VECTOR SHEAVES ASSOCIATED WITH PRINCIPAL SHEAVES

Efstathios Vassiliou

Abstract

In the framework of Abstract Differential Geometry, especially that dealing with vector sheaves (as expounded in [8]) and principal sheaves (initiated by [9]), we show that to a given principal sheaf $(\mathcal{P}, \mathcal{G}, X, \pi)$ together with a representation $\varphi : \mathcal{G} \to \mathcal{G}L(n, \mathcal{A})$, we associate a vector sheaf $(\mathcal{E}, X, p)$. If φ is compatible with the representations of $\mathcal{G}$ and $\mathcal{G}L(n, \mathcal{A})$ into appropriate sheaves of Lie algebras, as well as with the Maurer-Cartan (or logarithmic) differentials of the same sheaves of groups, then every connection on $\mathcal{P}$ induces an $\mathcal{A}$-linear connection on $\mathcal{E}$. An example is provided by the principal sheaf of frames of a vector sheaf.

AMS subject classification: 53C05, 58A40, 18F20
Key words: Principal and vector sheaves, connections

0 Introduction

Gauge theories are, roughly speaking, built on principal bundles (P, G, X, π) and their connections. This is a consequence of the fact that observations and measurements in physics lead to certain sections of a parametrized group, in general non abelian. However, Lie groups and principal bundles are quite complicated objects and one is looking for a reduction of the non-commutative framework to a commutative one, the latter being described by a vector bundle. This can be often achieved by an appropriate representation of G into a vector space (in this respect we refer to [1]).

The aim of this note is to examine the analogous situation in the context of *Abstract Differential Geometry*. As a matter of fact, the present author has initiated a research program devoted to the geometry of principal sheaves (see [9]–[12]) influenced by the geometry of vector sheaves expounded in [8]. These abstractions are

Editor Gr.Tsagas *Proceedings of the Conference of Applied Differential Geometry - General Relativity and The Workshop of Global Analysis, Differential Geometry and Lie Algebras, 2000*, 204-214

developed in a completely *algebrotopological* setting, without any differentiability, in spite of the wide use of the adjective "differential" accompanying various terms in order to remind the analogy with the classical geometry of ordinary (smooth) fiber bundles.

In the present abstract approach we consider a principal sheaf $(\mathcal{P}, \mathcal{G}, X, \pi)$ and a representation of the form $\varphi : \mathcal{G} \longrightarrow \mathcal{GL}(n, \mathcal{A})$, where $\mathcal{A}$ is a sheaf of unital, commutative and associative algebras. Thus $\mathcal{A}^n$ is the vector sheaf in which the structure sheaf of groups $\mathcal{G}$ is represented. We show that such a representation leads to a vector sheaf $(\mathcal{E}, X, p)$ associated with $\mathcal{P}$ (Section 2). In the sequel (Section 3), under some additional assumptions pertaining to the compatibility of φ with the Maurer-Cartan (or logarithmic) differentials of $\mathcal{P}$ and $\mathcal{GL}(n, \mathcal{A})$, as well as with the representations of the latter into certain sheaves of Lie algebras, we prove that the connections on $\mathcal{P}$ (in the sense of [9]) induce $\mathcal{A}$-connections on $\mathcal{E}$ (in the sense of [8]). The converse is not always true unless extra conditions are imposed on φ. An example is provided by the principal *sheaf of frames* of a given vector sheaf (already studied in [10]), in which case we have the trivial representation of $\mathcal{GL}(n, \mathcal{A})$.

Since the notations and terminology used throughout are not yet standard, the preliminary Section 1 contains a brief account of the material essentially needed,in order to make the note as self sufficient as possible, referring for details to the relevant literature.

1 Preliminaries

1. Our setting is based on a fixed *algebraized space* $(X, \mathcal{A})$, where X is a topological space and $\mathcal{A}$ a sheaf (over X) of *unital, commutative* and *associative* $\boldsymbol{K}$*-algebras* $(\boldsymbol{K} = \boldsymbol{R}, \boldsymbol{C})$. For instance, in the classical case of a real smooth manifold X, we take $\mathcal{A} = \mathcal{C}^\infty_X$, the sheaf of germs of smooth functions on X. For other examples we refer to [8, Chapter 10].

To such an algebraized space we also attach a *differential triad* $(\mathcal{A}, d, \Omega^1)$, where Ω^1 is an $\mathcal{A}$-module (over X) and $d : \mathcal{A} \longrightarrow \Omega^1$ a derivation of $\mathcal{A}$; that is, a $\boldsymbol{K}$-linear morphism satisfying the *Leibniz condition*

$$d(s \cdot t) = s \cdot d(t) + t \cdot d(s);$$

for any (local) sections $s, t \in \mathcal{A}(U)$ and $U \subseteq X$ open. Note that in the previous formula we have identified a sheaf with the sheaf of germs of its sections, a convenient fact which will be often used below.

In the classical case, Ω^1 is nothing but the sheaf of germs of smooth 1-forms on X. In the abstract (algebraic-topological) framework we are dealing with, differential triads always exist by Kähler's theory of differentials (for details [7], [8, Chapter 11, Sections 5–6]).

2. Among the objects of prime interest here are principal sheaves, originally considered (in a different context) by A. Grothendieck [4]. More precisely, a *principal* *sheaf* over X is described by a quadruple $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$, where π is the projection of $\mathcal{P}$ on X and $\mathcal{G}$ is a sheaf of groups representing simultaneously the *structure sheaf* and the *structural type* of $\mathcal{P}$. This means that there exists a (right) action $\mathcal{P} \times_X \mathcal{G} \longrightarrow \mathcal{P}$, as well as an open covering $\mathcal{U} = \{U_\alpha | \alpha \in I\}$ of X together with local $\mathcal{G}$-equivariant isomorphisms $\phi_\alpha : \mathcal{P}|_{U_\alpha} \longrightarrow \mathcal{G}|_{U_\alpha}$.

However, in order to built up an abstract differential geometry on $\mathcal{P}$, in particular a gauge theory, we enrich the structure sheaf with two additional properties. In fact, we assume that $\mathcal{G}$ is a sheaf of groups of *Lie-type*, by which we mean that:

i) There exists a *representation* (: a continuous morphism of sheaves of groups) $\varrho : \mathcal{L} \longrightarrow \mathcal{A}ut(\mathcal{L})$ of $\mathcal{G}$ in an $\mathcal{A}$-module of Lie algebras $\mathcal{L}$;

ii) There exists a morphism (of sheaves of sets) $\partial : \mathcal{G} \longrightarrow \Omega^1 \otimes_{\mathcal{A}} \mathcal{L}$, called *Maurer-Cartan* or *logarithmic differential*, such that

$$\partial(s \cdot t) = \varrho(t^{-1}).\partial(s) + \partial(t),$$

for every $s, t \in \mathcal{G}(U)$ and $U \subseteq X$ open. The first term of the right-hand side of the previous formula denotes the result of the natural action of $\mathcal{G}$ on $\Omega^1 \otimes_{\mathcal{A}} \mathcal{L}$ induced by ϱ. To be more explicit, for any $g \in \mathcal{G}$ and any decomposable element $\omega \equiv \theta \otimes u \in \Omega^1 \otimes_{\mathcal{A}} \mathcal{L}$, we set

$$\varrho(g).\omega \equiv (1 \otimes \varrho(g)).\omega := \theta \otimes \varrho(g)(u), \tag{1}$$

where 1 here denotes the identity of Ω^1. We extend this action by linearity to arbitrary elements.

$\mathcal{P}$ admits a family of *natural (local) sections*

$$s_\alpha := \psi_\alpha \circ \mathbf{1}|_{U_\alpha} \in \mathcal{P}; \quad \alpha \in I,$$

where $\mathbf{1}$ is the unit section of $\mathcal{G}$ (: $\mathbf{1}(x)$ is the unit of the fiber $\mathcal{G}_x$).

As an example we take the sheaf $\mathcal{P}$ of germs of smooth sections of a principal fibre bundle (P, G, X, p). It is a principal sheaf with structure sheaf $\mathcal{G}$ the sheaf of germs of smooth G-valued maps on X. $\mathcal{G}$ is of Lie-type with $\mathcal{L}$ being now the sheaf of germs of smooth maps on X with values in the Lie algebra of G. In this case ϱ and ∂ are obtained by the sheafification of the adjoint representation and the total (logarithmic) differential respectively. For complete details we refer e.g to [9, 12]

3. A typical abstract example of a sheaf of groups of Lie-type, which will play an important role in the sequel, is the sheaf $\mathcal{G}L(n, \mathcal{A})$ generated by the complete presheaf of groups $U \mapsto GL(n, \mathcal{A}(U))$, U running in the topology of X. Hence,

$$\mathcal{G}L(n, \mathcal{A})(U) \cong GL(n, \mathcal{A}(U)) \cong Lis_{\mathcal{A}|_U}(\mathcal{A}^n|_U, \mathcal{A}^n|_U). \tag{2}$$

Now $\mathcal{L} \equiv M_n(\mathcal{A})$, the sheaf generated by the complete presheaf of Lie algebras $U \mapsto M_n(\mathcal{A}(U))$; thus

$$M_n(\mathcal{A})(U) \cong M_n(\mathcal{A}(U)) \cong \mathcal{A}^{n^2}(U), \tag{3}$$

for every open $U \subseteq X$.

There exists an (adjoint) representation $\mathcal{A}d : \mathcal{GL}(n, \mathcal{A}) \longrightarrow \mathcal{A}ut(M_n(\mathcal{A}))$ obtained as follows: Let U be any open subset of X. We define the morphism of sections

$$\mathcal{A}d^U : \mathcal{GL}(n, \mathcal{A})(U) \longrightarrow \mathcal{A}ut((M_n(\mathcal{A})))(U) \cong Aut(M_n(\mathcal{A})|_U, M_n(\mathcal{A})|_U)$$

by requiring that, for any $g \in \mathcal{GL}(n, \mathcal{A})(U)$, $\mathcal{A}d^U(g)$ to be the automorphism generated by the automorphisms of presheaves

$$\left(\mathcal{A}d^U(g)\right)_V : M_n(\mathcal{A})(V) \longrightarrow M_n(\mathcal{A})(V) : \; a \mapsto g \cdot a \cdot g^{-1}; \quad a \in M_n(\mathcal{A})(V),$$

for all open $V \subseteq U$, with the identifications (2) and (3) being applied here.

The corresponding Maurer-Cartan differential $\widetilde{\partial} : \mathcal{G} \longrightarrow \Omega^1 \otimes_{\mathcal{A}} M_n(\mathcal{A})$ is given by $\partial(a) := a^{-1} \cdot \boldsymbol{d}(a)$, for every $a \in M_n(\mathcal{A}(U))$ and $U \subseteq X$ open, where $\boldsymbol{d} : \mathcal{GL}(n, \mathcal{A}) \longrightarrow \Omega^1 \otimes_{\mathcal{A}} M_n(\mathcal{A})$ is the extension of d (of the initial differential triad); i.e., $\boldsymbol{d}(a) := (da_{ij})$, for every $a = (a_{ij}) \in GL(n, \mathcal{A}(U))$.

4. The last fundamental notion immediately needed in the next section is that of a *vector sheaf*. This is a sheaf $\mathcal{E} \equiv (\mathcal{E}, X, p)$ which is a locally free $\mathcal{A}$-module (over X). Hence, there exist an open cover, say, $\mathcal{U} = \{U_\alpha | \alpha \in I\}$ and $\mathcal{A}|_{U_\alpha}$-isomorphisms $\psi_\alpha : \mathcal{A}^n|_{U_\alpha} \longrightarrow \mathcal{E}|_{U_\alpha}$. The complete study of vector sheaves and their geometry is the content of [8].

2 Associated sheaves

In this section we fix a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ and a representation of the form

$$\varphi : \mathcal{G} \longrightarrow \mathcal{GL}(n, \mathcal{A}).$$

We shall construct a vector sheaf of rank n, associated with $\mathcal{P}$. To this end, for each open $U \subseteq X$, we consider the quotient set $Q(U) := \mathcal{P}(U) \times \mathcal{A}^n(U) \,/\, \mathcal{G}(U)$ determined by the equivalence relation

$$(s, a) \sim (t, b) \quad \Longleftrightarrow \quad \exists!\, g \in \mathcal{G}(U) : t = s \cdot g \,,\, b = \varphi(g^{-1}) \cdot a,$$

for every $s, t \in \mathcal{P}(U)$ and $a, b \in \mathcal{A}^n(U)$.

Running now U in the topology of X, we obtain a (not necessarily complete) presheaf $U \mapsto Q(U)$ generating the quotient sheaf

$$\mathcal{E} := \mathcal{P} \times_X \mathcal{A}^n \,/\, \mathcal{G},$$

with base X and a projection p defined in the obvious way. This is, by definition, *the sheaf associated with $\mathcal{P}$ by the representation φ.*

With regard to the previous construction one may consult [3]. We note that the last quotient can be also constructed, in an equivalent way, by defining (fiber-wise) on $\mathcal{P} \times_X \mathcal{A}^n$ an analogous (global) equivalence relation (see [4]).

Lemma 1 *$(\mathcal{E}, X, p)$ is a sheaf locally isomorphic to $\mathcal{A}^n$ with corresponding cocycle $(G_{\alpha\beta}) = (\varphi(g_{\alpha\beta})) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$, where $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ is the cocycle of the principal sheaf $\mathcal{P}$.*

Proof. Fix a $U_\alpha \in \mathcal{U}$. For every open $V \subseteq U_\alpha$ we define the map

$$\psi_V^\alpha : \mathcal{A}^n(V) \ni f \mapsto [s_\alpha|_V, f] \in Q(V),$$

where $s_\alpha \in \mathcal{P}(U_\alpha)$ is the natural section over U_α.

It is immediate that ψ_V^α is 1-1. On the other hand, for a given $[\sigma, h] \in Q(V)$, the section $\varphi(g) \cdot h \in \mathcal{A}^n(V)$, with g determined by $\sigma = s_\alpha|_V \cdot g$, gives that $\psi_V^\alpha(\varphi(g) \cdot h) = [\sigma, h]$, which implies that ψ_V^α is onto. In this way we obtain a morphism $\{\psi_V^\alpha\}_{V \subseteq U_\alpha}$ between the presheaves $V \mapsto \mathcal{A}^n(V)$ and $V \mapsto Q(V)$, generating an isomorphism (of sheaves of sets) $\psi_\alpha : \mathcal{A}^n|_{U_\alpha} \xrightarrow{\cong} \mathcal{E}|_{U_\alpha}$. This shows the first claim of the statement.

By definition, $G_{\alpha\beta} = \psi_\alpha^{-1} \circ \psi_\beta$, where now both the isomorphisms are restricted on appropriate sheaves over $U_{\alpha\beta} := U_\alpha \cap U_\beta$ (for simplicity we omit explicit expressions like $\psi_\alpha|_{U_{\alpha\beta}}$). Hence, $G_{\alpha\beta}$ is generated by $(\psi_V^\alpha)^{-1} \circ \psi_V^\beta$, for all open $V \subseteq U_{\alpha\beta}$. As a result, for every $h \in \mathcal{A}^n(V)$, we check that

$$\left((\psi_V^\alpha)^{-1} \circ \psi_V^\beta\right)(h) = (\psi_V^\alpha)\left([s_\beta|_V, h]\right) = \varphi(g_{\alpha\beta}|_V) \cdot h.$$

Using the identification (2), we obtain $(\psi_V^\alpha)^{-1} \circ \psi_V^\beta = \varphi(g_{\alpha\beta}|_V)$. We prove the second claim by taking all open $V \subseteq U_{\alpha\beta}$. □

Theorem 1 *$\mathcal{E} \equiv (\mathcal{E}, X, p)$ is a vector sheaf (of rank n).*

Proof. Each isomorphism (of sheaves of sets) $\psi_\alpha : \mathcal{A}^n|_{U_\alpha} \longrightarrow \mathcal{E}|_{U_\alpha}$ induces (fiber-wise) on $\mathcal{E}|_{U_\alpha}$ the operations

$$\Sigma_\alpha : \mathcal{E}|_{U_\alpha} \times_{U_\alpha} \mathcal{E}|_{U_\alpha} \longrightarrow \mathcal{E}|_{U_\alpha}\, ; \; \Pi_\alpha : \mathcal{A}^n|_{U_\alpha} \times_{U_\alpha} \mathcal{E}|_{U_\alpha} \longrightarrow \mathcal{E}|_{U_\alpha},$$

respectively given by

$$\Sigma_\alpha(u, v) \equiv u + v := \psi_\alpha\left(\psi_\alpha^{-1}(u) + \psi_\alpha^{-1}(v)\right), \psi_\alpha^{-1}(v))$$

$$\Pi_\alpha(a \cdot u) \equiv a \cdot u := \psi_\alpha\left(a \cdot \psi_\alpha^{-1}(u)\right),$$

for every $u, v \in \mathcal{E}_x$ and $a \in \mathcal{A}_x$ with $x \in U_\alpha$.

Since $\Sigma_\alpha = \psi_\alpha \circ \overline{\Sigma} \circ (\psi_\alpha, \psi_\alpha)$ and $\Pi_\alpha = \psi_\alpha \circ \overline{\Pi} \circ (\psi_\alpha, \psi_\alpha)$, where $\overline{\Sigma}$ and $\overline{\Pi}$ are the respective (continuous) operations of the $\mathcal{A}$-module $\mathcal{A}^n$, appropriately restricted over U_α, it follows that Σ_α and Π_α are also continuous morphisms giving on $\mathcal{E}|_{U_\alpha}$ the structure of an $\mathcal{A}|_{U_\alpha}$-module such that ψ_α is an $\mathcal{A}|_{U_\alpha}$-linear isomorphism. This determines the desired local structure of $\mathcal{E}$.

The previous local operations globalize to corresponding continuous operations on $\mathcal{E}$ since $\Sigma_\alpha = \Sigma_\beta$ and $\Pi_\alpha = \Pi_\beta$ on the overlappings. Indeed, for any $(u, v) \in$

$\mathcal{E}|_{U_{\alpha\beta}} \times_{U_{\alpha\beta}} \mathcal{E}|_{U_{\alpha\beta}}$, using the identification (2) and the previous Lemma, we have that

$$\begin{aligned}\Sigma_\beta(u,v) &= (\psi_\alpha \circ G_{\alpha\beta})\big(\psi_\beta^{-1}(u) + \psi_\beta^{-1}(v)\big)\\ &= (\psi_\alpha \circ \psi_\beta^{-1})(u) + (\psi_\alpha \circ \psi_\beta^{-1})(u)\\ &= \psi_\alpha\big(\psi_\alpha^{-1}(u) + \psi_\alpha^{-1}(v)\big) = \Sigma_\alpha(u,v)\end{aligned}$$

and similarly for the multiplications. Therefore, $\mathcal{E}$ becomes an $\mathcal{A}$-module. □

For the sake of completeness, we examine the relationship between the (global) sections of $\mathcal{E}$ and certain morphisms corresponding to the classical tensorial maps. In fact, a morphism (of sheaves of sets) $f : \mathcal{P} \longrightarrow \mathcal{A}^n$ is said to be *tensorial* if

$$f(s \cdot g) = \varphi(g^{-1}) \cdot f(s)\,; \quad (s,g) \in \mathcal{P}(U) \times \mathcal{G}(U),$$

for every open $U \subseteq X$. Clearly, the product of the right-hand side is well defined by the obvious action of $\mathcal{GL}(n,\mathcal{A})$ on the left of $\mathcal{A}^n$. As a result, we prove

Theorem 2 *Tensorial morphisms $f : \mathcal{P} \longrightarrow \mathcal{A}^n$ correspond bijectively to global sections of $\mathcal{P}$.*

Proof. Let f be a given tensorial morphism. For a $U_\alpha \in \mathcal{U}$, we set $\sigma_\alpha := [s_\alpha, f(s_\alpha)]$ (recall that s_α is the natural section of $\mathcal{P}$ over U_α and f is now the induced morphism of sections). Since $\sigma_\alpha \in (\mathcal{P}(U_\alpha) \times \mathcal{A}^n(U_\alpha)/\sim) \subset \mathcal{E}(U_\alpha)$, we obtain a family of local sections (σ_α) of $\mathcal{E}$. However, over $U_{\alpha\beta}$, we have that

$$\sigma_\beta = [s_\alpha \cdot g_{\alpha\beta}, \varphi(g_{\alpha\beta}^{-1}) \cdot f(s_\alpha)] = [s_\alpha, f(s_\alpha)] = \sigma_\alpha;$$

hence we can define a global section $\sigma \in \mathcal{E}(X)$ by setting $\sigma|_{U_\alpha} := \sigma_\alpha$.

Conversely, let $\sigma \in \mathcal{E}(X)$ be given a section. For an open $U \subseteq X$, we define the map $f_U : \mathcal{P}(U) \longrightarrow \mathcal{A}^n(U)$ by requiring that

$$f_U(s)|_{U\cap U_\alpha} := \varphi(g_\alpha^{-1}) \cdot \psi_\alpha^{-1}(\sigma|_{U\cap U_\alpha}), \tag{4}$$

for every $s \in \mathcal{P}(U)$ and with $g_\alpha \in \mathcal{G}(U_\alpha)$ determined by $s|_{U\cap U_\alpha} = s_\alpha|_{U\cap U_\alpha} \cdot g_\alpha$. We check that f_U is defined by gluing the restrictions given by (4), for all $U_\alpha \in \mathcal{U}$. Indeed, for $U_\beta \in \mathcal{U}$, we have the analogous expression

$$f_U(s)|_{U\cap U_\beta} := \varphi(g_\beta^{-1}) \cdot \psi_\beta^{-1}(\sigma|_{U\cap U_\beta}), \tag{5}$$

with $g_\beta \in \mathcal{G}(U_\beta)$ satisfying $s|_{U\cap U_\beta} = s_\beta|_{U\cap U_\beta} \cdot g_\beta$. Therefore, over $U \cap U_\alpha \cap U_\beta$, $g_\alpha = g_{\alpha\beta} \cdot g_\beta$. Omitting, for simplicity the explicit mention of the restrictions on $U \cap U_\alpha \cap U_\beta$ of the sections involved, we see that (see also Lemma 1)

$$\varphi(g_\beta^{-1}) \cdot \psi_\beta^{-1}(\sigma) = \varphi(g_\alpha^{-1} \cdot g_{\alpha\beta}) \cdot G_{\beta\alpha} \cdot \psi_\alpha^{-1}(\sigma) = \varphi(g_\alpha^{-1}) \cdot \psi_\alpha^{-1}(\sigma),$$

which proves that (4) and (5) coincide on $U \cap U_\alpha \cap U_\beta$ and f_U is well defined by the gluing process.

Finally, for any $s \in \mathcal{P}(U)$ and $g \in \mathcal{G}(U)$, we have that

$$f_U(s \cdot g)|_{U \cap U_\alpha} = \varphi(g|_{U\cap U_\alpha} \cdot g_\alpha^{-1}) \cdot \psi_\alpha^{-1}(\sigma|_{U \cap U_\alpha}) = \varphi(g^{-1})|_{U\cap U_\alpha} \cdot f_U(s)|_{U\cap U_\alpha},$$

for every $U_\alpha \in \mathcal{U}$; thus $f_U(s \cdot g) = \varphi(g^{-1}) \cdot f_U(s)$. Varying U in the topology of X, we obtain a morphism of presheaves generating a tensorial morphism f and the proof is now complete. □

Remark 1 In all the previous construction it is not necessary to assume that $\mathcal{G}$ is a sheaf of groups of Lie-type (see Paragraph 1.2), a fact which will be needed in the study of connections below.

3 Connections on associated sheaves

In this section we consider a principal sheaf $\mathcal{P}$ with structure sheaf $\mathcal{G}$ of Lie-type. We recall that (see [9]) a *connection* on $\mathcal{P}$ (or gauge potential, in the terminology of [1]) is a morphism of sheaves of sets $D : \mathcal{P} \longrightarrow \Omega^1 \otimes_{\mathcal{A}} \mathcal{L}$ satisfying

$$D(s \cdot g) = \rho(g_{\alpha\beta}^{-1}).D(s) + \partial(g), \tag{6}$$

for any $s \in \mathcal{P}(U)$, $g \in \mathcal{G}(U)$ and $U \subseteq X$ open.

A connection D is equivalently determined by the family of local sections

$$\omega_\alpha := D(s_\alpha) \in (\Omega^1 \otimes_{\mathcal{A}} \mathcal{L})(U_\alpha); \quad \alpha \in I,$$

which are called, following the classical terminology, *the local connection forms* (or local gauge potentials) of D. They satisfy the (local) gauge transform

$$\omega_\beta = \rho(g_{\alpha\beta}^{-1}).\omega_\alpha + \partial(g_{\alpha\beta}) \tag{7}$$

on each $U_{\alpha\beta} \neq \emptyset$ (see [9, Theorem 5.4]).

On the other hand (see [8, Vol. II, Chapter 6, Section 3]), an $\mathcal{A}$-*connection* on a vector sheaf $\mathcal{E}$ (of rank n) is a $\boldsymbol{K}$-linear morphism $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$ satisfying the *Leibniz-Koszul* condition

$$\nabla(a \cdot s) = a \cdot \nabla(s) + s \otimes d(a), \tag{8}$$

for every $a \in \mathcal{A}(U)$, $s \in \mathcal{E}(U)$ and $U \subseteq X$ open.

Equivalently (see also [8, Chapter 7]), ∇ is fully determined by corresponding local connection forms as follows: For each U_α, the $\mathcal{A}(U_\alpha)$-module $\mathcal{E}(U_\alpha)$ admits a natural basis $e^\alpha := (e_1^\alpha, \ldots, e_n^\alpha)$ with

$$e_i^\alpha(x) := \psi_\alpha(0_x, \ldots, 1_x, \ldots, 0_x); \qquad x \in U_\alpha,$$

where 0_x and 1_x (in the i-th entry) are the zero and unit element of the stalk $\mathcal{A}_x$ respectively. Evaluating now ∇ on the sections of the basis, we obtain the expressions

$$\nabla(e_j^\alpha) = \sum_{i=1}^{n} e_i^\alpha \otimes \theta_{ij}^\alpha\,; \qquad 1 \leq\ j\ \geq n,$$

with $\theta_{ij}^\alpha \in \Omega^1(U_\alpha)$, forming thus a matrix $(\theta_{ij}^\alpha) \in M_n(\Omega^1(U_\alpha))$, for every $\alpha \in I$. In virtue of (3), we check that

$$(\Omega^1 \otimes_{\mathcal{A}} M_n(\mathcal{A}))(U_\alpha) \cong \Omega^1(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} M_n(\mathcal{A}(U_\alpha)) \cong M_n(\Omega^1(U_\alpha)); \tag{9}$$

hence, (θ_{ij}^α) can be identified with a section $\theta_\alpha \in (\Omega^1 \otimes_{\mathcal{A}} M_n(\mathcal{A}))(U_\alpha)$. The sections $(\theta_\alpha)_{\alpha\in I}$, are the *local connection forms of* ∇ and satisfy the analog of (7), namely

$$\theta_\beta = \mathcal{A}d(G_{\alpha\beta}^{-1}).\theta_\alpha + \widetilde{\partial}(G_{\alpha\beta}), \tag{10}$$

where $(G_{\alpha\beta})$ is the cocycle of $\mathcal{E}$. This is a consequence of (8) and routine, though tedious, calculations.

We come now to the following basic

Definition 1 A representation $\varphi : \mathcal{G} \longrightarrow \mathcal{G}L(n,\mathcal{A})$ is said to be of *Lie-type* if there exists a morphism of sheaves of Lie algebras $\overline{\varphi} : \mathcal{L} \longrightarrow M_n(\mathcal{A})$ such that the following conditions hold:

$$\widetilde{\partial} \circ \varphi = (1 \otimes \overline{\varphi}) \circ \partial$$

$$\overline{\varphi} \circ \varrho(g) = \mathcal{A}d(\varphi(g)) \circ \overline{\varphi}; \quad g \in \mathcal{G},$$

where, for simplicity, we have set $1 = id|\Omega^1$.

Clearly, the previous conditions express the compatibility of φ and $\overline{\varphi}$ with the Maurer-Cartan differentials of $\mathcal{G}$ and $\mathcal{G}L(n,\mathcal{A})$, as well as with the their representations ϱ and $\mathcal{A}d$. For a more general situation see also [11, Definition 3.6]. Note that in the classical case $\overline{\varphi}$ is the morphism of Lie algebras induced by the differential of φ and the above conditions are always true.

Theorem 3 *Let $\varphi : \mathcal{G} \longrightarrow \mathcal{G}L(n,\mathcal{A})$ be a representation of Lie-type. Then, every connection on $\mathcal{P}$ induces an $\mathcal{A}$-linear connection on the associated vector sheaf $\mathcal{E}$.*

Proof. For a given connection $D \equiv (\omega_\alpha)$ on $\mathcal{P}$, we set

$$\theta_\alpha := (1 \otimes \overline{\varphi})(\omega_\alpha), \quad \alpha \in I. \tag{11}$$

Then, in virtue of (1), Lemma 1 and Definition 1, equality (6) implies that

$$\begin{aligned}
\theta_\beta &= (1 \otimes \overline{\varphi})\big((1 \otimes \varrho(g_{\alpha\beta})).\omega_\alpha + \partial(g_{\alpha\beta})\big) \\
&= \big(1 \otimes \mathcal{A}d(\varphi(g_{\alpha\beta}^{-1})) \circ \overline{\varphi}\big).\omega_\alpha + (\overline{\partial} \circ \varphi)(g_{\alpha\beta}) \\
&= \big(1 \otimes \mathcal{A}d(G_{\alpha\beta}^{-1})\big).\theta_\alpha + \overline{\partial}(G_{\alpha\beta}) \\
&\equiv \mathcal{A}d(G_{\alpha\beta}^{-1}).\theta_\alpha + \overline{\partial}(G_{\alpha\beta}),
\end{aligned}$$

which proves (10) and yields, in turn, an $\mathcal{A}$-linear connection ∇ on $\mathcal{E}$. For the sake of completeness we outline the construction of ∇, referring for details to [8, 10]. First, for each $\alpha \in I$, we define the map $\nabla^{\alpha} : \mathcal{E}|_{U_\alpha} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 |_{U_\alpha}$ by setting

$$\nabla^{\alpha}(s) := \sum_{i=1}^{n} e_i^{\alpha} \otimes \big(\partial(s_i^{\alpha}) + \sum_{j=1}^{n} s_j^{\alpha} \cdot \theta_{ij}^{\alpha}\big),$$

for every $s = \sum_{i=1}^{n} s_i^{\alpha} \cdot e_i^{\alpha} \in \mathcal{E}(U_\alpha)$ with $s_i^{\alpha} \in \mathcal{A}(U_\alpha)$. Recall that $\theta_\alpha \equiv (\theta_{ij}^{\alpha})$, after the identifications (9). The compatibility condition (10) implies that $\nabla^{\alpha} = \nabla^{\beta}$ on $\mathcal{A}(U_{\alpha\beta})$, hence we obtain a global connection ∇. □

An immediate consequence of (11) is the following

Corollary 1 *If $\overline{\varphi} : \mathcal{L} \longrightarrow M_n(\mathcal{A})$ is an isomorphism, then the connections of $\mathcal{P}$ are in bijective correspondence with the $\mathcal{A}$-linear connections of its associated vector sheaf $\mathcal{E}$.*

Example Let $\mathcal{E}$ be now a *given* vector sheaf of rank n with a local structure as in Paragraph 1.4. We denote by $\mathcal{B}$ the basis of topology on X containing all the open $V \subseteq X$ such that $V \subseteq U_\alpha$, for some $U_\alpha \in \mathcal{U}$, and consider the (complete) presheaf $\mathcal{B} \ni V \mapsto Iso_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V)$, where the last space is the group of $\mathcal{A}|_V$-linear isomorphisms. This generates a principal sheaf $\mathcal{P}(\mathcal{E}) \equiv (\mathcal{P}(\mathcal{E}), \mathcal{G}L(n,\mathcal{A}), X, \pi)$, called *the sheaf of frames* of $\mathcal{E}$.

We recall that (see [10]) there is a natural action of $\mathcal{G}L(n,\mathcal{A})$ on the right of $\mathcal{P}(\mathcal{E})$ induced by the partial actions

$$Iso_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) \times \mathcal{G}L(n,\mathcal{A})(V) \longrightarrow Iso_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) : (f,g) \mapsto f \cdot g \equiv f \circ g$$

by employing, of course, the identification (2). The local structure is described as follows: First we define the local $\mathcal{G}L(n,\mathcal{A})$-equivariant isomorphism

$$\Phi_\alpha^V : \mathcal{P}(\mathcal{E})(V) \longrightarrow \mathcal{G}L(n,\mathcal{A})(V) : f \mapsto \psi_\alpha^{-1} \circ f,$$

for every open $V \in U_\alpha$. Hence, varying V in U_α, we obtain an equivariant morphism $\Phi_\alpha : \mathcal{P}(\mathcal{E})|_{U_\alpha} \xrightarrow{\simeq} \mathcal{G}L(n,\mathcal{A})|_{U_\alpha}$ and similarly for all $\alpha \in I$.

The natural sections $\sigma_\alpha \in \mathcal{P}(\mathcal{E})(U_\alpha)$, with respect to $\mathcal{U}$, are now given by

$$\sigma_\alpha := \Phi_\alpha^{-1}(id|\mathcal{A}^n(U_\alpha)) = \psi_\alpha. \tag{12}$$

The previous considerations lead now to

Corollary 2 *Every vector sheaf $\mathcal{E}$ is associated with its principal sheaf of frames $\mathcal{P}(\mathcal{E})$, with respect to the trivial representation of $\mathcal{G}L(n,\mathcal{A})$. Hence, the $\mathcal{A}$-linear connections on $\mathcal{E}$ correspond bijectively to the connections on $\mathcal{P}(\mathcal{E})$.*

Proof. By the general construction discussed in Section 2, the vector sheaf, say, $\mathcal{F}$ associated with $\mathcal{P}(\mathcal{E})$, is generated by the presheaf

$$\mathcal{B} \ni V \mapsto \mathcal{P}(\mathcal{E})(V) \times \mathcal{A}^n(V)/\sim,$$

defined by the trivial representation (: $\varphi = id|\mathcal{G}L(n, \mathcal{A})$). Though we are restricted on a basis of topology, instead of the whole topology of X, the final result remains unaffected. Following the proof of [10, Proposition 4.3], for any $V \in \mathcal{B}$ with $V \subseteq U_\alpha$, we consider the map

$$F_V : \mathcal{P}(\mathcal{E})(V) \times \mathcal{A}^n(V)/\sim \;\longrightarrow\; \mathcal{E}(V) : [f, a] \mapsto f \circ a.$$

We show that F_V is a well defined bijection. Varying V in $\mathcal{B}$, we obtain an isomorphism $F : \mathcal{F} \longrightarrow \mathcal{E}$. It is also an isomorphism of $\mathcal{A}$-modules. Indeed, if we denote by $\Psi_\alpha : \mathcal{A}^n|_{U_\alpha} \longrightarrow \mathcal{F}|_{U_\alpha}$ the isomorphisms describing the local structure of $\mathcal{F}$, then (12) implies that

$$(F \circ \Psi_\alpha)(a) = F([\sigma_\alpha, a]) = \sigma_\alpha \circ a = \psi_\alpha(a); \quad a \in \mathcal{A}^n(U_\alpha),$$

with F and ψ_α denoting now the induced morphisms between sections. By the procedure used repeatedly so far, we see that $F = \psi_\alpha \circ \Psi_\alpha^{-1}$. This, along with the definition of the module operations on $\mathcal{F}$ (see Theorem 2), completes the claim about F. The rest of the proof is clear. □

Remark 2 In the previous Corollary we recover, by a different approach, some of the results of [10], notably Proposition 4.3 and Theorem 5.5.

Acknowledgment. This work has been partially supported by the University of Athens Research Grant 70/4/3410.

References

[1] D. Bleecker, *Gauge Theory and Variational Principles*, Addison-Wesley, Reading Mass., 1981.

[2] N. Bourbaki, *Groupes et Algèbres de Lie*, Chapitres 2–3, Hermann, Paris, 1972.

[3] R. Godement, Topologie Algébrique et Theorie des Faisceaux, Hermann, Paris, 1973.

[4] A. Grothendieck, *A General Theory of Fibre Spaces with Structural Sheaf*, Kansas Univ., 1958.

[5] S. Kobayashi–K. Nomizu, *Foundations of Differential Geometry*, Vol. I. Interscience, New York, 1963.

[6] A. Mallios, *On an abstract form of Weil's integrality theorem*, Note di Matematica **12** (1992), 167–202.

[7] A. Mallios, *The de Rham-Kähler complex of the Gel'fand sheaf of a topological algebra*, J. Math. Anal. Appl. **175** (1993), 143–168.

[8] A. Mallios, *Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry* , Vol. I–II. Kluwer, Dordrecht, 1998.

[9] E. Vassiliou, *Connections on principal sheaves*, New Developments in Differential Geometry, Budapest 1996 (J. Szenthe Ed.), pp. 459–484, Kluwer, Dordrecht, 1998.

[10] E. Vassiliou, *On Mallios' $\mathcal{A}$-connections as connections on principal sheaves*, Note di Matematica **14** (1994), 237–249 (1996).

[11] E. Vassiliou, *Transformations of sheaf connections*, Balkan J. Geom.Appl. **1** (1996), 117–133.

[12] E. Vassiliou, *From principal connections to connections on principal sheaves*, Anal. Stiint. Univ. "Al. I Cuza" Iasi **42** (1996) (supl.), 149-160.

Efstathios Vassiliou
Institute of Mathematics
University of Athens
Panepistimiopolis
Athens 157 84, Greece

E-mail address: evassil@cc.uoa.gr