad = \quad \delta_{uv}.$$

Then $\bar{S}$ has the form

$$(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \xrightarrow{\bar{S}}$$
$$(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i), H^i(x^i, y^{(1)i}, \ldots,$$
$$y^{(k-1)i}, p_i) - s^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)).$$

We define $H : T^{k*}M \to I\!R$ using the formula

$$H(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \quad = \quad p_j(H^j(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) - \tag{4}$$
$$s^j(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)) -$$
$$L(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, H^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)).$$

It is easy to see that H is globally defined on $T^{k*}M$. In order to prove that the vertical hessian of H is non-degenerate and also that the vertical bundle morphism is an isometry, it suffices to prove that

$$\left(\frac{\partial H^2}{\partial p_u \partial p_v}\left(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \frac{\partial L}{\partial y^i}\left(x^i, y^{(1)i}, \ldots, y^{(k)i}\right)\right)\right) \quad =$$
$$\left(\frac{\partial L^2}{\partial y^u \partial y^v}\left(x^i, y^{(1)i}, \ldots, y^{(k)i}\right)\right)^{-1}.$$

This can be obtained by a straightforward computation, as follows. Using formula (2), we obtain $\frac{\partial H}{\partial p_j}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = H^j(x^i, y^{(1)i}, \ldots,$ $y^{(k-1)i}, p_i)$, then using the relations (3) and (2), the above formula follows. It is easy to see that the vertical hessian of the hamiltonian does not depend on the section s. □

An inverse construction is performed in the sequel. Starting from a hamiltonian, a lagrangian on T^kM can be constructed.

Given a hamiltonian $H : \widetilde{T^{k*}M} \to I\!R$ and a section s of T^kM, the *Legendre* transformation* is the fibered manifold morphism $\mathcal{H} : \widetilde{T^{k*}M} \to \widetilde{T^kM}$ defined by the

local formula $(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i) \xrightarrow{\mathcal{H}}$
$(x^i, y^{(1)i}, \dots, y^{(k-1)i}, \frac{\partial H}{\partial p_i}(x^j, y^{(1)j}, \dots, y^{(k-1)j}, p_j) + s^i(x^j))$. If the hamiltonian is regular, then the Legendre* transformation is a diffeomorphism.

The Legendre* transformation defines an $\mathcal{H}$-morphism of the vertical vector bundles $V\widetilde{T^{k*}M} \to V\widetilde{T^kM}$ (called the *vertical Legendre* morphism*) and expressed in local coordinates by $(x^i, y^{(1)i}, \dots, y^{(k-1)i}, p_i, P_i) \to (x^i, y^{(1)i}, \dots, y^{(k-1)i}, \frac{\partial H}{\partial p_i}(x^j, y^{(1)j},$ $\dots, y^{(k-1)j}, p_j) + s^i(x^i, y^{(1)i}, \dots, y^{(k-1)i}),\ P^u \frac{\partial^2 H}{\partial p_i \partial p_u}(x^j, y^{(1)j}, \dots, y^{(k-1)j}, p_j))$.

Theorem 1.2 *Let $s : M \to \widetilde{T^kM}$ be an affine section and $H : T^{k*}M \to I\!R$ be a non-degenerate hamiltonian.*

Then there is a lagrangian $L : T^kM \to I\!R$ of order k on M such that the vertical Legendre morphism is an isometry and the vertical hessian of L does not depend on the section s.*

Proof. The proof is analogous to the proof of Theorem 1.1. In fact we reverse the order of H and L in the construction of H in the formula (4). We denote by $\mathcal{L} = \mathcal{H}^{-1} : T^kM \to T^{k*}M$ the inverse of the Legendre* transformation. It has the local form

$$(x^i, y^{(1)i}, \dots, y^{(k)i}) \xrightarrow{\mathcal{L}} (x^i, y^{(1)i}, \dots, y^{(k-1)i}, L_i(x^j, y^{(1)j}, \dots, y^{(k)j})),$$

where

$$\begin{aligned}
&L_i(x^j, y^{(1)j}, \dots, y^{(k-1)j}, \frac{\partial H}{\partial p_j}(x^u, y^{(1)u}, \dots, y^{(k-1)u}, p_u) + \\
&s^j(x^u, y^{(1)u}, \dots, y^{(k-1)u})) = p_i \text{ and} \\
&\frac{\partial H}{\partial p_i}(x^j, y^{(1)j}, \dots, y^{(k-1)j}, L_j(x^u, y^{(1)u}, \dots, y^{(k)u})) + \\
&s^i(x^j, y^{(1)j}, \dots, y^{(k-1)j}) = y^i.
\end{aligned}$$

One defines $H : \widetilde{T^{k*}M} \to I\!R$ using the formula

$$\begin{aligned}
&L(x^i, y^{(1)i}, \dots, y^{(k)i}) = \\
&L_i(x^j, y^{(1)j}, \dots, y^{(k)j}) \left(y^i - s^i(x^j, y^{(1)j}, \dots, y^{(k-1)j})\right) - \\
&H(x^i, y^{(1)i}, \dots, y^{(k-1)i}, L_i(x^j, y^{(1)j}, \dots, y^{(k)j})).
\end{aligned} \tag{5}$$

The proof follows in the same manner as the proof of Theorem 1.1. □

2 Induced hamiltonians on submanifolds

Besides the theory of Lagrange and Finsler submanifolds, which is studied by many authors, (see the Bibliography), an attempt to study the Hamilton submanifolds is performed in [18, 11], using an arbitrary section of the natural projection of the cotangent bundles. In [30] we have shown that there is a distinguished section, which depends only on the Hamiltonian. It solves a problem from [18, 11], concerning the possibility to induce in an intrinsic way a hamiltonian on a submanifold. Following a similar ideea, we show that an analogous result in the higher order.

If $E \xrightarrow{\pi} M$ is an affine bundle then an *affine subbundle* of E is an affine bundle $E' \xrightarrow{\pi'} M'$ such that $E' \subset E$ and $M' \subset M$ are submanifolds, π' is the restriction of π and the affine structure on the fibers of E' is induced by the affine structure on the fibers of E.

Consider $M' \subset M$ a submanifold and denote by $i : M' \to M$ the inclusion. Consider some coordinates on M, along M', adapted to the submanifold M'. It means that the coordinates have the form $(x^i)_{i=\overline{1,m}} = (x^\alpha)_{\alpha=\overline{1,m'}} \cup (x^{\bar{\alpha}})_{\bar{\alpha}=\overline{m'+1,m}}$ and the points in M' are characterized by $x^{\bar{\alpha}} = 0$, $(\forall)\bar{\alpha} = \overline{m'+1,m}$. Using these coordinates on T^kM, the inclusion $i^k : T^kM' \to T^kM$ has the local form $(x^\alpha, y^{(1)\alpha}, \ldots, y^{(k)\alpha}) \to (x^\alpha, x^{\bar{\alpha}} = 0, y^{(1)\alpha}, y^{(1)\bar{\alpha}} = 0, \ldots, y^{(k)\alpha}, y^{(k)\bar{\alpha}} = 0)$.

We consider also a section $s : T^{k-1}M \to T^kM$ which, in general, may not restricts to a section $s' : T^{k-1}M' \to T^kM'$. If the section $s : T^{k-1}M \to T^kM$ restricts to a section $s' : T^{k-1}M' \to T^kM'$ we say that s is *adapted* to the submanifold M.

There are some local coordinates (x^α) on M' and $(x^\alpha, y^{(1)\alpha}, \ldots, y^{(k)\alpha})$ on T^kM' which extend to local coordinates $(x^i) = (x^\alpha, x^{\bar{\alpha}})$ on M and $(x^i, y^\alpha) = (x^\alpha, x^{\bar{\alpha}}, y^{(1)\alpha}, y^{(1)\bar{\alpha}}, \ldots, y^{(k)\alpha}, y^{(k)\bar{\alpha}})$ on T^kM respectively, such that the points in M' and in T^kM' are characterized by the conditions $x^{\bar{\alpha}} = 0$ and $x^{\bar{\alpha}} = y^{(1)\bar{\alpha}} = \cdots = y^{(k)\bar{\alpha}} = 0$ respectively. ($i, j, k, \ldots = \overline{1,m}$, $m = \dim M$, $\alpha, \beta, \ldots = \overline{1,m'}$, $\bar{\alpha}, \bar{\beta}\bar{v}, \ldots \in \overline{m'+1,m}$, $m' = \dim M'$.

We consider also local coordinates $(x^\alpha, , y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, p_a)$ on $T^{k*}M'$ and $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) = (x^\alpha, , y^{(1)\alpha}, y^{(1)\bar{\alpha}}, \ldots, y^{(k-1)\alpha}, y^{(k-1)\bar{\alpha}}, p_a, p_{\bar{\alpha}})$ on $T^{k*}M$, which are adapted to the vector bundle structures and to the submanifolds structures. The local form of the section s is $(x^i, y^{(1)i}, \ldots, y^{(k-1)i}) \xrightarrow{s} (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, s^i(x^i, y^{(1)i}, \ldots, y^{(k-1)i}))$, where $s^{\bar{a}}(x^u, 0) = 0$.

The local form of the Legendre* transformation $\mathcal{H}$ is

$$\begin{aligned}(x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i) \to \qquad & (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, \frac{\partial H}{\partial p_i}(x^j, y^{(1)j}, \ldots, \\ & y^{(k)j}, p_j) + s^i(x^i, , y^{(1)i}, \ldots, y^{(k-1)i})),\end{aligned}$$

and we denote $\dfrac{\partial H}{\partial p_i}(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j) = H^i(x^j, y^{(1)j}, \ldots, y^{(k-1)j}, p_j)$. The local forms of the inclusions $i : M' \to M$, $I : T^kM' \to T^kM$ and of the canonical projection $I^* : T^{k*}M \to T^{k*}M'$ are

$$(x^\alpha) \xrightarrow{i} (x^\alpha, 0), (x^\alpha, y^{(1)\alpha}, \ldots, y^{(k)\alpha}) \xrightarrow{I} (x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k)\alpha}, 0),$$

and

$$(x^\alpha, x^{\bar{\alpha}}, y^{(1)\alpha}, y^{(1)\bar{\alpha}}, \ldots, y^{(k-1)\alpha}, y^{(k-1)\bar{\alpha}}, p_a, p_{\bar{\alpha}}) \xrightarrow{I^*} (x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, p_a),$$

respectively.

Let $H : \widetilde{T^{k*}M} \to I\!R$ be a regular hamiltonian, thus the Legendre* transformation $\mathcal{H} : \widetilde{T^{k*}M} \to \widetilde{T^kM}$ is a diffeomorphism. We denote by $(x^i, y^{(1)i}, \ldots, y^{(k)i}) \to (x^i, y^{(1)i}, \ldots, y^{(k-1)i}, L_i(x^i, y^{(1)i}, \ldots, y^{(k)i}))$ the local form of $\mathcal{L} = \mathcal{H}^{-1} : \widetilde{T^kM} \to \widetilde{T^{k*}M}$, the inverse of the Legendre* transformation.

We have that $W' = \mathcal{L} \circ I(\widetilde{T^kM'})$ is a submanifold of $\widetilde{T^{k*}M}$.

Proposition 2.1 *The restriction of I^* to W', $I^*_{|W'} : W' \to \widetilde{T^{k*}M'}$ is a diffeomorphism.*

Proof. We have: $\mathcal{L}$ is a diffeomorphism, I^* is a surjective submersion and I is an injective immersion. The local form of $I^* \circ \mathcal{L} \circ I$ is $(x^\alpha, y^{(1)\alpha}, \ldots, y^{(k)\alpha}) \to (x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, L_\alpha(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k)\alpha},\ 0))$, thus it is a local diffeomorphism. In fact $I^* \circ \mathcal{L} \circ I$ is a diffeomorphism, since it sends the fibre $\widetilde{T^kM'}_x$ in the fibre $\widetilde{T^{k*}M'}_x$ for every $x \in T^{k-1}M'$ and $\mathcal{L}$ is a diffeomorphism when it is restricted to the fiber, thus $I^*_{|W'}$ is also a diffeomorphism. □

Taking into account of the local form of the Legendre* transformation and of the local coordinates, it follows that the points of the submanifold W' have as coordinates $(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}, p_\alpha, Q_{\bar{a}}(x^\alpha, y^{(1)\alpha}, \ldots,$
$y^{(k-1)\alpha}, p_\alpha))$ in $\widetilde{T^{k*}M}$, where

$$\frac{\partial H}{\partial p_{\bar{\alpha}}}(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}, p_\alpha, Q_{\bar{\alpha}}(x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, p_\alpha)) + s^{\bar{\alpha}}(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}) = 0. \tag{6}$$

Differentiating this equation with respect to p_α, we get:

$$\frac{\partial^2 H}{\partial p_\alpha \partial p_{\bar{\alpha}}} + \frac{\partial^2 H}{\partial p_{\bar{\beta}} \partial p_{\bar{\alpha}}} \cdot \frac{\partial Q_{\bar{\beta}}}{\partial p_\alpha} = 0.$$

Denoting by $h^{\alpha\beta} = \dfrac{\partial^2 H}{\partial p_\alpha \partial p_\beta}$, we suppose that the matrix $\tilde{h} = (h^{\bar{\alpha}\bar{\beta}})_{\bar{\alpha},\bar{\beta}=\overline{m'+1,m}}$ is non-degenerate; if this condition holds, we say that the Hamiltonian is *non-degenerate* along the affine subbundle E' (notice that this condition automatically holds when the vertical hessian of the Hamiltonian defines a positive quadratic form). Considering the inverse $\tilde{h}^{-1} = (\tilde{h}_{\bar{\alpha}\bar{\beta}})_{\bar{\alpha},\bar{\beta}=\overline{m'+1,m}}$, it follows that

$$\frac{\partial Q_{\bar{\beta}}}{\partial p_\alpha} = -h^{\alpha\bar{\alpha}}\tilde{h}_{\bar{\alpha}\bar{\beta}}. \tag{7}$$

Denote $\bar{I} = I^{*-1}_{|W'} : \widetilde{T^{k*}M'} \to W' \subset \widetilde{T^{k*}M}$. Using the above constructions, we obtain the following result.

Theorem 2.1 *The map $\bar{I}$ is a section of I^* which depends only on H and s. If the section s is adapted, then the map $\bar{I}$ depend only on the hamiltonian H.*

We define $H' = H \circ \bar{I} : \widetilde{T^{k*}M'} \to I\!R$ and we consider the vertical Hessian of H':

$$\left(\frac{\partial^2 H'}{\partial p_\alpha \partial p_\beta} (x^\gamma, y^{(1)\gamma}, \ldots, y^{(k-1)\gamma}, p_\gamma) \right)_{\alpha,\beta=\overline{1,m'}}$$

in every point of $\widetilde{T^{k*}M'}$.

Proposition 2.2 *a) If the Hamiltonian H is non-degenerate along the submanifold W', then H' is a regular Lagrangian.*

b) If the Hamiltonian H has a positive definite metric along the submanifold W', then H' is a regular Lagrangian with a positive defined metric.

Proof. We use local coordinates. We have $H'(x^\alpha, y^{(1)\alpha}, \ldots, y^{(k-1)\alpha}, p_\alpha)$ $= H(x^\alpha, 0, y^{(1)\alpha}, 0, \ldots, y^{(k-1)\alpha}, 0, p_\alpha, Q_{\bar{\alpha}}(x^\beta, y^{(1)\beta}, \ldots, y^{(k-1)\beta}, p_\beta))$. Using formula (6) it follows that:

$$\frac{\partial H'}{\partial p_\alpha}(x^\beta, y^{(1)\beta}, \ldots, y^{(k-1)\beta}, p_\beta) =$$
$$\frac{\partial H}{\partial p_\alpha}(x^\beta, 0, y^{(1)\beta}, 0, \ldots, y^{(k-1)\beta}, 0, p_\beta, Q_{\bar{\beta}}(x^\gamma, y^{(1)\gamma}, \ldots, y^{(k-1)\gamma}, p_\gamma)).$$

Differentiating this formula with respect to p_β, then using formula (7), we get:

$$\frac{\partial^2 H'}{\partial p_\alpha \partial p_\beta} = \frac{\partial^2 H}{\partial p_\alpha \partial p_\beta} + \frac{\partial Q_{\bar{\alpha}}}{\partial p_\beta} \frac{\partial^2 H}{\partial p_{\bar{\alpha}} \partial p_\alpha} = h^{\alpha\beta} - h^{\bar{\alpha}\alpha} \tilde{h}_{\bar{\alpha}\bar{\beta}} h^{\beta\bar{\beta}}.$$

We use now the following Lemma of linear algebra.

Lemma 2.1 *Let A be a symmetric matrix of dimension p, B a symmetric and non-degenerated matrix of dimension q and C a $p \times q$ matrix such that the symmetric matrix $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix}$ of dimension $p+q$ is non-degenerate. Denote $\begin{pmatrix} A & C \\ C^t & B \end{pmatrix}^{-1} = \begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix}$, where X, Y and Z have the same dimensions as the matrices A, B and C respectively.*

Then the matrix $A - C \cdot B^{-1} C^t$ is invertible and its inverse is X.

Turning back to the proof of the Proposition 2.2, consider the matrix $h = (h^{ij}) = \begin{pmatrix} h^{\alpha\beta} & h^{\bar{\alpha}\beta} \\ h^{\alpha\bar{\beta}} & h^{\bar{\alpha}\bar{\beta}} \end{pmatrix}$. Using the Lemma 2.1, it follows that the matrix

$$\left(h^{\alpha\beta} - h^{\bar{\alpha}\alpha} \tilde{h}_{\bar{\alpha}\bar{\beta}} h^{\beta\bar{\beta}} \right)_{\alpha,\beta=\overline{1,m'}}$$

is invertible and its inverse is $(h_{\alpha\beta})$, where $\begin{pmatrix} h_{\alpha\beta} & h_{\bar{\alpha}\beta} \\ h_{\alpha\bar{\beta}} & h_{\bar{\alpha}\bar{\beta}} \end{pmatrix} = \begin{pmatrix} h^{\alpha\beta} & h^{\bar{\alpha}\beta} \\ h^{\alpha\bar{\beta}} & h^{\bar{\alpha}\bar{\beta}} \end{pmatrix}^{-1}$. $\square$

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