The Effect of Tsallis Statistics on the Black Body Spectrum

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Abstract

By applyng Tssalis statistics to each model of thermal black body radiation we show how gigh frequency modifications to the Planck distributions may arise. We also demonstrate that Tsallis statistics does not affect the equation of state $\left(P = \frac{\varepsilon}{3}\right)$ for thermal radiation and how a high frequency cut-off the modified Planck distribution occurs naturally because of non-extensive statistics.

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1 Introduction

In the past decades there has been a flowering of interest in modifications to conventional statistics brought about by studies in multi-fractals. All of the well-known results of both classical statistical mechanics and quantum statistical mechanics are based on an apriori equi-probability in phase base (classical statistics) and the exclusion principle or Bose-Fermi symmetry in quantum statistics. However, when non-Markovian memory effects and long range interactions are present, Tsallis demonstrated that the idea of a multi-fractal structure is a better representation of the statistical system are distributed. Here self-similary and scale invariance replace the principles applied in previous applications of statistical mechanics. Subsequent applications of Tsallis statistics to the solar plasma, a generalized H theorem, the fluctuation dissipation theorem, the Langevin and Fokker-Planckequation, the equipartition theorem, the Ising chain, paramagnetic systems, and the Planck radiation law, have lead to the usual treatments and provisional tests for the new statistics. Limits on the non-extensive statistics parameter can be set in cosmology by finding out how the new statistics effects the primordial helium abundance. In previous notes we have applied the new statistics to a two level system, to the two state paramagnetic system, and the Debye theory of specific heats at low temperatures. In the following motivated

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by discussions of [18, 19] we study how the Planck radiation law is altered in the new statistics using a model that allows the non-extensive parameter to depend on the frequency of the oscillators. For this particular model we show that the equation of state $\left(\frac{\varepsilon}{3}\right)$ is unaltered in cosmology and secondly high frequency corrections to the Planck radiation law lead to a high frequency cut-off for the energy density versus frequency. We show that any experimental measurements of such a high frequency cut-off at given T can lead to a numerical estimate of the non-extensive parameter q.

2 Corrections to the Planck Spectra and Tsallis Statistics

We begin by writing the entropy of N particles within the context of Tsallis statistics as [24]

$$S = \frac{kN}{q-1}\left(\sum P_i - \sum P_i^q\right), \left(P_i = \frac{N_i}{N}\right) \tag{2.1}$$

q=non-extensive statistics parameters.

Varyng equation (2.1) with respect to N_i using the constraints

$$\begin{aligned} \sum N_i &= const., \\ \sum N_i \varepsilon_i &= const., \end{aligned}$$

(with Lagrange multipliers, $\frac{\mu}{\tau}, \frac{-1}{\tau}$), we find

$$N_i = Ne^{-1}\, e^{\frac{\mu_0-\varepsilon_i}{\tau}} + \alpha N e^{-1}\left[\frac{e^{\frac{\mu_0-\varepsilon_i}{\tau}}}{2} + \frac{\mu_1}{\tau}e^{\frac{\mu_0-\varepsilon_i}{\tau}} - e^{\frac{\mu_0-\varepsilon_i}{\tau}}\frac{(\mu_0-\varepsilon_i)^2}{2\tau^2}\right]. \tag{2.2}$$

In (2.2) we use $\alpha = q-1, \mu = \mu_0 + \alpha\mu_1 + \alpha^2\mu_2$ and calculated the perturbations in $N(q-1)$. Also from [24]:

$$\mu_1 = -\frac{\tau}{2} + \frac{1}{2\tau}\frac{\sum e^{\frac{\mu_0-\varepsilon_i}{\tau}}(\mu_0-\varepsilon_i)^2}{\sum e^{\frac{\mu_0-\varepsilon_i}{\tau}}} - e^{\frac{\mu_0-\varepsilon_i}{\tau}}\frac{(\mu_0-\varepsilon_i)^2}{2\tau^2}. \tag{2.3}$$

Substituting (2.3) into (2.2) gives

$$\begin{aligned} N_i &= Ne^{-1}\, e^{\frac{\mu_0-\varepsilon_i}{\tau}} + \\ & \alpha N e^{-1}\left[\frac{1}{2\tau^2}e^{\frac{\mu_0-\varepsilon_i}{\tau}}\frac{\sum e^{\frac{\mu_0-\varepsilon_i}{\tau}}(\mu_0-\varepsilon_i)^2}{\sum e^{\frac{\mu_0-\varepsilon_i}{\tau}}} - e^{\frac{\mu_0-\varepsilon_i}{\tau}}\frac{(\mu_0-\varepsilon_i)^2}{2\tau^2}\right]. \end{aligned} \tag{2.4}$$

Now for the average energy per particle we have

$$\begin{aligned} \langle\varepsilon\rangle &= \frac{\sum N_i\varepsilon_i}{N} = \sum Ne^{-1}\varepsilon_i e^{\frac{\mu_0-\varepsilon_i}{\tau}} + \\ & \alpha e^{-1}\left[\frac{\frac{1}{2\tau^2}\sum\varepsilon_i e^{\frac{\mu_0-\varepsilon_i}{\tau}}\sum(\mu_0-\varepsilon_i)^2 e^{\frac{\mu_0-\varepsilon_i}{\tau}}}{\sum e^{\frac{\mu_0-\varepsilon_i}{\tau}}} - \frac{1}{2\tau^2}\sum\varepsilon_i(\mu_0-\varepsilon_i)^2 e^{\frac{\mu_0-\varepsilon_i}{\tau}}\right] \end{aligned} \tag{2.5}$$

Also from [24]

$$e^{\frac{\mu_0}{\tau}} = \frac{e}{\sum e^{\frac{\mu i}{\tau}}}. \tag{2.6}$$

Using equations (2.6) and (2.5) we now evaluate the average energy of oscillator of frequency ω (which represents the average energy in a mode ω of the electromagnetic field). From (2.6) we calculate exactly

$$\left(\varepsilon_n = \left(n + \frac{1}{2}\right) nh\omega\right) e^{\frac{\mu_0}{\tau}} = \frac{e}{e^{\frac{-h\omega}{2\tau}} \sum e^{\frac{-nh\omega}{\tau}}} = ee^{\frac{h\omega}{2\tau}} \left(1 - e^{\frac{-h\omega}{\tau}}\right). \tag{2.7}$$

or

$$\mu_0 = \tau + \frac{h\omega}{2} + \tau \ln_e \left(1 - e^{\frac{-h\omega}{\tau}}\right).$$

Using (2.5) we may evaluate the average energy of an oscillator (after inserting $e^{\frac{\mu_0}{\tau}}$ from (2.6) as

$$\begin{aligned}\varepsilon \quad = \quad & \frac{\sum \varepsilon_i N_i}{N} = \frac{\sum \varepsilon_i e^{\frac{-\varepsilon_i}{\tau}}}{\sum_i e^{\frac{-\varepsilon_i}{\tau}}} + \\ & \alpha e^{\frac{h\omega}{2\tau}} \left(1 - e^{\frac{-h\omega}{\tau}}\right) \cdot \left[\frac{\left(\sum \varepsilon_i e^{\frac{-\varepsilon_i}{\tau}}\right) \sum \left(\mu_0^2 - 2\mu_0\varepsilon_i + \varepsilon_i^2\right) e^{\frac{-\varepsilon_i}{\tau}}}{2\tau^2 \sum_i e^{\frac{-\varepsilon_i}{\tau}}} - \right. \\ & \left. \frac{1}{2\tau^2} \sum \left(\mu_0^2 - 2\mu_0\varepsilon_i + \varepsilon_i^2\right) e^{\frac{-\varepsilon_i}{\tau}}\right].\end{aligned} \tag{2.8}$$

Since we have chosen the high frequency domain we may evaluate the sums $\sum \varepsilon_i^2 e^{\frac{-\varepsilon_i}{\tau}}$ and $\sum \varepsilon_i^3 e^{\frac{-\varepsilon_i}{\tau}}$ by integrating over n. For the sums $\sum \varepsilon_n^2 e^{\frac{-\varepsilon_n}{\tau}}$, $\sum e^{\frac{-\varepsilon_n}{\tau}}$ we may evaluate them exactly $\left(\varepsilon_n = \left(n + \frac{1}{2}\right) h\omega\right)$ in (2.8) we have $(i = n)$:

$$\begin{aligned}\sum \varepsilon_n^2 e^{\frac{-\varepsilon n}{\tau}} \quad & \rightarrow \quad \int \left(n + \frac{1}{2}\right)^2 (h\omega)^2 e^{-\left(n+\frac{1}{2}\right)\frac{h\omega}{\tau}} dn \rightarrow \\ & \qquad (h\omega)^2 \int x^2 e^{\frac{-xh\omega}{\tau}} dx, \left(x = n + \frac{1}{2}\right) \\ \text{and } \sum \varepsilon_n^3 e^{\frac{-\varepsilon n}{\tau}} \quad & \rightarrow \quad (h\omega)^3 \int x^2 e^{\frac{-xh\omega}{\tau}} dx.\end{aligned} \tag{2.9}$$

Using (2.9), the expression $\sum e^{\frac{-\varepsilon n}{\tau}} = \frac{e^{\frac{h\omega}{2\tau}}}{1-e^{\frac{h\omega}{2\tau}}}$ and $\sum \varepsilon_n e^{\frac{-\varepsilon n}{\tau}} = \left(\frac{h\omega}{2} + \frac{h\omega}{e^{\frac{h\omega}{\tau}} - 1}\right) \frac{e^{\frac{-h\omega}{2\tau}}}{1-e^{\frac{-h\omega}{\tau}}}$ in (2.8) we find

$$\langle\varepsilon\rangle = \frac{h\omega}{2} + \frac{h\omega}{\left(e^{\frac{h\omega}{\tau}} - 1\right)} + \alpha e^{\frac{h\omega}{2\tau}} \left(1 - e^{\frac{-h\omega}{\tau}}\right) \left[\ \frac{1}{2\tau} \left(\frac{h\omega}{2} + \frac{h\omega}{\left(e^{\frac{h\omega}{\tau}} - 1\right)}\right) \alpha_{11-}\alpha_{22}\ \right],$$

where

$$\alpha_{11} = -2\mu_0 \left(\frac{h\omega}{2} + \frac{h\omega}{\left(e^{\frac{h\omega}{\tau}-} - 1\right)} \right) - \frac{e^{\frac{-h\omega}{2\tau}}}{1 - e^{\frac{-h\omega}{\tau}}} + (h\omega)^2 \int_{\frac{1}{2}}^{\infty} x^2 e^{\frac{-xh\omega}{\tau}} dx,$$

$$\begin{aligned} \alpha_{22} &= -\frac{1}{2\tau^2} \left(\mu_0^2 \left(\frac{h\omega}{2} + \frac{h\omega}{\left(e^{\frac{h\omega}{\tau}-} - 1\right)} \right) \frac{e^{\frac{-h\omega}{2\tau}}}{1 - e^{\frac{-h\omega}{\tau}}} - \right. \\ & \left. 2\mu_0 (h\omega)^2 \int_{\frac{1}{2}}^{\infty} x^2 e^{\frac{-xh\omega}{\tau}} dx + (h\omega)^3 \int_{\frac{1}{2}}^{\infty} x^3 e^{\frac{-xh\omega}{\tau}} dx \right). \end{aligned}$$

In (2.10) we insert the value of μ_0 in (2.7). For the integrals $\int x^2 e^{\frac{-xh\omega}{\tau}} dx$, $\int x^3 e^{\frac{-xh\omega}{\tau}} dx$ we have:

$$\begin{aligned} \int x^2 e^{\frac{-xh\omega}{\tau}} dx &= e^{\frac{-xh\omega}{\tau}} \left(\frac{1}{8} \left(\frac{\tau}{h\omega} \right) + \left(\frac{\tau}{h\omega} \right)^2 + \left(\frac{\tau}{h\omega} \right)^3 \right), \\ \int x^3 e^{\frac{-xh\omega}{\tau}} dx &= e^{\frac{-xh\omega}{\tau}} \left(\frac{1}{8} \left(\frac{\tau}{h\omega} \right) + \frac{3}{4} \left(\frac{\tau}{h\omega} \right)^2 + 3 \left(\frac{\tau}{h\omega} \right)^3 + 6 \left(\frac{\tau}{h\omega} \right)^3 \right). \end{aligned} \tag{2.11}$$

In (2.10) we study the high frequency behavior $\left(\frac{\tau}{h\omega} >> 1\right)$, using the expression for μ_0 in (2.7), also taking the leading correction to (2.10) for $h\omega >> \tau$, we find

$$\langle \varepsilon \rangle = \frac{h\omega}{2} + \frac{h\omega}{e^{\frac{h\omega}{\tau}} - 1} - \alpha \frac{(h\omega)^3}{8\tau^2} + \frac{\alpha}{8\tau} (h\omega)^2. \tag{2.12}$$

In (2.12), if we neglect the vacuum term $\frac{h\omega}{2}$ we have

$$\langle \varepsilon \rangle = \frac{h\omega}{\left(e^{\frac{h\omega}{\tau}} - 1\right)} - \frac{1}{8} \alpha \frac{(h\omega)^3}{\tau^2} + \frac{\alpha}{8\tau} (h\omega)^2. \tag{2.13}$$

The equation (2.13) has the curious property that at a certain critical frequency $\langle \varepsilon \rangle = 0$, letting $\frac{h\omega}{\tau} = x$, we for $\langle \varepsilon \rangle = 0$, $\left(\frac{1}{\left(e^{\frac{h\omega}{\tau}} - 1\right)} \cong e^{\frac{h\omega}{\tau}} \right)$

$$e^{-x} = \frac{1}{8} \alpha x^2 - \frac{\alpha x}{8}. \tag{2.14}$$

Thus the solution to the transcendental equation (2.14) gives

$$x_c = c = \frac{h\omega_c}{\tau} = \frac{h\omega_c}{kT},$$

or

$$\omega_c = \frac{c\,(kT)}{h} = 2\pi v_c \,(c = const.)\,.$$

Thus the spectrum will cut-of at ω_c. We secondly choose α to be dependent on ω at a given τ $(\tau = kT)$, such that

$\alpha = 0$ for $\frac{h\omega}{\tau} = \frac{h\omega}{kT} < 1$,

$\alpha = \alpha$ for $\frac{h\omega}{\tau} > 1$.

In this case the total energy per unit volume is using (2.13) (neglecting vacuum energy):

$$\begin{aligned} U &= \int \frac{h\vartheta}{\left(e^{\frac{h\vartheta}{kT}} - 1\right)} \left(\frac{8\pi\vartheta^2 d\vartheta}{c^3}\right) + \int \frac{h\vartheta}{\left(e^{\frac{h\vartheta}{kT}} - 1\right)} \left(\frac{8\pi\vartheta^2 d\vartheta}{c^3}\right) - \\ &\quad \frac{\alpha}{8k^2T^2} \int (h\vartheta)^3 \frac{8\pi\vartheta^2 d\vartheta}{c^3} + \frac{\alpha}{8kT} \int (h\vartheta)^2 \frac{8\pi\vartheta^2 d\vartheta}{c^3}. \end{aligned} \tag{2.15}$$

In (2.15) we have used (2.13) for $\langle\varepsilon\rangle$ for $\frac{h\vartheta}{kT} > 1$. We now change integration variables to $x = \frac{h\vartheta}{kT}$ giving

$$U = \left[\begin{array}{c} \left(\frac{k}{h}\right)^4 h \left(\frac{8\pi}{c^3}\right) \int \frac{x^3 dx}{e^x - 1} + \left(\frac{k}{h}\right)^4 h \left(\frac{8\pi}{c^3}\right) \int \frac{x^3 dx}{e^x - 1} \\ -\frac{\alpha}{8k^2} \left(\frac{8\pi}{c^3}\right) \left(h^3\right) \left(\frac{h}{k}\right)^6 \int x^5 dx + \frac{\alpha}{8k} \left(\frac{8\pi}{c^3}\right) \left(h^2\right) \left(\frac{h}{k}\right)^5 \int x^4 dx \end{array} \right] T^4. \tag{2.16}$$

Thus, (2.16) will give a modified Stefan Boltzmann constant with $U = kT^4$ for the energy per unit volume. It also leads to $P = \frac{U}{3}$ and will not alter the evolution of the scale factor during the radiation era. Of course, we have chose a specific model $\alpha = q - 1$ which was

$\alpha = 0$ for $\frac{h\vartheta}{kT} < 1$ (T fixed),

$\alpha = \alpha$ for $\frac{h\vartheta}{kT} > 1$.

The unique feature of our discussion is that independent of the model for how α varies for high v (2.13) holds and leads to a cut-off the Planck spectrum for $v > v_c$.

3 Conclusion

The results above suggest modification to the Planck distribution occur at high v that would lead to distinct features in thermal radiation that if red-shifted down would produce a cut-off even in the low temperature phase of the universe. Anomolies due to heavy neutrino decay and Higgsino decay have previously been suggested to explain spikes and irregular patterns in the CBR. Also the mixing of the photon with the para-photon has also been suggested to explain certain features of the CBR not associated with the primordial density fluctuations generated during inflation. However, the cut-off predicted by (2.13) tells us that at very high v (small α) the spectra abruptly to 0. In observations of gamma ray burst anomolies, there is evidence that a high energy cut-off exists. Whrtever this is the result the dynamics of particle interactions creating the bursts or anomalous effects in the photons producedis a question that clearly has no answer at this time. Hovewer, this unanswered question suggests that radiation in the high frequency region emitted by astrophysical source should be studied to see if it exhibits any of the spectral features suggested by (2.13) lending evidence to a modification of the statistics of photons. Lastly, if α turns out to be temperature dependent this would lead to modifications of the total energy $U(T)$ that would deviate from the T^4 dependence in (2.16) . Such effects would also produce nonstandard cosmological evolution of the scale factor during the radiation era that might have an effect on the time scale for transition from the radiation era to the matter dominated era

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