DENSITY OF LIPSCHITZ FUNCTIONS

Themistokles Andreou

Abstract

In this note some results are presented on approximation of continuous function on metric spaces by Lipschitz functions.

AMS Subject Classification: 46J10

Key words: Banach spaces, Lipschitz functions, Approximation of functions, Density of Lipschitz functions.

1 Introduction

Let S be a metric space, and E a (real or complex) normed linear space. Let $BC(S,E)$ be the space of bounded continuous functions $f : S \to E$, supplied with the uniform norm. Now if S be a locally compact metric space, let $C_{00}\left(S,E\right)$ be the subset of $BC(S,E)$, consisting of all f in $BC(S,E)$ that vanish in a neighborhood of infinity, i.e. for some compact subset L of S, (depending on f), $f(s) = 0$ for all $s \in S - L$. Let $C_0\left(S,E\right)$ be the subset of $BC(S,E)$, consisting of all $f \in BC(S,E)$ that vanish at infinity, i.e. for every positive number ε, there is a compact subset K of S (depending on f and ε), such that $\|f(s)\| < \varepsilon$ for all $s \in S - K$, and let $Lip_{00}\left(S,E\right)$ be the subset of $C_{00}\left(S,E\right)$ consisting of Lipschitz functions.

All these sets $C_{00}\left(S,E\right)$, $C_0\left(S,E\right)$ and $Lip_{00}\left(S,E\right)$, supplied with the uniform norm, are normed linear spaces and the following inclusions hold:

$$Lip_{00}\left(S,E\right) \subset C_{00}\left(S,E\right) \subset C_0\left(S,E\right).$$

When $E = K$,i.e. the space of real or complex numbers, we write $BC(S)$, $C_{00}\left(S\right)$, $C_0\left(S\right)$ and $Lip_{00}\left(S\right)$ respectively.

The following result is proved in [2].

Lemma 1 *Let S be a compact metric space and $\{A_i : 1 \leq i \leq n\}$ a finite open covering of S. Then there exists a Lipschitz - partition of unity of S, subordinated to $\{A_i : 1 \leq i \leq n\}$.*

In the present note, following [1] and [2], we give a few results on the density of Lipschitz function.

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2001,* 1-4

2 Lipschitz functions

Let S be a locally compact metric space and E a normed linear space.

Theorem 2 *$Lip_{00}(S,E)$ is dense in $C_0(S,E)$.*

Proof. Let h be in $C_{00}(S,E)$; then there is a compact set L, such that $h(s)=0$ for all $s\in S-L$. By [2], given $\varepsilon>0$, there is a Lipschitz function $H:L\to E$, i.e. a function H in $Lip_{00}(S,E)$, such that

$$\|h-H\|<\frac{\varepsilon}{2}. \tag{1}$$

Now let f be in $C_0(S,E)$. By definition for a given positive ε, the set

$$K=\left\{s\in S:\|f(s)\|\geq\frac{\varepsilon}{2}\right\}$$

is a compact subset of S. It is well known that there is a function $g\in C_{00}(S)$, such that $g(s)=1$ for every $s\in K$ and $0\leq g\leq 1$ on S.

Taking the function $h=gf\in C_{00}(S,E)$ we have

$$\|f(s)-H(s)\|\leq\|f(s)-h(s)\|+\|h(s)-H(s)\|.$$

But

$$\|f(s)-h(s)\|=\|f(s)-g(s)f(s)\|=\|f(s)\|\,|1-g(s)|.$$

For $s\in K$ we have $|1-g(s)|=0$, and for $s\in S-K$, $\|f(s)\|<\frac{\varepsilon}{2}$ and $|1-g(s)|\leq 1$ so that, for every $s\in S$:

$$\|f(s)-h(s)\|<\frac{\varepsilon}{2}. \tag{2}$$

From (1) and (2) follows that for every $s\in S$

$$\|f(s)-H(s)\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

and $H\in Lip_{00}(S,E)$, i.e. $Lip_{00}(S,E)$ is dense in $C_0(S,E)$.

3 Products in metric spaces

For $i=1,2,...,n$, let S_i be a metric space and g_i be a K-valued function on S_i. Let $S=S_1\times S_2\times...\times S_n$ be the Cartesian product of all S_i. The product $g=g_1\times g_2\times...\times g_n$ is a K-valued function on S defined by $g(s)=g_1(s_1)g_2(s_2)...g_n(s_n)$ for every $s=(s_1,s_2,...,s_n)$, where $s_i\in S_i$, $i=1,2,...,n$.

We set $\mathop{\times}\limits_{i=1}^{n} V_i$ for the set of all functions of the form $g_1\times g_2\times...\times g_n$, where g_i is in the function space V_i, $i=1,2,...,n$.

It is well known that S is a (locally) compact metric space when S_i, $i=1,2,...,n$ are (locally) compact metric spaces.

We prove the following

Theorem 3 *Let S_i, $i = 1, 2, ..., n$ be compact metric spaces and $S = S_1 \times S_2 \times ... \times S_n$. The set of all linear combinations of functions of the form $g_1 \times g_2 \times ... \times g_n$, where g_i be a K-valued Lipschitz function on S_i, $i = 1, 2, ..., n$, is dense in $BC(S)$.*

Proof. It is sufficient to establish the theorem for $n = 2$. The general case then follows by induction.

Let f be in $BC(S)$, where $S = S_1 \times S_2$ is compact. Given $\varepsilon > 0$, there is an open covering $\{A_i : 1 \leq i \leq p\}$ of S_1 and an open covering $\{B_j : 1 \leq j \leq q\}$ of S_2 such that $s_1, s_2 \in A_i$ and $t_1, t_2 \in B_j$ imply

$$\|f(s_1, t_1) - f(s_2, t_2)\| < \varepsilon. \tag{3}$$

Let a_{ij} be a value of f in $A_i \times B_j$, $1 \leq i \leq p$, $1 \leq j \leq q$ and let $\{g_{1i} : 1 \leq i \leq p\}$ be a Lipschitz-partition of unity of S_1, subordinated to $\{A_i : 1 \leq i \leq p\}$, and $\{g_{2j} : 1 \leq j \leq q\}$ be a Lipschitz-partition of unity of S_2, subordinated to $\{B_j : 1 \leq j \leq q\}$.

Define the function $F : S \to K$ by

$$F(s,t) = \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} g_{1i}(s) g_{2j}(t), \quad s \in S_1 \quad t \in S_2;$$

then

$$\begin{aligned} \|f(s,t) - F(s,t)\| &= \left\| f(s,t) - \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} g_{1i}(s) g_{2j}(t) \right\| = \\ &= \left\| \left(\sum_{i=1}^{p} \sum_{j=1}^{q} f(s,t) g_{1i}(s) g_{2j}(t)\right) - \sum_{i=1}^{p} \sum_{j=1}^{q} a_{ij} g_{1i}(s) g_{2j}(t) \right\| = \\ &= \left\| \sum_{i=1}^{p} \sum_{j=1}^{q} g_{1i}(s) g_{2j}(t) (f(s,t) - a_{ij}) \right\| \leq \\ &\leq \sum_{i=1}^{p} \sum_{j=1}^{q} g_{1i}(s) g_{2j}(t) \, \|f(s,t) - a_{ij}\| < \varepsilon \end{aligned}$$

This follows from the fact that

$$\sum_{i=1}^{p} \sum_{j=1}^{q} g_{1i}(s) g_{2j}(t) = \sum_{i=1}^{p} g_{1i}(s) \sum_{j=1}^{q} g_{2j}(t) = 1$$

and from

$$g_{1i}(s) g_{2j}(t) \, \|f(s,t) - a_{ij}\| < \varepsilon.$$

This estimate is verified by noticing, from (3), that $\|f(s,t) - a_{ij}\| < \varepsilon$ if $(s,t) \in A_i \times B_j$, and that $g_{1i}(s) g_{2j}(t) = 0$ if $(s,t) \notin A_i \times B_j$.

The proof is completed by noticing that $g_{1i} \times g_{2j}$ is a product of K-valued Lipschitz functions $g_{1i} : S_1 \to K$ and $g_{2j} : S_2 \to K$.

4 Products in locally compact metric spaces

When S_i, $i = 1, 2, ..., n$ are locally compact metric spaces, we have the following result.

Theorem 4 *Let S_i, $i = 1, 2, ..., n$ be locally compact metric spaces and $S = S_1 \times S_2 \times ... \times S_n$. The set of all linear combinations of functions of the form $g_1 \times g_2 \times ... \times g_n$, where $g_i \in Lip_{00}(S_i)$, is dense in $C_0(S)$.*

Proof. We first show that the set of all linear combinations of functions of the form $g_1 \times g_2 \times ... \times g_n$, $g_i \in Lip_{00}(S_i)$ is dense in $Lip_{00}(S)$.

Let f be in $Lip_{00}(S)$ and $L \subset S$ a compact set such that $f(s) = 0$ for all $s \in S - L$. Since f is continuous on L, by theorem 3, can be arbitrarily uniformly approximated by a finite sum of functions of the form $g_1 \times g_2 \times ... \times g_n$, where $g_i : L_i \to K$ is a Lipschitz function on L_i, and L_i is the projection of L on S_i; hence g_i is in $Lip_{00}(S_i)$; i.e. the set of all linear combinations of functions in $\underset{i=1}{\overset{n}{\times}} Lip_{00}(S_i)$ is dense in $Lip_{00}(S)$.

By theorem 2, $Lip_{00}(S)$ is dense in $C_0(S)$, hence the set of all linear combinations of functions in $\underset{i=1}{\overset{n}{\times}} Lip_{00}(S_i)$ is dense in $C_0(S)$.

References

[1] J. Dieudonné, *Sur les fonctions continues numeriques défini dans un produit de deux espaces compacts*, Comptes Rendus Acad. Sc., Paris, t. 205(1937) 593-595.

[2] G. Georganopoulos, *Sur l' approximation des fonctions continues par de fonctions lipschitziennes*, Comptes Rendus Acad. Sc., Paris, 264(1967) 319-321.

Author's address:

Themistokles Andreou
Technological Education Institute Terma Magnisias
62124 Serres, GREECE
E-mail: tandr@teiser.gr