TIME DEPENDENT HAMILTON OPERATOR OF AN RLC CIRCUIT WITH A SOURCE

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Abstract

In this paper we find a Hermitian time-dependent Hamilton operator which describes an RLC circuit with a source. This operator is a special case of the Caldirola operator which describes dissipation. By using the Heisenberg equations of motion we obtain the time evolution of the corresponding annihilation and creation operators of the photons in the circuit, which describe displaced squeezed states.

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1 Introduction

Recently, diverse authors [1, 2] tried to quantize an RLC circuit with a source. In [1], a quantization for a circuit with a source is proposed and the fluctuation of the charge and the magnetic flux of the circuit in several quantum states is studied. Also in [2] with the basis of the equation of motion for an RLC circuit with sourvce, have discussed the energy fluctuation of the circuit by using a fluctuation dissipation theorem. The significance of such studies and applications in several technological branches are discussed in detail in [1, 2]; therefore we shall not discuss them here.

Our main problem is to find a time-dependent Hermitian operator describing dissipation. Such an operator is the known Caldirola operator [3], i.e.

$$\mathcal{H} = \frac{p^2}{2m} e^{-\gamma t} + V(q,t)e^{\gamma t} \tag{1}$$

with the commutator $[q,p] = i\hbar$ and γ is the friction coefficient. Operators of the form (1) have been studied by several authors [4, 7]. Three recent papers by Schuch [8]

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are interesting for linear and nonlinear Schrödinger equations for dissipative systems with broken time-reversal symmetries.

For the special case

$$V(q,t) = \frac{m}{2}\omega^2 q^2 - \mathcal{E}(t)q, \tag{2}$$

the operator (1) yields

$$\mathcal{H} = \frac{p^2}{2m}\,e^{-\gamma t} + (\frac{m}{2}\omega^2 q^2 - \mathcal{E}(t)q)e^{\gamma t} \tag{3}$$

and the Heisenberg equations of motion take the form

$$\frac{dq}{dt} = \frac{pe^{-\gamma t}}{m}, \qquad \frac{dp}{dt} = -m\omega^2 q e^{\gamma t} + \mathcal{E}(t)e^{\gamma t}. \tag{4}$$

From the above equations we obtain

$$m\frac{d^2q}{dt^2} + m\gamma\,\frac{dq}{dt} + m\omega^2 q = \mathcal{E}(t) \tag{5}$$

and according to Zki-Ming et al [1], the above equation coincides with the equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \mathcal{E}(t) \tag{6}$$

of an RLC circuit with a source, by the transformation

$$p \to \Phi,\ q \to q,\ m \to L,\ \gamma \to \frac{R}{L},\ m\omega^2 \to \frac{1}{C},$$

where L, R, C and $\mathcal{E}$ are the inductance, resistance, capacitance and the electromotive force of the circuit, respectively. By using the charge q and the magnetic flux Φ, with the commutator $[q,\Phi] = i\hbar$, the Hamilton operator (3) takes the form

$$\mathcal{H} = \frac{\Phi^2}{2L}e^{-\frac{R}{L}\,t} + (\frac{1}{2C}q^2 - \mathcal{E}(t)q)e^{\frac{R}{L}\,t} \tag{7}$$

and the corresponding Heisenberg equations of motion are written

$$\frac{dq}{dt} = \frac{\Phi}{L}e^{-\frac{R}{L}t}, \quad \frac{d\Phi}{dt} = (-\frac{1}{C}\,q + \mathcal{E}(t))e^{\frac{R}{L}t}. \tag{8}$$

In the following we shall study the equation (6), i.e.

$$\frac{d^2q}{dt^2} + \gamma\frac{dq}{dt} + \omega^2 q(t) = \frac{1}{L}\mathcal{E}(t) = F(t), \tag{9}$$

where $\gamma = \frac{R}{L}$ and $\omega^2 = \frac{1}{LC}$.

2 Solution of the equation (9)

A partial solution of the eq. (9) has the form

$$q_1(t)=\int_{-\infty}^{\infty}\frac{F(t')}{\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\omega^2}\,\delta(t-t')dt'=\frac{1}{2\pi}\int\int_{-\infty}^{\infty}\frac{F(t')e^{i\omega'(t-t')}}{-\omega'^2+i\gamma\omega'+\omega^2}dt'd\omega \quad (10)$$

$$=\int_{-\infty}^{\infty}\frac{\widetilde{F}(\omega')e^{i\omega' t}}{-\omega'^2+i\gamma\omega'+\omega^2}d\omega',$$

where

$$\widetilde{F}(\omega')=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(t')e^{-i\omega' t'}dt'=\frac{1}{2\pi L}\int_{-\infty}^{\infty}e^{-i\omega' t'}\mathcal{E}(t')dt' \quad (11)$$

is the Fourier transform function. The general solution of eq. (9) is given by

$$q(t)=q_1(t)+e^{-\frac{\gamma}{2}t}(A\cos\Omega t+B\sin\Omega t) \quad (12)$$

and

$$\Phi(t)=Le^{\gamma t}\frac{dq}{dt}=L\Omega e^{\frac{\gamma}{2}t}[\frac{\dot{q}_1(t)}{\Omega}e^{\frac{\gamma}{2}t}-(\sin\Omega t+\frac{\gamma}{2\Omega}\cos\Omega t)A+(\cos\Omega t-\frac{\gamma}{2\Omega}\sin\Omega t)B], \quad (13)$$

where

$$\Omega^2=\omega^2-\frac{\gamma^2}{4}=\frac{1}{LC}-\frac{R^2}{4L^2}. \quad (14)$$

Here we have the following cases:

1. the case of weak external friction $\omega>\frac{\gamma}{2}$;
2. the case of critical external friction $\omega=\frac{\gamma}{2}$;
3. the case of strong external friction $\omega<\frac{\gamma}{2}$.

For the first case and t=0 we obtain for the constants A and B

$$A=q(0)-q_1(0), \quad (15)$$

$$B=\frac{\Phi(0)}{L\Omega}+\frac{\gamma}{2\Omega}(q(0)-q_1(0)), \quad (16)$$

where the solutions (12) and (13) take the following forms:

$$q(t)=q_1(t)+e^{-\frac{\gamma t}{2}}[(\cos\Omega t+\frac{\gamma}{2\Omega}\sin\Omega t)(q(0)-q_1(0))+(\frac{\Phi(0)}{L\Omega}-\frac{\dot{q}_1(0)}{\Omega})\sin\Omega t], \quad (17)$$

$$\Phi(t)=e^{\frac{\gamma t}{2}}[(\cos\Omega t-\frac{\gamma}{2\Omega}\sin\Omega t)\Phi(0)-\frac{L\Omega^2}{\Omega}(q_0-q_1(0))\sin\Omega t$$

$$+L(\dot{q}_1(t)e^{\frac{\gamma t}{2}} - \dot{q}_1(0)(\cos\Omega t - \frac{\gamma}{2\Omega}\sin\Omega t))]. \tag{18}$$

The above operators q(t) and $\Phi(t)$ satisfy the relation

$$[q(t), p(t)] = [q(0), \Phi(0)] = i\hbar \tag{19}$$

and the uncertainty principle holds, i.e.

$$\Delta q \Delta\Phi \geq \frac{\hbar}{2}. \tag{20}$$

From the relations

$$a(t) = \frac{1}{\sqrt{2\hbar}}[\sqrt{L\omega}e^{\frac{\gamma t}{2}}q(t) + i\frac{e^{\frac{-\gamma t}{2}}p(t)}{\sqrt{L\omega}}], \tag{21}$$

$$a^+(t) = \frac{1}{\sqrt{2\hbar}}[\sqrt{L\omega}e^{\frac{\gamma t}{2}}q(t) - i\frac{e^{\frac{-\gamma t}{2}}p(t)}{\sqrt{L\omega}}], \tag{22}$$

where $a(t)$ and $a^+(t)$ are the annihilation and creation operators of the photons in the circuit and satisfy the commutation relations $[a(t), a^+(t)] = 1$. Substituting $e^{\frac{\gamma t}{2}}q(t)$ and $e^{\frac{-\gamma t}{2}}\Phi(t)$ from eq. (17), (18) in (21), (22) and after some algebra, we obtain

$$a(t) = (\cos\Omega t - i\frac{\omega}{\Omega}\sin\Omega t)a(0) + \frac{\gamma}{2\Omega}\sin\Omega t : a^+(0) + b(t), \tag{23}$$

$$a^+(t) = (\cos\Omega t + i\frac{\omega}{\Omega}\sin\Omega t)a^+(0) + \frac{\gamma}{2\Omega}\sin\Omega t : a(0) + b^*(t), \tag{24}$$

where

$$b(t) = \sqrt{\frac{L\Omega}{2\hbar}}\{[q_1(t)e^{\frac{\gamma t}{2}} - q_1(0)(\cos\Omega t - \frac{\gamma}{2\Omega}\sin\Omega t) - \frac{\dot{q}_1(0)}{\Omega}\sin\Omega t]$$

$$+i[\frac{\omega}{\Omega}\sin\Omega t : q_1(0) + \frac{1}{\omega}(\dot{q}_1(t) - \dot{q}_1(0)(\cos\Omega t - \frac{\gamma}{2\Omega}\sin\Omega t)]\}. \tag{25}$$

From the above results we see that the operators $a(t)$ and $a^+(t)$ satisfy the commutation relation

$$[a(t), a^+(t)] = [a(0), a^+(0)] = 1. \tag{26}$$

The time evolution of the operators $a(t)$ and $a^+(t)$ leads to operators of the Yuen [9] type, i.e.

$$a(t) = \mu(t)a(0) + \nu(t)a^+(0) + b(t), \tag{27}$$

$$a^+(t) = \mu^*(t)a^+(0) + \nu^*(t)a(0) + b^*(t), \tag{28}$$

where

$$\mu(t) = \cos\Omega t - i\frac{\omega}{\Omega}\sin\Omega t, \qquad \nu(t) = \nu^*(t) = \frac{\gamma}{2\Omega}\sin\Omega t, \tag{29}$$

with the relation

$$|\mu(t)|^2 - |\nu(t)|^2 = 1 \tag{30}$$

and describe displaced squeezed number states.

From the above results we see that by considering the hermitian time-dependent Hamilton operator (7) and the time evolution of the operators $a(t)$ and $a^+(t)$, we are led to nonclassical effects and mainly to displaced squeezed number states that we shall study briefly in the following.

3 Squeezing effect

The squeezed states are nonclassical states, which were first introduced and studied in the field of quantum optics with the ultimate aim to obtain a reduced fluctuation in one field quadrature, at the expence of an increased fluctuation in the other [10], leading to an increase in the signal-to-noise ratio in suitable experiments ranging from optical communication to the detection of gravitational radiation.

From the operators (27) and (28) we distinguish two cases. The first case is without external source, i.e. $\mathcal{E}(t) = 0$, and the second case is for $\mathcal{E} \neq 0$.

According to Yuen [9] the uncertainty principle for the first case is given by

$$(\Delta q \Delta \Phi) \geq \frac{\hbar}{2}|\mu + \nu|.|\mu - \nu|. \tag{31}$$

Substituting $\mu(t)$ and $\nu(t)$ from eq.(29) and after some algebra we obtain

$$\Delta q \Delta \Phi \geq \frac{\hbar}{2}\sqrt{1 + \frac{\gamma^2}{\Omega^2}(1 + \frac{\gamma^2}{4\Omega^2})\sin^4 \Omega t} \geq \frac{\hbar}{2}, \tag{32}$$

which is a periodical function of time. For $\Omega t = \pi n, : n = 0, \pm 1, \pm 2, ...$ the above relation coincides with the minimal uncertainty product MUP.

The case $\mathcal{E} \neq 0$ has been studied in [10] for generating displaced squeezed number states (in the harmonic approximation) of a driven time-dependent Hamiltonian with an $SU(1,1) \oplus h(4)$ algebraic structure. The authors of ref.10, by using the evolution operator method have found the exact wavefunction $|\psi(t)>$ (formula (24) of ref.10), the number operator

$$N(t)|\psi(t)>= a^+(t)a(t)|\psi(t)>= n|\psi(t)> \tag{33}$$

the uncertainty principle and the exact forms of the occupation probabilities both for displaced squeezed number and vacuum states. More details one can find in ref.10.

4 Conclusion

In this paper we have found the Hermitian time-dependent Hamilton operator which describes the RLC circuit with a source. This operator is a special case of the Caldirola operator which describes dissipative systems. From the Heisenberg equations of motion we obtain the time evolution of the charge and the magnetic flux operators and also the time evolution of the corresponding annihilation and creation operators of the photons in the circuit, which are Yuen [9] operators and describe displaced squeezed number and vacuum states, for which the Heisenberg uncertainty relation holds.

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