ON THE KILLING TENSOR FIELDS ON A COMPACT RIEMANNIAN MANIFOLD

Gr. Tsagas

Abstract

Let (M, g) be a compact Remannian manifold of dimension n. The aim of the present paper is to study the dimension of $K^q(M, IR)$ in the connection with the Riemannian metric g on M.

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Key words: Riemannian manifold, Killing tensor field. Riemannian metric, harmonic q-form and Killing q-form.

1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension n. Let $K^q(M, IR)$, where $q = 2, ..., n-1$, be the vector space of Killing tensor fields of order q on M. The study of the dimension of $K^q(M, IR)$ is an important problem. This importance comes from the fact there is a connection between q-harmonic forms and Killing tensor fields of order q. Let $H^q(M, IR)$ be the vector space of harmonic q-forms. It is known that $\dim(H^q(M, IR)) = b_q$ is the q-Betti number of M, which is topological invariant. It is still open if $\dim(K^q(M, IR))$ for $q = 2, ..., n-1$ is also a topological invariant. The aim of the present paper is to study this problem. We also improve Yano's results [16].

The whole paper contains three sections. Each of them is analyzed as follows. In the second section we study differential operators of cross sections of a fibre bundle over a compact Riemannian manifold M. The Killing tensor fields of order q can be considered as special cross sections of the fibre bundle $\Lambda^q(T(M))$ over M. The space of Killing tensor fields $K^q(M, IR)$ of order q with the connection of the Riemannian metric g on M is studied in the last section. These results are an improvement Yano's results [16].

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2 An isomorphism between covariant and contravariant antisymmetric tensors

Let (M, g)be a compact Riemannian manifold of dimension n without boundary. We denote by $\Lambda^q(T(M))$ and $\Lambda^q(T^*M)$ the fibre bundles of antisymmetric covariant tensor fields of order q and antisymmetric contravariant tensor fields of order q respectively on the manifold M. It is known that the vector space $\Lambda^q(T^*M)$ coincide with the vector space $\Lambda^q(M)$ of exterior q-forms.

We must notice that each exterior q-form ω is a cross section of $\Lambda^q(T^*M) = \Lambda^q(M)$. The same is true for each element $\lambda \in \Lambda^q(T(M))$. The Laplace operator Δ is a second order elliptic differential operator on $C^\infty(\Lambda^q(M))$, that is

$$\Delta = d\delta + \delta d : C^\infty(\Lambda^q(M)) \to C^\infty(\Lambda^q(M)),$$

$$\Delta = d\delta + \delta d : \alpha \to \Delta(\alpha) = d\delta(\alpha) + \delta d(\alpha), \alpha \in C^\infty(\Lambda^q(M)),$$

where α an exterior q-form and d, δ are first order differential operator defined by

$$d : C^\infty(\Lambda^q(M)) \to C^\infty\left(\Lambda^{q+1}(M)\right),$$

$$\delta : C^\infty(\Lambda^q(M)) \to C^\infty\left(\Lambda^{q-1}(M)\right).$$

These differential operators are related by

$$< \alpha.\delta\beta >=< d\alpha.\beta >, \forall\alpha \in C^\infty(\Lambda^q(M)), \forall\beta \in C^\infty\left(\Lambda^{q+1}(M)\right),$$

where $<>$ is the global inner product on $C^\infty(\Lambda^q(M))$. The local inner product is defined by..

$$< \alpha.\gamma >_1 = \alpha_{i_1,...,i_q}\beta^{i_1,...,i_q} = g^{j_1 i_1 \cdots} g^{j_q i_q}\alpha_{i_1,...,i_q}\beta_{j_1,...,j_q}.$$

Let $(x_1, ..., x_n)$ be a local coordinate system on the chart (U, φ) and let $(e_1, ..., e_n)$ be the associated local frame in M, that is

$$e_1 = \frac{\partial}{\partial x_1}, ..., e_n = \frac{\partial}{\partial x_n}.$$

If α is a q-form, which is a cross section of $\Lambda^q(M)$, that is $\alpha \in C^\infty(\Lambda^q(M))$, then α with respect to the local coordinate system can be expressed by

$$\alpha\left(e_{i_1}, e_{i_2}, ..., e_{i_q}\right) = \alpha_{i_1,...,i_q}.1 \leq i_1 < i_2 < ... < i_q \leq n.$$

The following formulas are known

$$(d\alpha)_{i_1 \ldots i_q j} = \frac{1}{q}\varepsilon^{k j_1 \ldots j_q}_{i_1 \ldots i_q j}\nabla_k \alpha_{j_1 \ldots j_q}\ , \tag{2.1}$$

$$(\delta\alpha)_{i_2 \ldots i_q} = \nabla_1 \alpha^1_{i_2 \ldots i_q}\ , \tag{2.2}$$

$$(\Delta\alpha)_{i_2...i_q} = \nabla^k\nabla_k\alpha_{i_2...i_q} + \frac{1}{(q-1)}\varepsilon^{kj_1...j_q}_{i_1...i_q}\left(\nabla_1\nabla_k\alpha^1_{i_2...i_q} - \nabla_k\nabla_1\alpha^1_{i_2...i_q}\right), \quad (2.3)$$

where

$$\varepsilon^{i_1...i_r}_{j_1...j_r} = \begin{cases} 1 \text{ if } (i_1...i_r) \text{ is even permutation of } (j_1...j_r), \\ -1 \text{ if } (i_1...i_r) \text{ is odd permutation of } (j_1...j_r), \\ 0 \text{ if } (i_1...i_r) \text{ is not a permutation of } (j_1...j_r). \end{cases}$$

The formula (2.3), by means of Ricci's formula, becomes:

$$\nabla_1\nabla_k\alpha^1_{i_2...i_q} - \nabla_k\nabla_1\alpha^1_{i_2...i_q} = R^1_{r1k}\alpha^r_{i_2...i_q} - \sum_{s=2}^{q} R^r_{i_s1k}\alpha^1_{i_2...i_{s-1}}ri_{s+1}...i_q \quad (2.4)$$

and after some estimates, takes the form:

$$(\Delta\alpha)\, i_1...i_q = \nabla^k\nabla_k\alpha_{i_1...i_q} + \frac{1}{(q-1)}\varepsilon^{kj_1...j_q}_{i_1...i_q}R_{kl}\alpha_{j_2...j_q} - \quad (1)$$

$$\frac{1}{2(q-2)}\varepsilon^{klj_3...j_q}_{i_1...i_q}R_{klmn}\alpha^{mn}_{j_2...j_q}\,. \quad (2)$$

If α is a q-form, then we have

$$\frac{1}{2}\Delta\left(|\alpha|^2\right) = (\alpha\nabla\alpha) - |\nabla\alpha|^2 - \frac{1}{(q-1)}L_q(\alpha), \quad (2.6)$$

where $q \geq 2$ and

$$|\nabla\alpha|^2 = \frac{1}{q}\nabla^k\alpha_{i_1...i_q}\nabla_k\alpha_{i_1...i_q}\,, \quad (2.7)$$

$$L_q(\alpha) = -(q-1)R_{klmn}\alpha^{kli_3...i_q}\alpha_{j_3...j_q} + 2R_{kl}\alpha^{ki_2...i_q}\alpha^1_{i_2...i_q}\,. \quad (2.8)$$

From (2.8) we can consider L_q as a quadratic form on the vector space $\Lambda^q(MIR)$, that is

$$L_q : \Lambda^q(MIR) \to IR, L_q : \alpha \to L_q(\alpha). \quad (2.9)$$

A q-form α is called a *Killing q-form* if its covariant derivative $\nabla\alpha$ is a $(q+1)$-form. This in local coordinate system $(x_1, ..., x_n)$ can be expressed as follows:

$$\nabla_j\alpha_{ii_2...i_q} + \nabla_i\alpha_{ji_2...i_q}\,, \quad (2.10)$$

which is equivalent to

$$q\nabla_j\alpha_{i_1i_2...i_q} + \nabla_{i_1}\alpha_{ji_2...i_q} + ... + \nabla_{i_q}\alpha_{i_1i_2...i_{q-1}} = 0. \quad (2.11)$$

If α is a Killing q-form, then from (2.11) we obtain

$$\nabla_j\alpha^j_{j_3...j_q} = 0. \quad (2.12a)$$

The Killing q-form α satisfies the equations

$$qg^{jk}\nabla_k\nabla_j\alpha_{i_1...i_q} + \sum_{s}^{1...q}\alpha_{i_1...i_{s-1}ri_{s+1}...i_q}R^r_{i_s} + \sum_{s<t}^{1...q}\alpha_{i_1...i_{s-1}ri_{s+1}...i_{t-1}}\mu i_{t+1}...i_q R^{r\mu}i_s i_t = 0. \quad (2.12b)$$

Hence if we consider the second order elliptic differential operator

$$D_q : C^\infty (\Lambda^q (M, IR)) \to C^\infty (\Lambda^q (M, IR)),$$

$$D_q : \alpha \to D_q \alpha,$$

where

$$(D_q \alpha)_{i_1 \dots i_q} = q g^{jk} \nabla_k \nabla_j \alpha_{j_1 \dots j_q} + \sum_{s}^{1 \dots q} \alpha_{j_1 \dots j_{s-1} r i_{s+1} \dots i_q} R^r_{is} + \sum_{s<t}^{1 \dots q} \alpha_{i_1 \dots i_{s+1} r i_{s+1} \dots i_{t-1} \mu i_{t+1} \dots i_q} . \tag{2.13}$$

Therefore the $\ker (D_q)$ of D_q, that is

$$\ker (D_q) = \{\alpha \in \Lambda^q (M, IR) / D_q(\alpha) = 0\}$$

consists of the Killing q-forms, whose space is denoted by $K_q (M, IR)$, that means $K_q (M, IR) = \ker (D_q)$.

Proposition 2.1 *There is an isomorphism between the vector spaces $AD_q (M, IR)$ and $AD^q (M, IR)$, where $AD_q (M, IR)$ and $AD^q (M, IR)$ are the vector spaces of antisymmetric covariant tensor fields of order q, that is q-forms, and antisymmetric contravariant tensor fields of order q respectively.*

Proof. Let (U, φ) be a chart of M with local coordinate system $(x_1, ..., x_n)$, If w is a q-form on M, then w has the following components

$$\left\{ w_{i_1 \dots i_q} / 1 \le i_1 < i_2 < ... < i_q \le n \right\}$$

with respect to the local coordinate system $(x_1, ..., x_n)$. We consider the following linear mapping

$$F : AD_q (M, IR) = \Lambda^q (M, IR) \to AD^q (M, IR),$$

$$F : w \to F(w),$$

whose component of $F(w)$ with respect to $(x_1, ..., x_n)$ are the following

$$F(w)^{i_1 \dots i_q} = g^{i_1 j_1 \dots} g^{i_q j_q} w_{i_1 \dots i_q} .$$

It can be easily proved that F is bijective. therefore the vector spaces $AD_q (M, IR)$ and $AD^q (M, IR)$ are isomorphic.. □

Remark 2.2 If w is a Killing q-form, then $F(w)$, which is an antisymmetric contravariant tensor field of order q, has the property $\nabla F(w) = 0$. An antisymmetric contravariant tensor field β of order q with the property $\nabla \beta = 0$. is called Killing tensor field of order q. Due to isomorphism F we can use the notion Killing tensor field of order q instead of Killing q-form and conversely.

3 The main result

The set of Killing tensor fields of order q is denoted by $K^q(M, IR)$, which is isomorphic onto $K_q(M, IR)$.

In this section we shall study the $\dim(K^q(M, IR))$ with respect to some properties of the Riemannian on g on M. If α is a Killing q-form, then

$$(\alpha\Delta\alpha) = (\Delta\alpha)_{i_1\cdots i_q}\alpha^{j_1\cdots j_q}, \tag{3.1}$$

which by means of (2.5) and after some estimates and taking under to consideration (2.8) we obtain

$$(\alpha\Delta\alpha) = \frac{(q+1)}{q}L_q(\alpha). \tag{3.2}$$

The equation (2.6) by means of (3.?) becomes

$$\frac{1}{2}\Delta\left(|\alpha|^2\right) = -|\nabla\alpha|^2 + \frac{(q+1)}{q}L_q(\alpha). \tag{3.3}$$

From the second order elliptic differential operator D_q we obtain and endomorphism $(D_q)_x$ of the fibre $\Lambda^q(M, IR)_x$ in x, that is

$$(D_q)_x : \Lambda^q(M, IR)_x \to \Lambda^q(M, IR)_x, \tag{3.4}$$

which satisfies the relation

$$< (D_q)_x u, v > = < u.(D_q)_x v >, \forall u, v \in \Lambda^q(M, IR),$$

where $< \cdot, \cdot >$ is the inner product on $\Lambda^q(M, IR)$ induced by the inner product on T^*M.

Now, we define

$$R(x) = Sup\left\{< (D_q)_x v, v > / v \in \Lambda^q(M, IR), < v, v >= 1\right\}, \tag{3.5}$$

$$R_{\max} = Sup\{R(x)/x \text{ in } M\}. \tag{3.6}$$

Now, we shall prove the theorem

Theorem 3.1 *Let $M(g)$ be a compact Riemannian manifold of dimension n. If $R(x) \leq 0$ and there exists an x_0 such that $R(x_0) < \{0\}$, then $K^q(M, IR)$*

If $R_{\max} = 0$, then $\dim K^q(M, IR) \leq 1 = rank\{\Lambda^q(M, IR)\}$

Proof. If we integrate (3.3) on the manifold M, we obtain

$$\int_M \left[-|\nabla\alpha|^2 + \frac{q+2}{2q}L_q(\alpha)\right] dM = 0. \tag{3.7}$$

From the inequalities

$$-|\nabla\alpha|^2 \leq 0 \tag{3.8}$$

and the assumptions that $R(x) \leq 0, \forall x \in M - \{0\}$ and $R(x_0) < 0$,which imply

$$L_q(x) \leq 0, \forall x \in M - \{0\} \text{ and } L_q(x_0) < 0, \tag{3.9}$$

we conclude that

$$\nabla\alpha \text{ and } \alpha/x = 0, \forall x \in M, \tag{3.10}$$

which yields

$$\alpha = 0.$$

This proves that $K^q(M, IR) = \{0\}$.

If $R_{\max} = 0$ then the formula (3.7) implies

$$\int_M \left[|\nabla\alpha|^2\right] dm = \frac{q+2}{q} \int_M L_q(\alpha) dM \leq 0. \tag{3.11}$$

which implies $|\nabla\alpha| = 0$, that means α is a parallel tensor field. Hence every Killing tensor field of order q on M is parallel. Since the maximal number of independent parallel Killing tensor fields on M is less or equal than the $rank(E)$, where E is the vector bundle of exterior q-forms, then we have

$$\dim(K^q(M, IR)) \leq 1 = rank\{\Lambda^q(M, IR)\}$$

□

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Author's address:

Gr. Tsagas
Division of Mathematics
Department of Mathematics and Physics
School of Thechnology
Aristotle University of Thessaloniki
Thessaloniki, 540 06, GREECE