PROPERTIES OF INVARIANT SUBMANIFOLDS IN A $F(3,\varepsilon)$-MANIFOLD

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Abstract

Two cases of invariant submanifolds are considered.

In the first case we get an induced almost complex structure or almost product structure, and in the second case we obtain an induced $\tilde{f}(3,\varepsilon)$-structure on the invariant submanifold in a $f(3,\varepsilon)$-manifold.

We shall give the condition for an invariant submanifold of special $f(3,\varepsilon)$-manifold to be minimal.

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1 Introduction

Let $\mathcal{N}^p$ be an p-dimensional C^∞ manifold imbedded differentially as a submanifold in a n-dimensional C^∞-Riemannian manifold $\mathcal{M}^n$. Let ϕ be an imbedding map $\phi : \mathcal{N}^p \to \mathcal{M}^n$, and $\phi_*(\equiv B)$ the Jacobian map of ϕ i.e. $B : T(\mathcal{N}^p) \to T(\mathcal{M}^n)$. Denoting by $T(\mathcal{N},\mathcal{M})$ the set of all vectors tangent to the submanifold $\phi(\mathcal{N}^p)$. It is known that $B : T(\mathcal{N}^p) \to T(\mathcal{N},\mathcal{M})$ is an isomorphism [2].

Take the C^∞ vector fields $\tilde{X}$ and $\tilde{Y}$ which are tangential to $\phi(\mathcal{N}^p)$. Let X and Y be the local C^∞ extension of X and Y respectively.

The restriction of $[X,Y]$ to $\phi(\mathcal{N}^p)$, i.e. $[X,Y]|_{\phi(\mathcal{N}_p)}$ is determined independently of the choise of these local extensions X and Y. We can write

$$[\tilde{X},\tilde{Y}] = [X,Y]|_{\phi(\mathcal{N}^p)}. \quad \text{Since } B \text{ is an isomorphism, we have}$$

$$[B\tilde{X},B\tilde{Y}] = B[\tilde{X},\tilde{Y}] \text{ for all } \tilde{X},\tilde{Y} \in T(\mathcal{N}^p).$$

We define the induced metric $\tilde{g}$ on $\mathcal{N}^m$ as follows

$$\tilde{g}(\tilde{X},\tilde{Y}) = g(B\tilde{X},B\tilde{Y}) \text{ for all } \tilde{X},\tilde{Y} \in T(\mathcal{N}^p), \tag{1}$$

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where g is the Riemannian metric in $\mathcal{M}^n$. It can be easily verified that $\tilde{g}$ is Riemanian metric in $\mathcal{N}^p$.

We assume that $\mathcal{M}^n$ is the manifold with structure tensor $f(3,\varepsilon)$ of rank $r \leq n$. $\mathbf{l} = \varepsilon f^2$ and $\mathbf{m} = I - \varepsilon f^2$, $(\varepsilon = \pm 1)$ are complementary projection operators corresponding to which L and M are complementary distributions of dimension r and $n - r$ respectively.

We have, as in [3]

$$\begin{aligned} &f^3 = \varepsilon f,\ (\varepsilon = \pm 1),\ f\mathbf{l} = \mathbf{l}f = f,\ f\mathbf{m} = \mathbf{m}f = 0, \\ &f^2\mathbf{l} = \mathbf{l}f^2 = \varepsilon\mathbf{l},\ f^2\mathbf{m} = \mathbf{m}f^2 = 0. \end{aligned} \tag{2}$$

2 Invariant submanifolds of $f(3,\varepsilon)$-structure manifold

Definition 2.1 $\mathcal{N}^p$ is said to be an invariant submanifold of $\mathcal{M}^n$ if the tangent space $T_u(\phi(\mathcal{N}^p))$ of $\phi(\mathcal{N}^p)$ is invariant by the linear mapping f at each point u of $\phi(\mathcal{N}^p)$ so that for each $\tilde{X} \in T(\mathcal{N}^p)$ we have $f(B\tilde{X}) = B\tilde{X}'$, for some $\tilde{X}' \in T(\mathcal{N}^p)$.

If we define a $(1,1)$ tensor field $\tilde{f}$ in $\mathcal{N}^p$ by $\tilde{f}(\tilde{X}) = \tilde{X}'$, as in [4], then we have

$$f(B\tilde{X}) = B(\tilde{f}\tilde{X}). \tag{1}$$

In the first case we assume that distribution M is never tangential to $\phi(\mathcal{N}^p)$ i.e. no vector field of type $\mathbf{m}X$, $X \in T(\phi(\mathcal{N}^p))$ is tangential to $\phi(\mathcal{N}^p)$. It shows that any vector field of type $\mathbf{m}X$ is independent of any vector field of the form $B\tilde{X}$, $\tilde{X} \in T(\mathcal{N}^p)$.

Applying f to (2.1) we have

$$f^2(B\tilde{X}) = B(\tilde{f}^2\tilde{X}). \tag{2}$$

Now we shall show that vector fields of type $B\tilde{X}$ belong to the distribution L in this case. If we suppose that $\mathbf{m}(B\tilde{X}) \neq 0$, then $\mathbf{m}(B\tilde{X}) = (I_{T(\mathcal{M}^n)} - \varepsilon f^2)(B\tilde{X}) = B\tilde{X} - \varepsilon f^2(B\tilde{X}) = B(\tilde{X} - \varepsilon\tilde{f}^2\tilde{X})$, which contrary to our assumption shows that $\mathbf{m}(B\tilde{X})$ is tangential to $\phi(\mathcal{N}^p)$. Therefore $\mathbf{m}(B\tilde{X}) = 0$.

Now since $\mathbf{l} = \varepsilon f^2$, from (2.2) and (1.2) we have $B(\tilde{f}^2\tilde{X}) = f^2(B\tilde{X}) = \varepsilon\mathbf{l}(B\tilde{X}) = \varepsilon(I_{T\mathcal{M}^n)} - \mathbf{m})(B\tilde{X}) = \varepsilon B\tilde{X} - \varepsilon\mathbf{m}(B\tilde{X})$, $B(\tilde{f}^2\tilde{X}) = B(\varepsilon\tilde{X})$, which in view of B being an isomorphism gives $\tilde{f}^2(\tilde{X}) = \varepsilon\tilde{X}$.

The tensor field $\tilde{f}$ in $\mathcal{N}^p$ defines an induced almost complex structure or almost product structure according as $\varepsilon = -1$ or $+1$.

Theorem 2.1 *An invariant submanifold $\mathcal{N}^p$ in a $f(3,\varepsilon)$-structure manifold $\mathcal{M}^n$ such that distribution M is never tangential to $\phi(\mathcal{N}^p)$ is an almost complex manifold or almost product manifold according as $\varepsilon = -1$ or $+1$.*

Let $\mathcal{M}^n$ a Riemannian metric g satisfying

$$g(X,Y) = g(fX,fY) + g(\mathbf{m}X,Y). \tag{3}$$

Theorem 2.2 *Let $\mathcal{N}^p$ be an invariant submanifold imbedded in an $f(3,\varepsilon)$-structure manifold $\mathcal{M}^n$ such that the distribution M is never tangential to $\phi(\mathcal{N}^p)$. If g denotes a Riemannian metric on $\mathcal{M}^n$ given by (2.3) than the induced metric $\tilde{g}$ and $\mathcal{N}^p$ defined by (1.1) is Hermitian.*

Proof. $\tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) = g(B\tilde{f}\tilde{X},B\tilde{f}\tilde{Y}) = g(fB\tilde{X},fB\tilde{Y}) =$
$= g(B\tilde{X},B\tilde{Y}) - g(\mathbf{m}B\tilde{X},B\tilde{Y}) = g(B\tilde{X},B\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}).$

In the second case we assume that the distribution M is always tangential to $\phi(\mathcal{N}^p)$. It follows, therefore, that $\mathbf{m}(B\tilde{X}) = B\tilde{X}^0$, where $\tilde{X}\in T(\mathcal{N}^p)$ for some $\tilde{X}^0\in T(\mathcal{N}^p)$.

Let us define a $(1,1)$ tensor field $\tilde{\mathbf{m}}$ in $\mathcal{N}^p$ such that $\tilde{\mathbf{m}}\tilde{X} = \tilde{X}^0$.

We can write $\mathbf{m}(B\tilde{X}) = B(\tilde{\mathbf{m}}\tilde{X})$. Define a $(1,1)$ tensor field $\tilde{\mathbf{l}}$ in $\mathcal{N}^p$ by $\tilde{\mathbf{l}} = \varepsilon\tilde{f}^2$.

Then $(B\tilde{\mathbf{l}}\tilde{X}) = B(\varepsilon\tilde{f}^2\tilde{X}) = \varepsilon B(\tilde{f}^2\tilde{X}) = \varepsilon f^2(B\tilde{X}) = \mathbf{l}(B\tilde{X})$. Thus we have $B(\tilde{\mathbf{l}}\tilde{X}) = \mathbf{l}(B\tilde{X})$.

Theorem 2.3 *The* $(1,1)$ *tensor fields* $\tilde{\mathbf{l}}$ *and* $\tilde{\mathbf{m}}$ *in* $\mathcal{N}^p$ *satisfy the following*

$$\tilde{\mathbf{l}}+\tilde{\mathbf{m}} = I_{(T(\mathcal{N}^p))},\ \tilde{\mathbf{l}}\tilde{\mathbf{m}} = \tilde{\mathbf{m}}\tilde{\mathbf{l}} = 0,\ \tilde{\mathbf{l}}^2 = \tilde{\mathbf{l}},\ \tilde{\mathbf{m}}^2 = \tilde{\mathbf{m}}. \tag{4}$$

Proof. We have $\mathbf{l}+\mathbf{m} = I_{(T(\mathcal{M}^p))}$ i.e. $(\mathbf{l}+\mathbf{m})(B\tilde{X}) = B\tilde{X}$. Thus we have

$$B(\tilde{\mathbf{l}}\tilde{X}) + B(\tilde{\mathbf{m}}\tilde{X}) = B\tilde{X},\ B(\tilde{\mathbf{l}}+\tilde{\mathbf{m}})(\tilde{X}) = B\tilde{X}.$$

Therefore $\tilde{\mathbf{l}}+\tilde{\mathbf{m}} = I$ in view of the fact that B is an isomorphism.

Similary we can prove other relations.

The relations (2.4) show that $\tilde{\mathbf{l}}$ and $\tilde{\mathbf{m}}$ are complementary projection operators in $\mathcal{N}^p$ given by $\tilde{\mathbf{l}} = \varepsilon\tilde{f}^2$ and $\tilde{\mathbf{m}} = I - \varepsilon\tilde{f}^2$. Moreover, from (2.1) we get $B(\tilde{f}^3\tilde{X}) = f^3(B\tilde{X}) = \varepsilon f(B\tilde{X}) = \varepsilon B(\tilde{f}\tilde{X})$. Thus we have $\tilde{f}^3 = \varepsilon\tilde{f}$ which shows that in this case $\tilde{f}$ defines an $\tilde{f}(3,\varepsilon)$-structure on $\mathcal{N}^p$ which we call induced $\tilde{f}(3,\varepsilon)$-structure. Further, from (1.1), (2.1) and (2.3) we have

$$\tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) + \tilde{g}(\mathbf{m}\tilde{X},\tilde{Y}) = g(B(\tilde{f}\tilde{X}),B(\tilde{f}\tilde{Y})) + g(B\tilde{\mathbf{m}}\tilde{X},B\tilde{Y}) =$$
$$= g(f(B\tilde{X}),f(B\tilde{Y})) + g(\tilde{\mathbf{m}}B\tilde{X},B\tilde{Y}) = g(B\tilde{X},B\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}),\ \text{i.e.}$$

$$\tilde{g}(\tilde{X},\tilde{Y}) = \tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) + \tilde{g}(\tilde{\mathbf{m}}\tilde{X},\tilde{Y}). \tag{5}$$

Theorem 2.4 *In invariant submanifold $\mathcal{N}^p$ imbedded in an $f(3,\varepsilon)$-structure manifold $\mathcal{M}^n$ such that the distribution M is always tangential to $\phi(\mathcal{N}^p)$ there exists an induced $\tilde{f}(3,\varepsilon)$-structure manifold which admits a similar Riemanian metric $\tilde{g}$ satisfying (2.5)*

Theorem 2.5 *The Nijenhuis tensor N and $\tilde{N}$ of $\mathcal{M}^n$ and $\mathcal{N}^p$ respectively are related, as in [1], by the following relation $N(B\tilde{Y},B\tilde{Y}) = B\tilde{N}(\tilde{X},\tilde{Y})$.*

We can easily verify the following relations

$$B\tilde{N}(\tilde{\mathbf{l}}\tilde{X},\tilde{\mathbf{l}}\tilde{Y}) = N(\mathbf{l}B\tilde{X},\mathbf{l}B\tilde{Y}) \qquad B\tilde{N}(\tilde{\mathbf{l}}\tilde{X},\tilde{\mathbf{l}}\tilde{Y}) = N(\mathbf{l}B\tilde{X},\mathbf{l}B\tilde{Y})$$
$$B\tilde{N}(\tilde{\mathbf{m}}\tilde{X},\tilde{\mathbf{m}}\tilde{Y}) = N(\mathbf{m}B\tilde{X},\mathbf{m}B\tilde{Y}) \qquad B\{\tilde{\mathbf{m}}\tilde{N}(\tilde{X},\tilde{Y}) = \mathbf{m}N(B\tilde{X},B\tilde{Y}).$$

If $\tilde{L}$ and $\tilde{M}$ denote the complementary distributions corresponding to the projection operators $\tilde{\mathbf{l}}$ and $\tilde{\mathbf{m}}$ in $\mathcal{N}^p$, then in view of the integrability conditions of the $\tilde{f}(3,\varepsilon)$-structure we can state the following theorems.

Theorem 2.6 *If L is integrable in $\mathcal{M}^n$ then $\tilde{L}$ is also integrable in $\mathcal{N}^p$. If M is integrable in $\mathcal{M}^n$ then $\tilde{M}$ is also integrable in $\mathcal{N}^p$.*

Theorem 2.7 *If the $f(3,\varepsilon)$-structure is integrable in $\mathcal{M}^n$ then the induced $\tilde{f}(3,\varepsilon)$-structure is also integrable.*

We call such a manifold a normal $f(3,\varepsilon)$-manifold.

3 Invariant submanifolds of special $f(3,\varepsilon)$-structure

In the special case let $dim\mathcal{M}^n = n$ be $2m+1$, $rankf = 2m$, $\varepsilon = -1$, then $\mathcal{M}^n$ is a contact Riemannian manifold with the structure tensor (f,ξ,η,g). Then they satisfy:

$$f^2 = -I + \eta \otimes \xi,\ f\xi = 0,\ \eta(\xi) = 0,\ g(fX,fY) = g(X,Y) - \eta(X)\eta(Y), \tag{1}$$

$$g(fX,Y) = d\eta(X,Y), \quad \eta(X) = g(\xi,X) \tag{2}$$

for any vector fields X and Y on $\mathcal{M}^n$.

$\mathcal{M}^n$ is called a K-contact Riemannian manifold if ξ is a Killing vector field. Then we have

$$\nabla_X \xi = fX,\ R(X,\xi)\xi = X - \eta(X)\xi, \tag{3}$$

where ∇_X denotes the Riemannian connection and R the Riemannian curvature tensor of $\mathcal{M}^n$. Moreover, if we have $R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi$, then $\mathcal{M}^n$ is called a normal contact metric manifold (Sasakian).

In this special case a submanifold $\mathcal{N}^p (p = 2l+1)$ of $\mathcal{M}^n$ is said to be invariant if (i) ξ is tangent to $\mathcal{N}^p$ everywhere on $\mathcal{N}^p$, (ii) $f(B\tilde{X})$ is tangent to $\mathcal{N}^p$ for any tangent vector $\tilde{X}$ to $\mathcal{N}^p$. An invariant submanfold $\mathcal{N}^p$ has the induced structure tensors $(\tilde{f},\tilde{\xi},\tilde{\eta},\tilde{g})$. Let $\tilde{\nabla}_X$ denote the Riemannian connection on $\mathcal{N}^p$ determined by the induced metric $\tilde{g}$ and $\tilde{R}$ denote the Riemannian curvature tensor of $\mathcal{N}^p$. Then Gauss-Weingarten formula is given by

$$\begin{aligned} &\nabla_{\tilde{X}}\tilde{Y} = \tilde{\nabla}_{\tilde{X}}\tilde{Y} + B(\tilde{X},\tilde{Y}), \quad \tilde{X},\tilde{Y} \in T(\mathcal{N}^p),\\ &\nabla_{\tilde{X}}N = -A_N(\tilde{X}) + D_{\tilde{X}}N, \quad \tilde{X} \in T(\mathcal{N}^p), \qquad N \in T(\mathcal{N}^p)^{\perp}, \end{aligned} \tag{4}$$

where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle. Both A and B are called the second fundamental form on $\mathcal{N}^p$ and satisfy $g(B(\tilde{X},\tilde{Y}),N) = \tilde{g}(A_N\tilde{X},\tilde{Y})$. The mean curvature vector H is defined as $H = \frac{1}{n}TrB$, TrB being defined by $TrB = \sum_i B(\tilde{e}_i,\tilde{e}_i)$ for an orthonormal frame $\{\tilde{e}_i\}$. If $H = 0$, then $\mathcal{N}^p$ is said to be minimal.

Lemma 3.1 *Let $\mathcal{N}^p$ be an invariant submanifold of a K-contact Riemannian manifold $\mathcal{M}^n$. Then its second fundamental form B satisfies $B(\tilde{X},\tilde{\xi}) = 0$ for any $\tilde{X} \in T(\mathcal{N}^p)$.*

Proof. Since ξ is tangent to $\mathcal{N}^p$ everywhere on $\mathcal{N}^p$, we see $\nabla_{\tilde{X}}\xi = \tilde{\nabla}_{\tilde{X}}\tilde{\xi} + B(\tilde{X},\tilde{\xi})$ on $\mathcal{N}^p$. On the other hand, by (3.3), $\nabla_{\tilde{X}}\xi$ is tangent to $\mathcal{N}^p$ for any $\tilde{X} \in T(\mathcal{N}^p)$, hence we have $B(\tilde{X},\tilde{\xi}) = 0$ for any $\tilde{X} \in T(\mathcal{N}^p)$.

For the second fundamental form B of an invariant submanifold $\mathcal{N}^p$ of a K-contact Riemannian manifold $\mathcal{M}^n$, we define its covariant derivative $\nabla_{\tilde{X}}B$, by

$$(\nabla_{\tilde{X}}B)(\tilde{Y},\tilde{Z}) = D_{\tilde{X}}(B(\tilde{X},\tilde{Z})) - B(\tilde{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}) - B(\tilde{Y},\tilde{\nabla}_{\tilde{X}}\tilde{Z}),\quad \tilde{X},\tilde{Y},\tilde{Z} \in T(\mathcal{N}^p). \tag{5}$$

Then by (3.4), we obtain

$$\begin{aligned} R(\tilde{X},\tilde{Y})\tilde{Z} &= \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} - A_{B(\tilde{Y},\tilde{Z})}(\tilde{X}) + A_{B(\tilde{X},\tilde{Z})}(\tilde{Y}) \\ &+ (\nabla_{\tilde{X}}B)(\tilde{Y},\tilde{Z}) - (\nabla_{\tilde{Y}}B)(\tilde{X},\tilde{Z}), \quad \tilde{X},\tilde{Y},\tilde{Z} \in T(\mathcal{N}^p). \end{aligned} \tag{6}$$

Theorem 3.1 *Any invariant submanifold $\mathcal{N}^p$ with induced structure tensors of a K-contact Riemannian manifold $\mathcal{M}^n$ is also K-contact.*

Proof. From Lemma 3.1, $\tilde{\xi}$ is a Killing vector field on $\mathcal{N}^p$, and by Lemma 3.1 and (3.6), we have

$$R(\tilde{X},\tilde{\xi})\tilde{\xi} = \tilde{R}(\tilde{X},\tilde{\xi})\tilde{\xi} + (\nabla_{\tilde{X}}B)(\tilde{\xi},\tilde{\xi}) - (\nabla_{\tilde{\xi}}B)(\tilde{X},\tilde{\xi}), \quad \tilde{X} \in T(\mathcal{N}^p).$$

On the other hand, from Lemma 3.1 (3.1) and (3.5), we get $(\nabla_{\tilde{X}}B)(\tilde{\xi},\tilde{\xi}) =$ $(\nabla_{\tilde{\xi}}B)(\tilde{X},\tilde{\xi}) = 0$. Therefore, $\tilde{R}(\tilde{X},\tilde{\xi})\tilde{\xi} = R(\tilde{X},\tilde{\xi})\tilde{\xi} = \tilde{X} - \tilde{\eta}(\tilde{X})\tilde{\xi}$, which shows that $\mathcal{N}^p$ is a K-contact Riemannian manifold.

Lemma 3.2 *Let $\mathcal{N}^p$ be an invariant submanifold of a K-contact Riemannian manifold $\mathcal{M}^n$. Then $R(\tilde{X},\tilde{\xi})\tilde{Y}$ is tangent to $\mathcal{N}^p$ if and only if $fB(\tilde{X},\tilde{Y}) = B(\tilde{X}, fY)$ for any $\tilde{X},\tilde{Y} \in T(\mathcal{N}^p)$.*

Proof. Since $\tilde{\xi}$ is a Killing vector field on $\mathcal{N}^p$ and $\mathcal{M}^n$, we have $R(\tilde{X},\tilde{\xi})\tilde{Y} =$ $(\nabla_{\tilde{X}}f)\tilde{Y}$ and $\tilde{R}(\tilde{X},\tilde{\xi})\tilde{Y} = (\tilde{\nabla}_{\tilde{X}}\tilde{f})\tilde{Y}$ for any $\tilde{X},\tilde{Y} \in T(\mathcal{N}^p)$. On the other hand, we have

$$\begin{aligned} \nabla_{\tilde{X}}(f\tilde{Y}) &= \tilde{\nabla}_{\tilde{X}}(\tilde{f}\tilde{Y}) + B(\tilde{X},\tilde{f}\tilde{Y}) = (\tilde{\nabla}_{\tilde{X}}\tilde{f})\tilde{Y} + \tilde{f}(\tilde{\nabla}_{\tilde{X}}Y) + B(\tilde{X},\tilde{f}\tilde{Y}), \\ \nabla_{\tilde{X}}(f\tilde{Y}) &= (\nabla_{\tilde{X}})\tilde{Y} + f(\nabla_{\tilde{X}}\tilde{Y}) = (\nabla_{\tilde{X}}f)\tilde{Y} + f(\tilde{\nabla}_{\tilde{X}}\tilde{Y}) + fB(\tilde{X},\tilde{Y}), \end{aligned}$$

hence we get $\tilde{R}(\tilde{X},\tilde{\xi})\tilde{Y} - R(\tilde{X},\tilde{\xi})\tilde{Y} = fB(\tilde{X},\tilde{Y}) - B(\tilde{X},\tilde{f}\tilde{Y})$. q.e.d.

Theorem 3.2 *An invariant submanifold $\mathcal{N}^{2l+1}$ of a K-contact Riemannian manifold $\mathcal{M}^{2m+1}$ is minimal if $R(\tilde{X},\tilde{\xi})\tilde{Y}$ is tangent to $\mathcal{N}^p$ for any vector fields $\tilde{X}$ and $\tilde{Y}$ on $\mathcal{N}^p$.*

Proof. By Lemma 3.1 and Lemma 3.2, we see

$$B(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) = f^2 B(\tilde{X},\tilde{Y}) = -B(\tilde{X},\tilde{Y}) + \eta(B(\tilde{X},\tilde{Y})) = -B(\tilde{X},\tilde{Y}).$$

Let us take a $\tilde{f}$-basis $(\tilde{V}_1,\ldots,\tilde{V}_l,\tilde{f}\tilde{V}_1,\ldots,\tilde{f}\tilde{V}_l,\tilde{\xi})$ for $T_x(\mathcal{N}^p)$. Then the mean curvature

$$H = \sum_{i=1}^{p=2l+1} [B(\tilde{f}\tilde{V}_i,\tilde{f}\tilde{V}_i) + B(\tilde{V}_i,\tilde{V}_i)] + B(\tilde{\xi},\tilde{\xi}) \text{ vanishes.} \qquad \text{q.e.d.}$$

Corollary 3.1 *An invariant submanifold imbedded in a normal contact metric manifold is minimal [4].*

By Lemma 3.1, (3.3) and (3.5), $B(\tilde{X},\tilde{f}\tilde{Y}) = -(\nabla_{\tilde{Y}}B)(\tilde{X},\tilde{\xi})$ for any $\tilde{X},\tilde{Y} \in T(\mathcal{N}^p)$, it follows that

$$B(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) = -(\nabla_{\tilde{Y}}B)(\tilde{\nabla}_{\tilde{X}}\tilde{\xi},\tilde{\xi}) = (\nabla_{\tilde{X}}\nabla_{\tilde{Y}}B)(\tilde{\xi},\tilde{\xi}) - B(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}),$$

therefore $B(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) = \frac{1}{2}(\nabla_{\tilde{X}}\nabla_{\tilde{Y}}B)(\tilde{\xi},\tilde{\xi})$. From this we obtain $B(\tilde{X},\tilde{Y}) = \frac{1}{2}(\nabla_{\tilde{f}\tilde{X}}\nabla_{\tilde{f}\tilde{Y}}B)(\tilde{\xi},\tilde{\xi})$, for any $\tilde{X},\tilde{Y} \in T(\mathcal{N}^p)$. Let $(\tilde{V}_1,\ldots,\tilde{V}_{2l+1})$ be a $\tilde{f}$-basis for $T_x(\mathcal{N}^p)$ such that $\tilde{V}_{l+i} = \tilde{f}\tilde{V}_i$, $\tilde{V}_{2l+1} = \tilde{\xi}$. Then the mean curvature H on $\mathcal{N}^p$ is written by

$$H = \frac{1}{2}\sum_{i=1}^{2l+1}(\nabla_{\tilde{V}_i}\nabla_{\tilde{V}_i}B)(\tilde{\xi},\tilde{\xi}). \qquad (7)$$

Hence we get

Theorem 3.3 *Let $\mathcal{N}^p$ be an invariant submanifold of a K-contact Riemannian manifold $\mathcal{M}^n$. Then $\mathcal{N}^p$ is minimal if and only if H in (3.7) vanishes.*

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