ON SOME COHOMOLOGIES ON SPACES ADMITTING FLAT LINEAR CONNECTIONS

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Abstract

In this paper two similar cohomologies $H^{p,j,k}$ on manifolds admitting flat linear connections are introduced. For the first cohomology it is used that $\Psi \circ \Psi = O$ for the operator Ψ introduced in [2] and in the second case it is used that $D \circ D = O$, where D is the differential operator which acts upon the tensorial forms.

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1 Introduction

The results of this paper are based on those of [2]. Although the results in [2] concern the complex case, they also holds for the real case. We give them below for the sake of completeness.

Definition 1. Let $\mathcal{V}$ and $\mathcal{V}'$ be vector spaces. For an arbitrary mapping $f : \mathcal{V}^{k-1} \to \mathcal{V}'$ $(k > 1)$ we define a mapping $\Psi f : \mathcal{V}^k \to \mathcal{V}'$ by

$$(\Psi f)(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_{k-1}, \mathbf{X}_k) = (-1)^{k-1} f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_{k-1})$$

$$-f(\mathbf{X}_2, \mathbf{X}_3, \cdots, \mathbf{X}_k) + \sum_{i=1}^{k-1} (-1)^{i+1} f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_i + \mathbf{X}_{i+1}, \cdots, \mathbf{X}_{k-1}, \mathbf{X}_k).$$

If $k = 1$, we define $\Psi f = O$.

Theorem 1. *For an arbitrary mapping $f : \mathcal{V}^{k-1} \to \mathcal{V}'$ it holds*

$$(\Psi \circ \Psi) f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k, \mathbf{X}_{k+1}) = O.$$

Note that $\Psi f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k) = O$ if f is a linear mapping.

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Theorem 2. *The general differentiable solution of the operator equation*

$$\Psi f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k, \mathbf{X}_{k+1}) = O$$

in the set of differentiable functions $\varphi : \mathcal{V}^{k-1} \to \mathcal{V}'$ $(k \geq 2)$ *is given by*

$$f(\mathbf{X}_1, \cdots, \mathbf{X}_k) = (\Psi F)(\mathbf{X}_1, \cdots, \mathbf{X}_k) + L(\mathbf{X}_1, \cdots, \mathbf{X}_k),$$

for an arbitrary differentiable function $F : \mathcal{V}^{k-1} \to \mathcal{V}'$ *and an arbitrary linear mapping* $L : \mathcal{V}^k \to \mathcal{V}'$ $(k \geq 2)$.

Example. If $k = 4$, the operator equation takes the form

$$(\Psi f)(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5) = O,$$

i.e. an explicit form will be given by the functional equation

$$f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) - f(\mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5) + f(\mathbf{X}_1 + \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5)$$

$$-f(\mathbf{X}_1, \mathbf{X}_2 + \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5) + f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 + \mathbf{X}_4, \mathbf{X}_5) - f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 + \mathbf{X}_5) = O.$$

The general differentiable solution of this functional equation is given by

$$\begin{aligned} f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) &= (\Psi F)(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) + L(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) \\ &= F(\mathbf{X}_1 + \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) + F(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 + \mathbf{X}_4) - F(\mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) \\ &\quad - F(\mathbf{X}_1, \mathbf{X}_2 + \mathbf{X}_3, \mathbf{X}_4) - F(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) + L(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4), \end{aligned}$$

where F is an arbitrary (differentiable) function and L is an arbitrary linear mapping.

In this paper two similar cohomologies on manifolds admitting flat linear connections are introduced. They differ from the ordinary cohomology [1].

2 Introducing a new cohomology on manifolds with flat linear connections

Theorem 1 can be used for introducing a new cohomology. Let M_n be an n-dimensional differentiable manifold which admits a flat linear connection ∇, i.e. with vanishing curvature tensor. We denote by T_x the tangent space at the point $x \in M_n$. Now we introduce (i, j, k)-forms and the corresponding cohomology groups $H^{i,j,k}$. We denote by $C_k^j(x)$ the vector space of linear mappings $(T_x)^k \to (T_x)^j$.

Definition 2. If for each $x \in M_n$, $f(x)$ is a mapping $f(x) : (T_x)^i \to C_k^j(x)$ such that

(i) $f(x)$ is a smooth mapping for each $x \in M_n$,

(ii) $f(x)$ depends smoothly on x,

(iii) $\nabla_{\mathbf{X}}(f(\mathbf{X}_1, \cdots, \mathbf{X}_i)) = O$ if $\nabla_{\mathbf{X}}(\mathbf{X}_1) = \cdots = \nabla_{\mathbf{X}}(\mathbf{X}_i) = O$,

then f is called a *smooth parallel* (i, j, k)*-form* on M_n.

We assume the convention to say smooth (i, j, k)-form or just (i, j, k)-form instead of smooth parallel (i, j, k)-form. The following lemma follows from the previous definition.

Lemma 1. *If f is a smooth (i,j,k)-form on M_n, then Ψf is a smooth $(i+1,j,k)$-form on M_n.*

Let $Z^{p,j,k}(M_n)$ be the vector space of closed (p,j,k)-forms, i.e.

$$Z^{p,j,k}(M_n) = \{f : f \text{ is a } (p,j,k)- \text{ form and } \Psi f = O\},$$

and $B^{p,j,k}(M_n)$ be the vector space of exact (p,j,k)-forms, i.e.

$$B^{p,j,k}(M_n) = \{\Psi f : f \text{ is a } (p-1,j,k)- \text{ form on } M_n\}.$$

According to Theorem 1, $B^{p,j,k}(M_n)$ is a vector subspace of $Z^{p,j,k}(M_n)$, and two closed (p,j,k)-forms f and g are said to be *congruent* if $f-g$ is an exact (p,j,k)-form. The set of the equivalence classes determines a cohomology group which will be denoted by $H^{p,j,k}$.

Note that the groups $H^{p,j,k}$ are well defined for a chosen flat linear connection ∇. We do not know whether the groups $H^{p,j,k}$ are invariant under the choice of the flat connection. We have assumed that the manifold is flat in order to define "parallel forms". If we omit the condition (iii) which uses the flat connection, then the corresponding cohomology group may happen to have infinite dimension.

In the special case when the manifold admits a flat Riemannian connection, i.e. with vanishing curvature and torsion tensor, the introduced cohomology can be described as follows. It is easy to prove that a manifold admits a flat Riemannian connection if and only if it admits an atlas of coordinate neighborhoods with Jacobi matrices with constant elements, i.e. affine differentiable manifolds. Assume that the atlas is such that all the elements of the corresponding Jacobi matrices are constant functions. For such chart $(U_\alpha,\varphi_\alpha)$ we consider the vector fields $\mathbf{X}_1^{(\alpha)},\cdots,\mathbf{X}_n^{(\alpha)}$ tangent to the parametric curves. It is easy to prove that for any vector field $\mathbf{X}$ on U_α, $[\mathbf{X}_1^{(\alpha)},\mathbf{X}] = \cdots = [\mathbf{X}_n^{(\alpha)},\mathbf{X}] = O$ if and only if $\mathbf{X}$ is a linear combination of $\mathbf{X}_1^{(\alpha)},\cdots,\mathbf{X}_n^{(\alpha)}$ with constant coefficients. Hence for any such two intersecting coordinate neighborhoods U_α and U_β it holds

$$[\mathbf{X}_i^{(\alpha)},\mathbf{X}_j^{(\beta)}] = O \qquad (1 \le i,j \le n).$$

We will denote by $\mathcal{L}$ the Lie derivative and we will use the chosen collection of vectors fields $\{\mathbf{X}^{(\alpha)}\}$ on M_n satisfying the previous property. We define a flat Riemannian connection on M_n by defining a connection with zero Christoffel symbols in each of these coordinate neighborhoods U_α. Using this flat connection, the parallel smooth (i,j,k)-forms are characterized by Definition 2, where condition (iii) takes the form

(iii) for each α

$$\mathcal{L}_{\mathbf{X}^{(\alpha)}}(f(\mathbf{X}_1,\cdots,\mathbf{X}_i)) = O \qquad \text{if} \qquad [\mathbf{X}^{(\alpha)},\mathbf{X}_1] = \cdots = [\mathbf{X}^{(\alpha)},\mathbf{X}_i] = O.$$

At the end we will consider some simple examples. We will consider the circle $\mathbf{S}^1$ and the torus $\mathbf{T} = \mathbf{S}^1 \times \mathbf{S}^1$. These are simple cases because there exist global linearly independent vector fields $\mathbf{X}_1,\cdots,\mathbf{X}_n$ $(n=1,2)$ such that $[\mathbf{X}_p,\mathbf{X}_q] = O$ for $1 \le p,q \le n$. The sphere $\mathbf{S}^3$ admits three linearly independent vector fields $\mathbf{X},\mathbf{Y},\mathbf{Z}$, but does not satisfy this property since the condition $[\mathbf{X},\mathbf{Y}]=[\mathbf{Y},\mathbf{Z}]=[\mathbf{Z},\mathbf{X}]=O$ cannot be satisfied everywhere.

1. Let M be the circle $\mathbf{S}^1$, and let us choose an arbitrary smooth tangent vector field $\mathbf{U}$ which is non-zero at each point. Then each vector field $\mathbf{X}$ is given by $\mathbf{X}=\alpha\mathbf{U}$ and $[\mathbf{X},\mathbf{U}]=O$ if and only if $\alpha=\text{const}$. Since $B^{0,j,k}$ is an empty set, $H^{0,j,k}=Z^{0,j,k}$. Further $\Psi f=O$ and $\mathcal{L}_{\mathbf{U}}f=0$ if and only if $f=\text{const}\cdot\mathbf{P}$, where $\mathbf{P}$ is a tensor field of type (j,k) with constant elements with respect to the chosen coordinates. Thus we obtain $H^{0,j,k}=\mathbf{R}^1$ for any $j,k\in\{0,1,\cdots\}$.

Further let $f(x):T_x\to C_k^j(x)$ be such that $\Psi f=O$, i.e. $f(\mathbf{X}_1+\mathbf{X}_2)=f(\mathbf{X}_1)+f(\mathbf{X}_2)$ and hence $f(\alpha\mathbf{X})=\alpha f(\mathbf{X})$. If $j=k=0$, then from $O=\mathcal{L}_{\mathbf{U}}(f(\mathbf{U}))=\mathbf{U}(f(\mathbf{U}))$ it follows that $f(\mathbf{U})$ is a constant C along the circle. Since $f(\alpha\mathbf{U})=\alpha f(\mathbf{U})=\alpha C$, we obtain that $H^{1,0,0}=\mathbf{R}$. If $j=1$, $C_0^1(x)$ is identified with T_x, and then $f(\mathbf{U})=a\cdot\mathbf{U}$ and (iii) implies

$$O=\mathcal{L}_{\mathbf{U}}(f(\mathbf{U}))=[\mathbf{U},f(\mathbf{U})]=[\mathbf{U},a\cdot\mathbf{U}],$$

and hence $a=\text{const}$ along the circle. Thus $H^{1,1,0}=\mathbf{R}$. Note that also $H^{1,j,k}=\mathbf{R}$ for $j,k\geq 0$.

2. Let M be the torus $\mathbf{S}^1\times\mathbf{S}^1$. Then there exist two linearly independent tangent vector fields $\mathbf{U}$ and $\mathbf{V}$ such that $[\mathbf{U},\mathbf{V}]=O$. We will calculate the groups $H^{i,j,0}$ for $i,j\in\{0,1\}$.

$H^{0,0,0}=\mathbf{R}$ analogously as for the circle.

In order to find $H^{1,0,0}$, we should consider the maps f such that $\Psi f=O$ and for each vector field $\mathbf{X}=\alpha\mathbf{U}+\beta\mathbf{V}$,

$$[\mathbf{U},\mathbf{X}]=O\qquad\text{and}\qquad[\mathbf{V},\mathbf{X}]=O$$

for any constants α and β. Since $\Psi f=O$, it holds $f(\alpha\mathbf{U}+\beta\mathbf{V})=\alpha f(\mathbf{U})+\beta f(\mathbf{V})$. Hence f satisfies the required conditions if and only if $f(\alpha\mathbf{U}+\beta\mathbf{V})=\alpha p+\beta q$, where $p=f(\mathbf{U})$ and $q=f(\mathbf{V})$ are arbitrary scalars. Thus $H^{1,0,0}=\mathbf{R}^2$.

In order to find $H^{0,1,0}$, we should find all tangent vector fields $\mathbf{f}$ such that

$$\mathcal{L}_{\mathbf{U}}\mathbf{f}=\mathcal{L}_{\mathbf{V}}\mathbf{f}=O,\qquad\text{i.e.}\qquad[\mathbf{U},\mathbf{f}]=[\mathbf{V},\mathbf{f}]=O,$$

and this is satisfied if and only if $\mathbf{f}=\alpha\mathbf{U}+\beta\mathbf{V}$ for arbitrary constants α and β, and hence $H^{0,1,0}=\mathbf{R}^2$.

In order to find $H^{1,1,0}$, we should find all maps $\mathbf{f}$ such that $\Psi\mathbf{f}=O$, i.e. $\mathbf{f}(\mathbf{X}+\mathbf{Y})=\mathbf{f}(\mathbf{X})+f(\mathbf{Y})$, and

$$[\mathbf{U},\mathbf{f}(\mathbf{U})]=[\mathbf{U},\mathbf{f}(\mathbf{V})]=[\mathbf{V},\mathbf{f}(\mathbf{U})]=[\mathbf{V},\mathbf{f}(\mathbf{V})]=O,$$

and this is satisfied if and only if

$$\mathbf{f}(\mathbf{U})=\alpha\mathbf{U}+\beta\mathbf{V}\quad\text{and}\quad\mathbf{f}(\mathbf{V})=\gamma\mathbf{U}+\delta\mathbf{V}$$

for arbitrary constants α,β,γ and δ, and hence $H^{1,1,0}=\mathbf{R}^4$.

Remark. Note that for the k-dimensional torus $(\mathbf{S}^1)^k$ it holds

$$H^{0,0,0}=\mathbf{R},\quad H^{0,1,0}=\mathbf{R}^k,\quad H^{1,0,0}=\mathbf{R}^k\quad\text{and}\quad H^{1,1,0}=\mathbf{R}^{k^2}.$$

3 Introducing another cohomology on manifolds with flat linear connections

The cohomology introduced in section 2 can be compared with the following one, which will be defined on manifolds admitting flat linear connections.

Let M_n be an n-dimensional differentiable manifold which admits a flat linear connection ∇. It is convenient to write the connection in local coordinates by Γ^p_{st}. For any tensorial i-form ω with values in the space of tensors of type (j,k), $D\omega$ is a tensorial $(i+1)$-form with values in the space of tensors of type (j,k). An i-form with values in the space of tensors of type (j,k) will be called (i,j,k)-form. Indeed, in local coordinates if ω is given by

$$\omega^{s_1\cdots s_j}_{t_1\cdots t_k} = A^{s_1\cdots s_j}_{t_1\cdots t_k \ell_1\cdots \ell_i} dx^{\ell_1}\wedge\cdots\wedge dx^{\ell_i},$$

then

$$(D\omega)^{s_1\cdots s_j}_{t_1\cdots t_k} = \Big[\partial A^{s_1\cdots s_j}_{t_1\cdots t_k\ell_1\cdots\ell_i}/\partial x^u + \sum_{p=1}^{j} A^{s_1\cdots s_{p-1}\lambda s_{p+1}\cdots s_j}_{t_1\cdots t_k\ell_1\cdots\ell_i}\Gamma^{s_p}_{\lambda u} -$$

$$-\sum_{q=1}^{k} A^{s_1\cdots s_j}_{t_1\cdots t_{q-1}\mu t_{q+1}\cdots t_k\ell_1\cdots\ell_i}\Gamma^{\mu}_{t_q u}\Big] dx^u\wedge dx^{\ell_1}\wedge\cdots\wedge dx^{\ell_i}.$$

By a direct calculation one can obtain the following identity

$$(DD\omega)^{s_1\cdots s_j}_{t_1\cdots t_k} = \frac{1}{2}\Big[\sum_{p=1}^{j} A^{s_1\cdots s_{p-1}\lambda s_{p+1}\cdots s_j}_{t_1\cdots t_k\ell_1\cdots\ell_i} R^{s_p}_{\lambda uv} -$$

$$-\sum_{q=1}^{k} A^{s_1\cdots s_j}_{t_1\cdots t_{q-1}\mu t_{q+1}\cdots t_k\ell_1\cdots\ell_i} R^{\mu}_{t_q uv}\Big] dx^u\wedge dx^v\wedge dx^{\ell_1}\wedge\cdots\wedge dx^{\ell_i}.$$

Using that the curvature tensor vanishes, we obtain that $DD\omega\equiv O$, i.e. $D\circ D = O$. Note that in the special case when $j=k=0$ the operator D coincides with the operator d for differential forms.

Using that $D\circ D = O$, we obtain a cohomology as follows. Let $Z^{p,j,k}(M_n)$ be the vector space of closed (p,j,k)-forms, i.e.

$$Z^{p,j,k}(M_n) = \{\omega : \omega \text{ is a } (p,j,k) - \text{ form and } D\omega = O\},$$

and $B^{p,j,k}(M_n)$ be the vector space of exact (p,j,k)-forms, i.e.

$$B^{p,j,k}(M_n) = \{D\omega : \omega \text{ is a } (p-1,j,k) - \text{ form on } M_n\}.$$

Since $D\circ D = O$, $B^{p,j,k}(M_n)$ is a vector subspace of $Z^{p,j,k}(M_n)$, and two closed (p,j,k)-forms ω and θ are said to be *congruent* if $\omega-\theta$ is an exact (p,j,k)-form. The set of the equivalence classes determines a cohomology group which will be denoted by $H^{p,j,k}$. Note that in the special case $j=k=0$ the cohomology groups $H^{p,0,0}$ are the cohomology groups from the de Rham cohomology. The de Rham cohomology is defined for any manifold because $d \circ d \equiv O$ for any connection on the manifold, not necessarily with zero curvature tensor.

The following question appears whether the cohomology groups $H^{p,j,k}$ from the previous section have the same dimensions as the cohomology groups $H^{p,j,k}$ defined in this section.

Note that the cohomology groups $H^{p,j,k}$ defined in this section can be introduced also on manifolds which are base manifolds of a vector bundle admitting flat connection. The construction is analogous to the previous one, and here we require that the tangent bundle admits a flat connection.

Example. The group $H^{0,j,k}$ coincides with $Z^{0,j,k}$ and that is the vector space of parallel tensor fields of type (j,k) on M_n. Thus $\dim H^{0,j,k}$ is equal to the number of (parallel) linearly independent tensor fields of type (j,k) on M_n.

References

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