LIE GROUP FOLIATED BY MINIMAL HYPERSURFACES

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Abstract

On $\mathbf{R}^4$, we consider a specific family of Riemannian metrics, previously used in complex geometry [1]. We show that these metrics are left invariant with respect to some Lie group multiplication. A 3-dimensional foliation with minimal leaves is discovered; in a particular case, one of these leaves is precisely the classical Heisenberg group.

AMS Subject Classification: 53C12
Key words: Heisenberg group, left invariant metrics, minimal foliations

1 Introduction

Invariant geometries on Lie groups are valuable laboratories for conjecture testing and examples providing. Usually, one starts with a specific Lie group G, we choose a basis of its Lie algebra $L(G)$ and define a left invariant metric g which orthonormalizes this basis. Properties of the Riemannian manifold (G, g) are strongly related to the algebraic properties of the Lie algebra $L(G)$, and this interplay is enlighting for both Geometry and Algebra.

By contrast, our left invariant metrics arrose in a very unespected way. In [1], a family of Hermitian metrics on $\mathbf{R}^4$ was found, with interesting properties from the complex geometry view point.(For example, with respect to a specific almost complex structure, these manifolds were proved to be globally conformal Kahler).

The aim of this paper is to show how these metrics may be considered as left invariant metrics on a nilpotent Lie group G. In §2, we recall the definition of the family of the metrics $g(k)$ on $\mathbf{R}^4$, and compute the main Riemannian tensor fields (curvature, Ricci, scalar curvature). With respect to our previous study ([4]), the mean curvature functions associated to the Ricci tensor are determined.

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2001,* 95-102

In §3, we find a family of Lie group structures G on $\mathbf{R}^4$, such that the previous metrics are left invariant with respect to it. These are 3-nilpotent Lie groups, obtained as semi-direct products of the real line with the Heisenberg group.

In §4, we find a 3-dimensional foliation, whose leaves are minimal in G. One of its leaves is the classical Heisenberg group.

2 The Riemannian geometry of $\tilde{\mathrm{R}}^4$

Let $\mathbf{R}^4$ be the standard 4-dimensional numerical space, with coordinates (x^1, x^2, x^3, x^4) and k an arbitrary real parameter. We consider the family of Riemannian metrics $\tilde{g}(k)$ on $\mathbf{R}^4$, defined by ([1])

$$\tilde{g}_{11} = \tilde{g}_{44} = 1 \quad , \quad \tilde{g}_{13} = -\tilde{g}_{24} = -kx^1 \quad , \quad \tilde{g}_{12} = \tilde{g}_{14} = 0 \quad , \quad g_{22} = 1 + k^2(x^1)^2,$$

$$\tilde{g}_{23} = kx^2 + k^3(x^1)^2x^2 \quad , \quad \tilde{g}_{33} = 1 + k^2\{(x^1)^2 + (x^2)^2 + k^2(x^1)^2(x^2)^2\} \quad , \quad \tilde{g}_{34} = k^2x^1x^2.$$

We denote by $\tilde{\mathbf{R}}^4(k)$ the Riemannian manifold $(\mathbf{R}^4, \tilde{g}(k))$. (When there is no danger of confusion, we will drop the index k).

We determine the non-vanishing coefficients of the curvature tensor

$$\tilde{R}_{1212} = \frac{1}{4}k^2\{1 + k^2(x^1)^2\} \quad , \quad \tilde{R}_{1213} = \frac{1}{4}k^3x^2\{1 + k^2(x^1)^2\} \quad , \quad \tilde{R}_{1214} = \frac{1}{4}k^3x^1,$$

$$\tilde{R}_{1223} = -\frac{1}{4}k^3x^1\{1 - k^2(x^1)^2\} \quad , \quad \tilde{R}_{1234} = -\frac{1}{4}k^4(x^1)^2 \quad ,$$

$$\tilde{R}_{1313} = -\frac{1}{4}k^2\{4 - k^2(x^2)^2 - k^4(x^1)^2(x^2)^2\},$$

$$\tilde{R}_{1314} = \frac{1}{4}k^4x^1x^2 \quad , \quad \tilde{R}_{1323} = -\frac{1}{4}k^4x^1x^2\{1 - k^2(x^1)^2\} \quad , \quad \tilde{R}_{1324} = -\frac{1}{2}k^2,$$

$$\tilde{R}_{1334} = -\frac{1}{4}k^3x^2\{2 + k^2(x^1)^2\} \quad , \quad \tilde{R}_{1414} = \tilde{R}_{2424} = \frac{1}{4}k^2 \quad , \quad \tilde{R}_{1423} = -\frac{1}{4}k^2\{2 - k^2(x^1)^2\},$$

$$\tilde{R}_{1434} = -\frac{1}{4}k^3x^1 \quad , \quad \tilde{R}_{2323} = -\frac{1}{4}k^2\{4 - k^2(x^1)^2\}\{1 + k^2(x^1)^2\},$$

$$\tilde{R}_{2334} = \frac{1}{4}k^3x^1\{2 - k^2(x^1)^2\} \; , \; \tilde{R}_{2434} = \frac{1}{4}k^3x^2 \; , \; \tilde{R}_{3434} = \frac{1}{4}k^4\{(x^1)^2 + (x^2)^2\}.$$

The non-vanishing components of the Ricci tensor are

$$\tilde{R}_{11} = -\tilde{R}_{44} = -\frac{1}{2}k^2 \quad , \quad \tilde{R}_{13} = \tilde{R}_{24} = \frac{1}{2}k^3x^1 \quad , \quad \tilde{R}_{22} = -\frac{1}{2}k^2\{1 - k^2(x^1)^2\},$$

$$\tilde{R}_{23} = -\frac{1}{2}k^3x^2\{1-k^2(x^1)^2\} \quad , \quad \tilde{R}_{33} = -\frac{1}{2}k^2\{4+k^2(x^1)^2+k^2(x^2)^2-k^4(x^1)^2(x^2)^2\},$$

$$\tilde{R}_{34} = \frac{1}{2}k^4x^1x^2.$$

Finally, the scalar curvature is

$$\tilde{\rho} = -\frac{5}{2}k^2.$$

Remarks. (i) Consider the characteristic polynomial of $\tilde{Ric}$ with respect to $\tilde{g}$ and denote by $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$ and $\tilde{H}_4$ the symmetric functions of its eigenvalues. These are the mean functions of $\tilde{Ric}$ ([4]). We compute

$$\tilde{H}_1 = -\frac{5}{8}k^2 \quad , \quad \tilde{H}_2 = \frac{3}{24}k^4 \quad , \quad \tilde{H}_3 = \frac{5}{32}k^6 \quad , \quad \tilde{H}_4 = -\frac{1}{4}k^8$$

Not only these mean functions, but also all the $\tilde{Ric}$ eigenvalues are constant:

$$-\frac{1}{2}k^2 \quad , \quad -\frac{1}{2}k^2 \quad , \quad -2k^2 \quad , \quad \frac{1}{2}k^2$$

Moreover, we have

Proposition 1. *The Ricci curvature is bounded on the unit sphere between* $-2k^2$ *and* $\frac{1}{2}k^2$.

Remark. (i) Even if the mean functions and the eigenvalues of $\tilde{Ric}$ are k- dependent, their sign is not.

(ii) The extremum values in Proposition 1 are attained effectively. This result gives examples of Riemannian 4-manifolds with Ricci curvature "pinched" (modulo a homothety) between -2 and $\frac{1}{2}$.

3 A family of 3-nilpotent Lie groups and their invariant geometry

Consider on $\mathbf{R}^4$ the following multiplication rule

(1)
$$(a^1, a^2, a^3, a^4)(b^1, b^2, b^3, b^4) = (a^1e^{kb^3} + b^1, a^2e^{-kb^3} + b^2, a^3 + b^3, a^4 + b^4 - ka^1b^2e^{kb^3}),$$

where k is a real parameter. We remark that this multiplication endowes $\mathbf{R}^4$ with a non-abelian Lie group structure (for $k \neq 0$). We will denote this Lie group by G. For $k = 0$ we recover the canonical abelian Lie group $\mathbf{R}^4$; in what follows, we will suppose implicitely that $k \neq 0$.

Proposition 2. *A basis of the Lie algebra* $L(G)$ *is furnished by the left invariant vector fields*

(2) $$X = \partial_1 \ , \ Y = \partial_2 - kx^1\partial_4 \ , \ Z = kx^1\partial_1 - kx^2\partial_2 + \partial_3 \ , \ T = \partial_4.$$

Proof. We denote by $L_a : G \to G$ the left translation defined by the element $a \in G$. Then we check that $(L_a)_*X = X$ and the same for Y, Z and T. So, they are left invariant vector fields.

Obviously, X, Y, Z, T are linearly independent, so they form a basis in the Lie algebra of G. $\Box$

Remark. (i) Another argument for the Proposition 1 consists in calculating the Lie brackets

(3) $$[X, Z] = kX \ , \ [Y, Z] = -kY \ , \ [X, Y] = -kT \ , \ [X, T] = [Y, T] = [Z, T] = 0.$$

Because the components of the Lie brackets are constant, they are exactly the *structure constants* of the Lie algebra $L(G)$, with respect to the basis $\{X, Y, Z, T\}$.

(ii) From (3) we derive $[L(G), [L(G), [L(G), L(G)]]] = 0$ and $[L(G), [L(G), L(G)]] \neq 0$. We have thus proved that G is a 3-nilpotent Lie group.

(iii) The Lie group G is the semi-direct product of the real line with the classical Heisenberg group. Indeed, consider the Heisenberg group $\mathcal{H}$ of triangular matrices

$$\begin{pmatrix} 1 & a^1 & a^4 \\ 0 & 1 & a^2 \\ 0 & 0 & 1 \end{pmatrix}$$

with the matrix multiplication. We identify such a matrix with the triple (a^1, a^2, a^4). Define the morphism $\tau : \mathbf{R} \to Aut(\mathcal{H})$,

$$[\tau(b)](a^1, a^2, a^4) = (a^1 e^{kb}, a^2 e^{-kb}, a^4).$$

Denote $\tau_b = \tau(b)$. Then the multiplication in $G = \mathbf{R} \times_\tau \mathcal{H}$ may be writen (modulo a permutation of the components) as

$$(a^3; a^1, a^2, a^3) \cdot (b^3; b^1, b^2, b^4) = (a^3 + b^3, \tau_{b^3}(a^1, a^2, a^4) \cdot (b^1, b^2, b^4)),$$

where the second multiplication is that from the Heisenberg group. So, G is the semi-direct product of $\mathbf{R}$ and $\mathcal{H}$, with respect to τ.

Proposition 3. *The unique left invariant metric on G, which makes $\{X, Y, Z, T\}$ orthonormal is $\tilde{g}$.*

Proof. It is enought to check that

$$\tilde{g}(X, X) = \tilde{g}(Y, Y) = \tilde{g}(Z, Z) = \tilde{g}(T, T) = 1,$$

$$\tilde{g}(X, Y) = \tilde{g}(X, Z) = \tilde{g}(X, T) = \tilde{g}(Y, Z) = \tilde{g}(Y, T) = \tilde{g}(Z, T) = 0.$$

As the coefficients of $\tilde{g}$ in this basis are constant, the metric is left invariant. The unicity follows from the fact that a left invariant (tensorial) geometric object is completely determined by its values on a left invariant basis of vector fields and 1-forms. $\square$

Remark. The Levi- Civita connection $\tilde{\nabla}$ and all the tensor fields $\tilde{R}$, $\tilde{Ric}$, $\tilde{\rho}$ determined in §2 are left invariant. This explains, for example, why the Ricci tensor eigenvalues (in particular, the scalar curvature) are constant.

4 A minimal 3-dimensional foliation

We denote by $\mathcal{D}$ the 3-dimensional distribution spanned by the vector fields X, Y and T defined in the Proposition 2. We have the following

Theorem 4. (i) *The distribution $\mathcal{D}$ is completely integrable and defines a minimal foliation of G. Each leaf has null Gauss curvature, but is not totally geodesic.*

(ii) *The leaf of $\mathcal{D}$ through the origin of $\mathbf{R}^4$ is the subgroup G_3 defined by the hyperplane $x^3 = 0$.*

(ii) *G_3 is isomorphic with the classical Heisenberg group.*

Proof. We remark that $\mathcal{D}$ is spanned also by $\{\partial_1, \partial_2, \partial_4\}$, so it is an integrable distribution. (Even if the basis $\{X, Y, T\}$ depends on the parameter k, the foliation $\mathcal{D}$ is independent of k). The leaf through $(0, 0, 0, 0)$ is the hyperplane G_3.

Let identify $(x, y, 0, t) = (x, y, t)$. The multiplication law from G, restricted to G_3, writes

$$(a^1, a^2, a^4)(b^1, b^2, b^4) = (a^1 + b^1, a^2 + b^2, a^4 + b^4 - ka^1b^2).$$

G_3 is a Lie subgroup of G, which may also be represented as a triangular matrix group with the usual multiplication. Indeed, a triple (a^1, a^2, a^4) may be identified with a matrix

$$\begin{pmatrix} 1 & a^1 & a^4 \\ 0 & 1 & a^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $k = -1$, G_3 coincides with the classical Heisenberg group. In general, the Lie algebra $L(G_3)$ has the following structure constants

$$[X, Y] = -kT \quad , \quad [X, T] = [Y, T] = 0$$

and the function sending T in kT and invariating X and Y is a Lie algebra isomorfism onto the Heisenberg algebra.

Let M be an integral manifold of the foliation $\mathcal{D}$. We show it is a minimal hypersurface in G, thus ending the proof of the theorem.

Denote by g the induced Riemannian metric on M. In the coordinates (x^1, x^2, x^4), the matrix of g writes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + k^2(x^1)^2 & kx^1 \\ 0 & kx^1 & 1 \end{pmatrix}.$$

The inverse matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -kx^1 \\ 0 & -kx^1 & 1 + k^2(x^1)^2 \end{pmatrix}.$$

The non-vanishing Christoffel coefficients are

$$\Gamma^1_{22} = -2\Gamma^2_{12} = 2\Gamma^4_{14} = -k^2x^1 \quad , \quad \Gamma^4_{12} = \frac{1}{2}k(1 - k^2(x^1)^2) \quad , \quad \Gamma^2_{14} = -\Gamma^1_{24} = \frac{1}{2}k.$$

We calculate the non-vanishing Riemann coefficients:

$$R_{1212} = \frac{1}{4}k^2\{(x^1)^2 - 3\} \ , \ R_{1214} = kx^1 \ , \ R_{1414} = kx^1 \ , \ R_{2424} = \frac{1}{4}k^3x^1.$$

From the Gauss formula, we deduce the second fundamental form of M, of matrix

$$\begin{pmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that the mean curvature functions of the hypersurface M are $H_1 = H_3 = 0$ and $H_2 = -\frac{1}{3}k^2$. So, M is minimal, without beeing totally geodesic; moreover, the Gauss curvature vanish. □

Remark. In the same coordinates (x^1, x^2, x^4), the components of the Ricci tensor of g are

$$R_{11} = -R_{44} = -\frac{1}{2}k^2 \quad , \quad R_{12} = R_{14} = 0 \quad ,$$

$$R_{22} = \frac{1}{2}k^2\{k^2(x^1)^2 - 1\} \quad , \quad R_{24} = \frac{1}{2}k^3x^1.$$

The eigenvalues of Ric in M are $t_1 = t_2 = -t_3 = -\frac{1}{2}k^2$ and their symmetric ("mean") functions are

$$\rho_1 = -\frac{1}{6}k^2 \ \ (\textit{the scalar curvature}) \quad , \quad \rho_2 = \frac{1}{12}k^4 \quad , \quad \rho_3 = -\frac{1}{8}k^6.$$

Moreover, we have

Proposition 4. *The Ricci curvature of M is between $-\frac{1}{2}k^2$ and $\frac{1}{2}k^2$.*

Remark. (i) The extremum values in Proposition 4 are attained effectively. This result gives examples of Riemannian 3-manifolds with Ricci curvature "pinched" (modulo a homothety) between $-\frac{1}{2}$ and $\frac{1}{2}$.

(ii) The behaviour of the Ricci eigenvalues (in the previous remark) was already known: in [2] it is proven that for *any* Riemannian left invariant metric on the Heisenberg group, the Ricci eigenvalues may differ only by sign, the Ricci quadratic form has signature (+,-,-) and the scalar curvature is strictly negative.

However, in our particular case, the proof is different: we used a byproduct of a submanifolds theory method instead a direct algebraic calculation in the Lie algebra.

(iii) All the proofs in our paper may be expressed in invariant form, replacing coordinate formalism by calculus in the Lie algebra of G, with left invariant vector fields. The second formalism may seem simpler, but we prefered to keep alive the original "flavour" of the metrics $\tilde{g}$, discovered quite far from the "quiet" Lie theory, in the "tumultous streams" of Complex Geometry.

References

[1] K. Matsumoto, G. T. Pripoae, *Certain 4-dimensional Almost Complex Manifold and their Submanifolds*, to appear in Rend. Sem. Matem. Messina.

[2] J. Milnor, *Curvature of left invariant metrics in Lie groups*, Advances in Math., 21 (1976) 293- 329.

[3] B. O'Neill, *Semi-Riemannian Geometry* , Academic Press, New York, 1983.

[4] G. Pripoae, *Inequalities for the mean curvatures of semi- Riemannian hypersurfaces*, Tensor N.S., 54 (1993) 179-184.

[5] J.B. Ripoll, *On hypersurfaces on Lie groups*, Illinois J. Math., 35, 1 (1991) 47-55.

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