ON CONSTITUTIVE EQUATIONS FOR RATE TYPE (COMPOSITE) MATERIALS

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Abstract

This work deals with the constitutive equations for rate type materials as well as their thermodynamics. We consider both the one-dymensional case and the three-dymensional case. We introduce the notion of relaxed state in energy and obtain results on the regular relaxed surfaces. Also, results regarding the instantaneous response of these materials are given. Finally, we define the notions of totally relaxed states and of totally relaxed regular surface and obtain results in these fields. A numerical study is given.

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1 Introduction

1. THE ONE-DIMENSIONAL CASE

Let $\mathbf{I} \subset \mathbb{R}$ be an interval.

Definition 1.1. *A one dimensional reference configuration for a body Ω is an interval $\mathcal{R}$, finite or infinite, of a real straight line, with a bijective correspondence between the points of $\mathcal{R}$ and those of Ω.*

A one dimensional movement of the body Ω is an application $\chi : \mathcal{R} \times \mathbf{I} \to \mathbb{R}$

$$x = \chi(X,t), \ X \in \mathcal{R}, \ t \in \mathbf{I}, \tag{1.1}$$

with the feature that for any fixed $t \in \mathbf{I}$, $\chi(\cdot,t) : \mathcal{R} \to \mathbb{R}$ is injective and continuous.

- *X is designed as the initial co-ordinate (or Lagrangian co-ordinate);*
- *x is designed as the actual co-ordinate (or Eulerian);*
- *t is time.*

Sometimes, it is admitted that $\chi \in C^1(\mathcal{R} \times \mathbb{R})$.
If $\chi \in C^1(\mathcal{R} \times \mathbb{R})$, we will define $v : \mathcal{R} \times \mathbf{I} \to \mathbb{R}$,

$$v(X,t) = \dot{x}(X,t) := \frac{\partial \chi}{\partial t}(X,t) \tag{1.2}$$

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and $\varepsilon : \mathcal{R} \times \mathbb{I} \to \mathbb{R}$,

$$\varepsilon(X,t) := \frac{\partial \chi}{\partial X}(X,t) \tag{1.3}$$

designed as the *velocity* and as the *deformation* of the (material) particle of the body, respectively.

We will also introduce the following notations:

$\sigma = \sigma(X,t)$ - *stress* which is the force per unit of area of the reference configuration;

$\theta = \theta(X,t)$ - *absolute temperature* (which is always positive);

$e = e(X,t)$ - *internal energy per unit of volume* within the referance configuration;

$\eta = \eta(X,t)$ - *entropy per unit of volume* within the reference configuration;

$\psi = \psi(X,t) = e - \eta\theta$ - *free energy per unit of volume* within the reference configuration;

$q = q(X,t)$ - *flux of heat through the unit of area* in the reference configuration (it is oriented after the inner normal);

$g = g(X,t) = \dfrac{\partial \theta}{\partial X}$ - *temperature gradient* in the reference configuration;

$b = massforce$;

$r = r(x,t)$ - *excess of heat* received from the exterior;

$\rho_0 = \rho_0(X,t)$ - *density* in the reference configuration.

The equations of conservation of impulse, of energy and the *Clausius - Duhem* inequality (see *Coleman and Gurtin* [5] (1965)) are of the form:

$$\frac{d}{dt}\left(\int_{X_1}^{X_2} \dot{x}\rho_0 dX\right) = \int_{X_1}^{X_2} b\rho_0 dX + \sigma(X_2,t) - \sigma(X_1,t), \tag{1.4$_1$}$$

$$\frac{d}{dt}\left(\int_{X_1}^{X_2} \rho_0\left(\frac{\dot{x}^2}{2} + e\right) dX\right) = \int_{X_1}^{X_2} (\dot{x}b + r)\rho_0 dX + \sigma(X_2,t)\dot{x}(X_2,t) -$$

$$-\sigma(X_1,t)\dot{x}(X_1,t) - q(X_2,t) + q(X_1,t), \tag{1.4$_2$}$$

$$\frac{d}{dt}\left(\int_{X_1}^{X_2} \eta dX\right) \geq \int_{X_1}^{X_2} \rho_0 \frac{r}{\theta} dX + \frac{q(X_1,t)}{\theta(X_1,t)} = \frac{q(X_2,t)}{\theta(X_2,t)}. \tag{1.4$_3$}$$

If the functions which apear in these relations are smooth enough, they can be rewrite over a differential form:

$$\rho_0(\ddot{x} - b) = \frac{\partial \sigma}{\partial X}, \tag{1.5$_1$}$$

$$\rho_0 \frac{\partial}{\partial t}\left(\frac{\dot{x}^2}{2} + e\right) = \rho_0(\dot{x}b + r) + \frac{\partial}{\partial X}(\sigma\dot{x}) - \frac{\partial q}{\partial X}, \tag{1.5$_2$}$$

$$\rho_0 \dot{\eta} \geq \frac{r}{\theta} - \frac{\partial}{\partial X}\left(\frac{q}{\theta}\right). \tag{1.5$_3$}$$

Definition 1.2. *We shall designate as* **wave** *throught the body Ω a smooth curve in the plane (X,t), denoted as $\mathcal{C} = \{(Y(t),t)/t \in \mathbf{I}\}, U(t) = \frac{dY}{dt}(t) = \dot{Y}(t)$, with the property that upon its crossing some of the magnitudes $v, \varepsilon, \sigma, e, \eta, \psi, \theta, q, g$ or their derivatives have discontinuities of the first sort*[1]*, and besides $\mathcal{C}$ they are continuous with respect to (X,t).*

Definition 1.3. *We designate as* **shock wave** *throughout the body Ω a curve denoted $\mathcal{C}$ with the property to render continuous the movement $\chi(X,t)$ on its crossing, but with at least one of the magnitudes $v, \varepsilon, \sigma, e, \eta, \psi, \theta, q, g$ having leaps*[2] *and besides $\mathcal{C}$ they are smooth functions of (X,t).*

In the case of a body Ω through which a shock wave $\mathcal{C}$ is propagated, assuming that b and r are continuous upon crossing $\mathcal{C}$, we get leak equations (see *Chen and Gurtin* [9] (1972)):

$$\rho_0 U[v] + [\sigma] = 0, \tag{1.6$_1$}$$

$$\rho_0 U[\frac{v^2}{2} + e] + [\sigma v] - [q] = 0, \tag{1.6$_2$}$$

$$U[\eta] \geq -[\frac{q}{e}] \tag{1.6$_3$}$$

and taking into account (1.2) and (1.3), we get:

$$U[\varepsilon] + [v] = 0. \tag{1.6$_4$}$$

The relations (1.6) are called *dynamic compatibility relations*, while the equality (1.6_4) is called *kinematic compatibility relations.*

Definition 1.4. *A smooth curve $\mathcal{C}$ with the equation $Y = \varphi(t)$ is called* **acceleration wave** *if upon its crossing the movment $\chi(X,t)$ is of the class C^1, the velocity $v(X,t)$ and deformation $\varepsilon(X,t)$ as well as θ, σ, ψ, and q are continuous functions, but their derivatives can assume discontinuities of the first sort, remaining continuous outside the curve $\mathcal{C}$.*

The continuity of the functions v and ε lead to the so-called kinematic states of compatibility that must be satisfied when crossing the curve $\mathcal{C}$:

$$\left[\frac{\partial v}{\partial t}\right] + c\left[\frac{\partial v}{\partial X}\right] = 0, \tag{1.7$_1$}$$

$$\left[\frac{\partial \varepsilon}{\partial t}\right] + c\left[\frac{\partial \varepsilon}{\partial X}\right] = 0, \tag{1.7$_2$}$$

where c is the slope of the curve $\mathcal{C}$ at point $(\varphi(t),t)$, that is $c = \frac{d\varphi}{dt}(t)$.

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[1] We say that a function $(X,t) \xrightarrow{f} f(X,t)$ has discontinuities of the first sort at the point (X,t) if for a fixed t, $\lim_{Y \nearrow X} f(Y,t)$ and $\lim_{Y \searrow X} f(X,t)$ exist and are finite

[2] If $f : A \subset \mathbb{R} \to \mathbb{R}$, we denote by $[f](t) := f(t+0) - f(t-0)$ - the leap.

Furthermore, owing to the fact that v and ε are partial derivatives of χ, we obtain the compatibilitiy state:

$$\left[\frac{\partial \varepsilon}{\partial t}\right] = \left[\frac{\partial v}{\partial X}\right]. \tag{1.8}$$

The dynamic states of compatibility are obtained from the equation (1.6), thus: these equations are written on either side of the curve $\mathcal{C}$, the limits of the two regions are calculated at a point on the curve $\mathcal{C}$, then the leap is calculated (that is, the difference of the two limits). The following relations are thus obtained:

$$\rho_0 \left[\frac{\partial v}{\partial t}\right] = \left[\frac{\partial \sigma}{\partial X}\right], \tag{1.9$_1$}$$

$$\rho_0 v \left[\frac{\partial v}{\partial t}\right] + \rho_0 \left[\frac{\partial \varepsilon}{\partial t}\right] = v \left[\frac{\partial \sigma}{\partial X}\right] + \sigma \left[\frac{\partial v}{\partial X}\right] - \left[\frac{\partial q}{\partial X}\right], \tag{1.9$_2$}$$

$$\rho_0 \left[\frac{\partial \eta}{\partial t}\right] + \frac{\partial}{\partial X}\left[\frac{q}{\theta}\right] \geq 0, \tag{1.9$_3$}$$

assuming further on that b and r are continuous functions upon crossing of the curve $\mathcal{C}$.

The leap states (1.7), (1.8), (1.9) thus obtained, form a system of linear and homogeneous algebraic equations, by the unknowns $\left[\frac{\partial v}{\partial t}\right], \left[\frac{\partial v}{\partial X}\right]$, etc. In order to discuss the states of existence of the acceleration wave, this system must be completed with constitutive equations, that is with relations between the magnitudes $\varepsilon, \theta, \sigma, e, \eta$ and q. For exemple, for the elastic-plastic bodies, the treatment of this problem can be found in *Rahmatulin and Demianov* [10].

2. The Three-Dimensional Case

Definition 1.5. *A configuration of reference of a body Ω is a domain $\mathcal{D} \subset \mathbb{R}^3$ with the property that there is a bijection $\varphi : \Omega \to \mathcal{D}$.*

A tree-dimensional movement of the body Ω is an application $\chi : \mathcal{D} \times \mathbf{I} \to \mathbb{R}^3$,

$$x = \chi(X,t),\ X \in \mathcal{D},\ t \in \mathbf{I}, \tag{1.10}$$

where $\mathbf{I} \subset \mathbb{R}$ is an interval, with the property that the partial function $\chi(\cdot,t) : \mathcal{D} \to \mathbb{R}^3$ is injective and continuous, $\forall t \in \mathbf{I}$.

The points $x = \chi(X,t) \in \Delta_t = \chi(\mathcal{D},t)$, $t \in \mathbf{I}$ are expressed with respect to the canonical base of the space $\mathbb{R}^3$ is called *actual configuration or configuration at the moment t* of the body Ω.

As in the one-dimensional case, we admit that the application $\chi \in C^2(\mathcal{D} \times \mathbf{I}; \mathbb{R}^3)$, except the cases when another smoothness hypotesis is mentioned.

The functions $v = \dot{x} = \frac{\partial \chi}{\partial t}(X,t)$ and $\ddot{x} = \dot{v} = \frac{\partial^2 \chi}{\partial t^2}(X,t)$ are designated as *the velocity* and, respectively, *the acceleration* of the material particle.

Remark 1.1. *Owing to the invertibility of the function $\chi(\cdot;t) : \mathcal{D} \to \Delta_t$, for any fixed t, the velocity and the acceleration can be also expressed with respect to the actual co-ordinate of the material point, thus:*

$$v(x,t) = \dot{x}\left(\chi^{-1}(x,t),t\right) \text{ şi } a(x,t) = \ddot{x}\left(\chi^{-1}(x,t),t\right) = \frac{dv}{dt}(x,t). \tag{1.11}$$

Definition 1.6. *The operator*

$$\frac{d}{dt} = \frac{\partial}{\partial t} + < v, \mathrm{grad_x} >= \frac{\partial}{\partial \mathrm{t}} + \mathrm{v}\frac{\partial}{\partial \mathrm{x_i}} = \frac{\partial}{\partial \mathrm{t}} + \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{v_i}\frac{\partial}{\partial \mathrm{x_i}}, \tag{1.12}$$

is called material derivative.

The magnitude

$$L(x,t) = \nabla_x v(x,t) = \left(\frac{\partial v_i}{\partial x_j}(x,t)\right)_{1\le i,j\le 3} = (L_{ij}(x,t))_{1\le i,j\le 3} \tag{1.13}$$

is called the velocity gradient or the deformation velocity, while the magnitude

$$F(X,t) = \nabla_x \chi(X,t) = \left(\frac{\partial \chi_i}{\partial X_j}(X,t)\right)_{1\le i,j\le 3} = (F_{ij}(X,t))_{1\le i,j\le 3} \tag{1.14}$$

is called the deformation gradient.

Proposition 1.1. *Between the two gradients there is an equality*

$$L(x,t) = \dot{F}() * F^{-1}, \tag{1.15}$$

where F^{-1} is the inverse of matrix F, while $*$ designates the usual product of two matrices.

As regards the forces acting over the body Ω, we will take the *nonpolar* case (see *Truesdell and Toupin* [1], sect. 200 or *Solomon L.* Chap. II), i.e. mass forcess $b(X,t)$ act on body, as well as contact forces, characterised by the tensor $T = T(X,t)$ of Cauchy (see *Truesdell and Toupin* [1]).

If n is a unitary vector, then $t_n = T * n$ is the force acting over the unit of area perpendicular to n in the current configuration.

The kinetic moment conservation equation leads to the equality

$$T = T^t \quad (T_{ij} = T_{ji}), \tag{1.16}$$

T^t is the transpose of T.

Analogically, the impulse conservation equation and the equation of energy, respectively, according to the actual co-ordinates, assume a differential form

$$\rho\dot{v} - \mathrm{div}_x T = \rho b \quad \left(\text{or by components } \rho\dot{v}_i - \frac{\partial T_{ij}}{\partial x_j} = \rho b_i, \;\; i,j = 1,2,3\right) \tag{1.17}$$

$$\rho\dot{e} - T * L + \mathrm{div}_x q = \rho r \quad \left(\text{sau } \;\rho\dot{e} - T_{ij}L_{ij} + \frac{\partial q_i}{\partial x_i} = \rho r\right), \tag{1.18}$$

where ρ is the actual mass density, e is the internal energy per unit of mass, q is the heat flux (the quantity of heat input per unit of time, through the actual unit of area, $r = r(x,t)$ is the excess heat per unit of time and the unit of mass (absorbed by the ω particle of Ω and furnished by radiation from the environment).

The mass conservation equation is given by one of the formulas

$$\rho_0 = J\rho \tag{1.19}$$

or

$$\frac{\partial v}{\partial t} + \mathrm{div}_x(\rho v) = 0, \tag{1.20}$$

where $J = \det F$ (the determinant of F).

The relation (1.19) is the mass conservation equation according to the reference configuration), while (1.20) is the mass conservation equation according to the actual co-ordinates.

If we introduce the *Piola-Kirchhoff* tensor (see *Coleman and Gurtin* [5])

$$S = \rho^{-1} T * (F^t)^{-1} = \frac{1}{\rho_0}\tilde{S}, \tag{1.21}$$

then the equation (1.18) can be written under the form

$$\rho\dot{e} - \rho S\dot{F} + \mathrm{div}_x q = \rho r. \tag{1.22}$$

The specific velocity γ for entropy production is defined by the equality

$$\rho\gamma = \rho\dot{\eta} - \left[\left(\frac{\rho r}{\theta}\right) - \mathrm{div}_x\left(\frac{\rho}{\theta}\right)\right], \tag{1.23}$$

where $\eta = \eta(X,t)$ is the entropy per unit of mass, while $\theta = \theta(X,t) > 0$ is absolute temperature.

The second law of thermodynamics or the *Causius-Duhem* inequality (see *Truesdell and Toupin* [1]) stats that

$$\gamma \geq 0. \tag{1.24}$$

By using the relations (1.22), (1.23) and (1.24) we obtain

$$\gamma = \dot{\eta} - \frac{\dot{e}}{\theta} + \theta^{-1}S\dot{F} - \frac{1}{\rho\theta^2}qg \geq 0, \tag{1.25}$$

where

$$g = \mathrm{grad}_x\theta, \quad qg = \sum_{i=1}^{3} q_i\frac{\partial\theta}{\partial x_i} = q_i\frac{\partial\theta}{\partial x_i}. \tag{1.26}$$

If we introduce free energy by the equality

$$\psi = e - \theta\eta, \tag{1.27}$$

then the inequality (1.25) becomes

$$\theta\gamma = -\dot{\psi} - \eta\dot{\theta} + S\dot{F} - (\rho\theta)^{-1}qg \geq 0. \tag{1.28}$$

Definition 1.7. *A regular (smooth) surface Σ, given through the implicit equation $\varphi(X,t) = 0$, $X \in \mathcal{D}$, $t \in \mathbf{I}$, is called* **acceleration wave** *for the body Ω if $\chi \in C^1$, $v, F, \theta, \tilde{S}, \psi, \eta$ and q are continuous over $\mathcal{D} \times \mathbf{I} \setminus \Sigma$ with possible assumptions of discontinuities of the first sort (leaps[3]) upon crossing of the surface Σ.*

Definition 1.8. *The magnitudes*

$$
\begin{aligned}
U(X,t) &= -\frac{1}{\|\mathrm{grad}_X\varphi\|}\frac{\partial\varphi}{\partial t}(X,t),\\
n(X,t) &= \frac{1}{\|\mathrm{grad}_X\varphi\|}\left(\frac{\partial\varphi}{\partial X_1}(X,t), \frac{\partial\varphi}{\partial X_2}(X,t), \frac{\partial\varphi}{\partial X_3}(X,t)\right)
\end{aligned}
\tag{1.29}
$$

are called **the propagation velocity of the acceleration wave** *and* **the direction of propagation of the acceleration wave,** *respectively.*

The leaps of the derivatives of the thermodynamic magnitudes $v, F, \theta, \tilde{S}, \psi, \eta$ and q cannot be independent. They are subject to three types of states:

- *the geometrical-kinetic compatibility states* express the fact that the leaps of the functions $\dfrac{\partial v}{\partial t}, \dfrac{\partial v}{\partial X}$, etc. are connected owing to the continuity of v and to the fact that v and F are partial derivatives of the same vectorial function χ. These compatibility conditions are:

$$
\begin{cases}
\left[\dfrac{\partial v_k}{\partial t}\right] = U^2 a_k, \quad \left[\dfrac{\partial F_{kl}}{\partial X_j}\right] = a_k n_l n_j, \quad \left[\dfrac{\partial F_{kl}}{\partial t}\right] = -U a_k n_l,\\
\left[\dfrac{\partial \theta}{\partial t}\right] = -U\nu, \quad \left[\dfrac{\partial \theta}{\partial X_j}\right] = \nu n_j, \quad \left[\dfrac{\partial \tilde{S}_{ij}}{\partial t}\right] = -U S_{ij}, \quad \left[\dfrac{\partial \tilde{S}_{ij}}{\partial X_k}\right] = S_{ij} n_k.
\end{cases}
\tag{1.30}
$$

The vector $a = (a_1, a_2, a_3)$ is called *mechanical amplitude of the wave*, v is a scallar called *thermal amplitude of the wave* while S_{ij} stands for *tension amplitude.*

- *the dynamic states of compatibility* are restrictions owing to the conservation equations. Since the functions b and r are assumed as continuous, we obtain:

$$
\begin{cases}
\rho_0\left[\dfrac{\partial v_i}{\partial t}\right] - \left[\dfrac{\partial \tilde{S}_{ij}}{\partial X_j}\right] = 0,\\
\rho_0\left[\dfrac{\partial e}{\partial t}\right] - \tilde{S}_{ij}\left[\dfrac{\partial F_{ij}}{\partial t}\right] + \left[\dfrac{\partial \tilde{q}_j}{\partial X_j}\right] = 0.
\end{cases}
\tag{1.31}
$$

- *the states dictated by the constitutive equations*, that is relations that must exist between $F, \theta, \mathrm{grad}_X\theta, \tilde{S}, \eta, \psi$ and q.

[3] We will denote by $[f(M_0)]$ the leap of function f at the point $M_0, M_0 \in \Sigma$, that is $[f(M_0)] = f^+(M_0) - f^-(M_0)$, where by $f^+(M_0)$ we denoted the limit of function f at the point M_0, when $M \to M_0, M_0$ being located in the sub-domain containing the positive normal at M_0 to Σ, while by $f^-(M_0)$ we denoted the limit in M_0, when $M \to M_0$ from the sub-doman containing the negative normal to Σ at the point M_0.

2 Thermodynamics of Rate Type Constitutive Equations

1. THE ONE-DIMENSIONAL CASE

Definition 2.1. *We will say that the thermodynamic states s as the time t of a material particle of the body Ω, which has the X co-ordinate in the reference configuration, is known if for any $(X,t) \in \mathcal{D} \times \mathbf{I}$, the magnitudes $\varepsilon, \theta, g, \sigma, \psi, \eta, k$ are known and we will denote:*

$$s = (\varepsilon, \theta, g, \sigma, \psi, \eta, k), \tag{2.1}$$

where

$$k = \frac{q}{\theta \rho_0}. \tag{2.2}$$

Obviously, s can be interpreted as a point in $\mathbb{R}^7$ whose co-ordinates depend on X and t.

Definition 2.2. *Let $f : [a,b] \to \mathbb{R}^n$ be a function. The function f is called riglated if at each point $t \in]a,b[$ one can find within $\mathbb{R}^n$ the lateral limits $f^+(t) := f(t+0), f^-(t) := f(t-0)$ and $f^+(a) := f(a+0), f^-(b) := f(b-0)$.*

The set of riglated functions shall be denoted by R°.

If $f \in R^\circ$ and if one can find within $\mathbb{R}^n$ the limits

$$f'_s(t_o) := \lim_{t \nearrow t_o} \frac{f(t) - f^-(t_o)}{t - t_o}, \quad f'_d(t_o) := \lim_{t \searrow t_o} \frac{f(t) - f^+(t_o)}{t - t_o}, \quad \forall t \in]a,b[,$$

$f'_d(a) := \lim_{t \searrow a} \dfrac{f(t) - f^+(a)}{t - a}$, $f'_s(b) := \lim_{t \nearrow b} \dfrac{f(t) - f^-(b)}{t - b}$ *and if $f'_s, f'_d \in R^o$, we will say that f belongs to the class R^1.*

The set of continuous functions that have as lateral derivatives riglated functions will be denoted by C^{01}, so $C^{01} = C^0 \cap R^1$.

Definition 2.3. *We call thermodynamic process of a duration $T_s \geq 0$ of the material partial $\omega \in \Omega$ a function $s : [0, T_s] \to \mathbb{R}^7$, $s(t) = \big(\varepsilon(t), \theta(t), g(t); \sigma(t), \psi(t), \eta(t), k(t)\big)$, of class C^{01}.*

The application $\tau = (\varepsilon, \theta, g) : [0, T_\tau] \to \mathbb{R}^3$ will be called a trajectory of duration T_τ of the process s.

We can have trajectories of duration $T_\tau \geq T_s$.

Definition 2.4. *We will say that a material particle ω of a body Ω* **assumes a rate type behaviour** *if there is a domain $A \subset \mathbb{R}^7$ and the continuous functions $a_i, b_i, c_i, d_i : A \to \mathbb{R}, i = \overline{1,4}$ with the property that for any process $s \in C^{01}$ starting from the state $s_0 \in A$, there is a duration $T_s > 0$ in a manner that:*

$$\begin{cases} \dot{\psi} = a_1(s)\dot{\varepsilon} + a_2(s)\dot{\theta} + a_3(s)\dot{g} + a_4(s), \\ \dot{\sigma} = b_1(s)\dot{\varepsilon} + b_2(s)\dot{\theta} + b_3(s)\dot{g} + b_4(s), \\ \dot{\eta} = c_1(s)\dot{\varepsilon} + c_2(s)\dot{\theta} + c_3(s)\dot{g} + c_4(s), \\ \dot{k} = d_1(s)\dot{\varepsilon} + d_2(s)\dot{\theta} + d_3(s)\dot{g} + d_4(s), \end{cases} \tag{2.3}$$

are satisfied for any $t \in [0, T_s]$, with the exception of a set at most numerable of points from $[0, T_s]$.

We will say that the body Ω has **a rate type behaviour** *if any of its particles of a body has rate type behaviour*

In other words, a particle of a body has a rate type behaviour if by knowing its state at the moment (time) t and its increments (ε, θ, g) we are able to determine its increments (σ, ψ, η, k) by means of the functions a_i, b_i, c_i, d_i, $i = \overline{1,4}$, that describe the material properties.

One can see that a trajectory $\tau(t) = (\varepsilon(t), \theta(t), g(t))$, $t \in [0, T_\tau]$ and the initial states $\sigma(0) = \sigma_0, \psi(0) = \psi_0, \eta(0) = \eta_0, k(0) = k_0$ are sufficient to cause the equations (2.3) to determine a unique thermodynamic process $s(t)$, for $t \in [0, T_s]$, $0 < T_s \leq T_\tau$ if the functions a_i, b_i, c_i, d_i are good enough.

Further on, we assume that the functions a_i, b_i, c_i, d_i are given in such manner that any fixed trajectory and any fixed initial states determine a unique thermodynamic process. It starts from the initial state $s_0 = (\varepsilon_0, \theta_0, g_0, \sigma_0, \psi_0, \eta_0, k_0)$, with $\varepsilon_0 = \varepsilon(0), \theta_0 = \theta(0), g_0 = g(0)$ and is generated by the trajectory $\tau(t)$.

Next, we will attempt to determine the restrictions that must be satisfied by the functions a_i, b_i, c_i, d_i in a manner to enable *Clausius-Duham* inequality

$$\dot{\psi} - \frac{1}{\rho_0}\sigma\dot{\varepsilon} + \eta\dot{\theta} + kg \leq 0 \tag{2.4}$$

to be fulfilled.

Theorem 2.1. *The constitutive equations* (2.3) *fulfil the inequality* (2.4) *for any process starting from a state $s_0 \in A$ and having a certain positive duration T_s if and only if the relations hold:*

$$\rho_0 a_1(s) = \sigma,\ a_2(s) = -\eta,\ a_3(s) = 0,\ \forall s \in A, \tag{2.5}$$

and

$$a_4(s) + kg \leq 0. \tag{2.6}$$

Proof. ($\Rightarrow$) We introduce the first equality (2.3) in the relation (2.4) and we select $s \in A$ in an arbitrary fashion but we set it. As $\dot{\varepsilon}, \dot{\theta}, \dot{g}$ can be arbitrarily selected, one can find a process of C^{01}, of a positive duration, to satisfy (2.3). Then, the relations (2.5) and (2.6) are immediately inferred.

($\Leftarrow$) Conversely, if the functions a_i are selected in a manner to satisfy the relations (2.5) and (2.6) the inequality (2.4) will take place. Jet, from the inequality (2.6) one cannot infer that $a_4(s) \leq 0$ and $kg \leq 0$. Nevertheless we have

$$a_4(\varepsilon, \theta, 0; \sigma, \psi, \eta, k) = 0 \text{ şi } a_4(\varepsilon, \theta, g; \sigma, \eta, 0) = 0. \tag{2.7}$$

□

Definition 2.5. *A state $s_0 = (\varepsilon_0, \theta_0, g_0 = 0; \sigma_0, \psi_0, \eta_0, k_0)$ is called* relaxed *(with respect to energy) if*

$$a_4(s_0) = 0. \tag{2.8}$$

Let $s_0 = (\varepsilon_0, \theta_0, g_0 = 0; \sigma_0, \psi_0, \eta_0, k_0)$ *and* $s_1 = (\varepsilon_1, \theta_1, g_1 = 0; \sigma_1, \psi_1, \eta_1, k_1)$ *be two states. We will say that* s_0 *is* **attracted** *by* s_1 *in energy, if*

$$\varepsilon_0 = \varepsilon_1, \ \theta_0 = \theta_1 \tag{2.9}$$

and if the admissible process $s(t)$ *that starts from state* s_0 *and is generated by the trajectory*

$$\tau(t) = (\varepsilon_0, \theta_0, g_0 = 0), \ \forall t \geq 0 \tag{2.10}$$

satisfies the equality

$$\psi(s(t)) = \psi_1, \tag{2.11}$$

for any $t \geq t_1 > 0$ *(*t_1 *being either finite or infinite).*

Theorem 2.2. *Let* s_0 *be attracted by* s_1 *in energy. Then*

$$\psi_1 \leq \psi_0 \tag{2.12}$$

Proof. By using $(2.7)_1$ and $(2.3)_1$, and by taking into account the fact that $\dot{\tau}(t) = 0$, $\forall t \geq 0$ (2.12) will result. □

Definition 2.6. *A set* $\Sigma \subset \mathbb{R}^7$ *is called* **regular energy relaxation surface** *if it satisfies the properties:*

(R_1) $\forall s = (\varepsilon, \theta, g = 0; \sigma, \psi, \eta, k) \in \Sigma$ *is a relaxed state;*

(R_2) *between the projection* D *of the domain* A *in the plane* (ε, θ) *and the surface* Σ *there is a one-to-one correspondence* $s_R : D \to \Sigma$ *defined by*

$$s = s_R(\varepsilon, \theta) := (\varepsilon, \theta, g = o; \sigma_R(\varepsilon, \theta), \psi_R(\varepsilon, \theta), \eta_R(\varepsilon.\theta), k_R(\varepsilon, \theta)); \tag{2.13}$$

(R_3) $\psi_R \in C^1$,

The direction $u = (u_1, u_2) \in \mathbb{R}^2$ *(with* $\|u\| = 1$*) is called* **relaxation direction** *in* $s_0 \in \Sigma$ *if there is an admissible process* $s(t)$ *such that*

$$s(0) = s_0, \ \dot{\tau}(0) = (u, 0) \tag{2.14}$$

for a t sufficiently low, $s(t)$ *is attracted (in energy) to* $s_R(\varepsilon(t), \theta(t))$.

Theorem 2.3. *Let* Σ *be a regular relaxation surface and u the relaxation direction. Then*

$$< \nabla\psi_R(\varepsilon_0, \theta_0) - f_R(\varepsilon_0, \theta_0), u > \leq 0. \tag{2.15}$$

If any direction is a relaxation direction, then

$$\nabla\psi_R(\varepsilon_0, \theta_0) = f_R(\varepsilon_0, \theta_0), \tag{2.16}$$

where

$$\begin{cases} \nabla\psi_R(\varepsilon_0, \theta_0) = \left(\dfrac{\partial\psi_R}{\partial\varepsilon}(\varepsilon_0, \theta_0), \dfrac{\partial\psi_R}{\partial\theta}(\varepsilon_0, \theta_0)\right), \\ f_R(\varepsilon_0, \theta_0) = \left(\dfrac{1}{\rho_0}\sigma_R(\varepsilon_0, \theta_0), -\eta_R(\varepsilon_0, \theta_0)\right) \end{cases} \tag{2.17}$$

Proof. Let $s(t)$ be the admissible process with the properties required by the proposition containing (2.14). Let

$$I = \nabla\psi_R(\varepsilon_0,\theta_0)w - f_R(\varepsilon_0,\theta_0)w = \lim_{t\searrow 0}\frac{\psi_R(\varepsilon(t),\theta(t)) - \psi(\varepsilon_0,\theta_0)}{t} - f_R(\varepsilon_0,\theta_0)\big(\dot\varepsilon(0),\dot\theta(0)\big).$$

Since s_0 is relaxed, $(2.3)_1, (2.5)_{1,2}$ and (2.13) will yield

$$f_R(\varepsilon_0,\theta_0)\big(\dot\varepsilon(0),\dot\theta(0)\big) = \dot\psi(0).$$

Then

$$I = \lim_{t\searrow 0}\left(\frac{\psi_R(\varepsilon(t),\theta(t)) - \psi_R(\varepsilon_0,\theta_0)}{t} - \frac{\psi(t)-\psi(0)}{t}\right)$$

and as $\psi_R(\varepsilon_0,\theta_0) = \psi(0)$, we wil have

$$I = \lim_{t\searrow 0}\frac{\psi_R(\varepsilon(t),\theta(t)) - \psi(t)}{t}.$$

Since $s(t)$ attracted $s_R(\varepsilon(t),\theta(t))$, then the theorem 2.2 yields

$$\psi_R(\varepsilon(t),\theta(t)) \le \psi(t)$$

so we will have $I \le 0$. □

3 Instantaneous Response for Rate Type Materials

1. The One-Dimensional Case

Let us have the system of differential forms

$$\begin{cases} d\psi = \dfrac{1}{\rho_0}\sigma d\varepsilon - \eta d\theta, \\ d\sigma = b_1(s)d\varepsilon + b_2(s)d\theta + b_3(s)dg, \\ d\eta = c_1(s)d\varepsilon + c_2(s)d\theta + c_3(s)dg, \\ dk = d_1(s)d\varepsilon + d_2(s)d\theta + d_3(s)dg. \end{cases} \tag{3.1}$$

Definition 3.1. *The system of differential forms* (3.1) *is* **called completely integrable**[4] *at a point* $s_0 \in A$ *if there is* $U_{s_0} \subset \mathbb{R}^3 \times A$ *a neighbourhood of* $(\varepsilon_0,\theta_0,g_0,s_0)$ *and the functions* $\psi_I, \sigma_I, \eta_I, k_I : U_{s_0} \to \mathbb{R}$,

$$\psi = \psi_I(\varepsilon,\theta,g,s_0),\ \sigma = \sigma_I(\varepsilon,\theta,g,s_0),\ \eta = \eta(\varepsilon,\theta,g,s_0),\ k = k_I(\varepsilon,\theta,g,s_0), \tag{3.2}$$

that verify the system (3.1) *for any curve* $\big(\varepsilon(t),\theta(t),g(t)\big)$ *of class* C^{01} *with* $t \ge 0$ *and sufficiently close to zero,* $\varepsilon(0) = \varepsilon_0, \theta(0) = \theta_0, g(0) = g_0$. *Furthermore* $\psi_I(\varepsilon_0,\theta_0,g_0,s_0) = \psi_0, \sigma_I(\varepsilon_0,\theta_0,g_0,s_0) = \theta_0, \eta_I(\varepsilon_0,\theta_0,g_0,s_0) = \eta_0, k_I(\varepsilon_0,\theta_0,g_0,s_0) = k_0$.

---

[4]see Hartman [8]

Definition 3.2. *We will say that the material possesses an* **instantaneous response to the state** $s_0 \in A$ *if the system of differential forms* (3.1) *is completely integrable.*

We will say that material possesses an instantaneous response at A *if it possesses an instantaneous response starting from any state* $s_0 \in A$.

Theorem 3.1. *If the functions* b_i, c_i, d_i, $i = \overline{1,3}$ *are of class* C^1 *on* A, *then the required and sufficient condition for the material of rate type behaviour to posses an instantaneous response to any state* $s_0 \in A$ *is that the following equalities be fulfilled:*

$$\frac{\partial b_1}{\partial g} + \frac{\partial b_1}{\partial k} d_3 = 0,\ \frac{\partial b_2}{\partial g} + \frac{\partial b_2}{\partial k} d_3 = 0,\ \frac{\partial c_1}{\partial g} + \frac{\partial c_1}{\partial k} d_3 = 0,\ \frac{\partial c_2}{\partial g} + \frac{\partial c_2}{\partial k} d_3 = 0; \tag{3.4}$$

$$\begin{cases} \dfrac{\partial b_1}{\partial \theta} + b_2 \dfrac{\partial b_1}{\partial \sigma} - \eta \dfrac{\partial b_1}{\partial \psi} + c_2 \dfrac{\partial b_1}{\partial \eta} + d_2 \dfrac{\partial b_1}{\partial k} = \dfrac{\partial b_2}{\partial \varepsilon} + b_1 \dfrac{\partial b_2}{\partial \sigma} + \sigma \dfrac{\partial b_2}{\partial \psi} + c_1 \dfrac{\partial b_2}{\partial \eta} + d_1 \dfrac{\partial b_2}{\partial k}, \\ \dfrac{\partial c_1}{\partial \theta} + b_2 \dfrac{\partial c_1}{\partial \sigma} - \eta \dfrac{\partial c_1}{\partial \psi} + c_2 \dfrac{\partial c_1}{\partial \eta} + d_2 \dfrac{\partial c_1}{\partial k} = \dfrac{\partial c_2}{\partial \varepsilon} + b_1 \dfrac{\partial c_2}{\partial \sigma} + \sigma \dfrac{\partial c_2}{\partial \psi} + c_1 \dfrac{\partial c_2}{\partial \eta} + d_1 \dfrac{\partial c_2}{\partial k}, \\ \dfrac{\partial d_1}{\partial \theta} + b_2 \dfrac{\partial d_1}{\partial \sigma} - \eta \dfrac{\partial d_1}{\partial \psi} + c_2 \dfrac{\partial d_1}{\partial \eta} + d_2 \dfrac{\partial d_1}{\partial k} = \dfrac{\partial d_2}{\partial \varepsilon} + b_1 \dfrac{\partial d_2}{\partial \sigma} + \sigma \dfrac{\partial d_2}{\partial \psi} + c_1 \dfrac{\partial d_2}{\partial \eta} + d_1 \dfrac{\partial d_2}{\partial k}, \end{cases} \tag{3.5}$$

$$\begin{cases} \dfrac{\partial d_1}{\partial g} + d_3 \dfrac{\partial d_1}{\partial k} = \dfrac{\partial d_3}{\partial \varepsilon} + b_1 \dfrac{\partial d_3}{\partial \sigma} + \sigma \dfrac{\partial d_3}{\partial \psi} + c_1 \dfrac{\partial d_3}{\partial \eta} + d_1 \dfrac{\partial d_3}{\partial k}, \\ \dfrac{\partial d_2}{\partial g} + d_3 \dfrac{\partial d_2}{\partial k} = \dfrac{\partial d_3}{\partial \theta} + b_2 \dfrac{\partial d_3}{\partial \sigma} - \eta \dfrac{\partial d_3}{\partial \psi} + c_2 \dfrac{\partial d_3}{\partial \eta} + d_2 \dfrac{\partial d_3}{\partial k}. \end{cases} \tag{3.6}$$

For proof, see *Hartman* [8], Chap. VI, part I.

The equalities $(3.3)_{1,2}$ tell us that in the case the material is endowed with an instantaneous response, the increases of tension and entropy are not dependent on the temperaturte gradient increases, while the equalities (3.4) show us that if $d_3 \neq 0$, then b_1, b_2, c_1, c_2 must or must not depens on g and k at the same time. Also, from (3.4) and (3.6) it can be inferred that if $d_3 = 0$, then b_1, b_2, c_1, c_2 are not depend on g.

Proposition 3.1. *If the conditions of Theorem 3.1 are satisfied, then the following equalities will take place:*

$$\begin{cases} \psi = \psi_I(\varepsilon, \theta, g, s_0), \\ \sigma = \sigma_I(\varepsilon, \theta, g, s_0) = \rho_0 \dfrac{\partial \psi_I}{\partial \varepsilon}(\varepsilon, \theta, g, s_0), \\ \eta = \eta_I(\varepsilon, \theta, g, s_0) = -\dfrac{\partial \psi_I}{\partial \theta}(\varepsilon, \theta, g, s_0), \\ b_1(\varepsilon, \theta, g, \sigma_I, \psi_I, \eta_I, k_I) = \dfrac{\partial \sigma_I}{\partial \varepsilon}(\varepsilon, \theta, g, s_0) = \rho_0 \dfrac{\partial^2 \psi_I}{\partial \varepsilon^2}(\varepsilon, \theta, g, s_0), \\ b_2(\varepsilon, \theta, g, \sigma_I, \psi_I, \eta_I, k_I) = \dfrac{\partial \sigma_I}{\partial \theta}(\varepsilon, \theta, g, s_0) = \rho_0 \dfrac{\partial^2 \psi_I}{\partial \theta \partial \varepsilon}(\varepsilon, \theta, g, s_0), \\ c_1(\varepsilon, \theta, g, \sigma_I, \psi_I, \eta_I, k_I) = \dfrac{\partial \eta_I}{\partial \varepsilon}(\varepsilon, \theta, g, s_0) = -\dfrac{\partial^2 \psi_I}{\partial \theta \partial \varepsilon}(\varepsilon, \theta, g, s_0), \\ c_2(\varepsilon, \theta, g, \sigma_I, \psi_I, \eta_I, k_I) = \dfrac{\partial \eta_I}{\partial \theta}(\varepsilon, \theta, g, s_0) = -\dfrac{\partial^2 \psi_I}{\partial \theta^2}(\varepsilon, \theta, g, s_0). \end{cases} \tag{3.7}$$

The proof is obvious.

Under the conditions of theorem 3.1, the Jacobian of the application

$$(\sigma_0, \psi_0, \eta_0, k_0) \to (\sigma_I(\varepsilon, \theta, g, s_0), \psi_I(\varepsilon, \theta, g, s_0), \eta_I(\varepsilon, \theta, g, s_0), k_I(\varepsilon, \theta, g, s_0))$$

is not zero for any (ε, θ, g) sufficiently close to $(\varepsilon_0, \theta_0, g_0)$ since for $\varepsilon = \varepsilon_0, \theta = \theta_0, g = g_0)$ this Jacobian is equal to 1.

We denote by B_0 the three-dimensional domain containing the point $(\varepsilon_0, \theta_0, g_0)$ for which the functions $\sigma_I, \psi_I, \eta_I, b_I$ are defined and the above mentioned Jacobian differs from zero.

Proposition 3.2. *If the material with a rate type behaviour possesses an instantaneous response and the conditions of theorem 3.1 are satisfied, then for any state $s_0 \in A$ and any trajectory $\tau = (\varepsilon, \theta, g) : [0, T] \to \mathbb{R}^3$, $\tau \in R^1$ such that $\tau(0) = (\varepsilon_0, \theta_0, g_0)$ and $\tau^+(0) \in B_0$, there is a thermodynamic process of duration $\tau_s > 0, s \in R^1$ satisfying the equations*

$$\begin{cases} \dot{\psi} - \dfrac{1}{\rho_0}\sigma\dot{\varepsilon} + \eta\dot{\theta} = a_4(s), \\ \dot{\sigma} - b_1(s)\dot{\varepsilon} - b_2(s)\dot{\theta} = b_4(s), \\ \dot{\eta} - c_1(s)\dot{\varepsilon} - c_2(s)\dot{\theta} = c_4(s), \\ \dot{k} - d_1(s)\dot{\varepsilon} - d_2(s)\dot{\theta} - d_3(s)\dot{g} = d_4(s), \end{cases} \tag{3.8}$$

for $t \in [0, T_s]$ except a set at mosty numerable of points, and at the discontinuity points, the equations are satisfied by the lateral limits of the functions and their derivatives.

Proof. We will use the method of constants' variation (see the case when the thermodynamic influence is neglijable). Let's consider the functions:

$$\begin{cases} \psi(t) = \psi_I(\varepsilon(t), \theta(t), \varepsilon_0, \theta_0, g_0, \bar{\sigma}, \bar{\psi}, \bar{\eta}, \bar{k}), \\ \sigma(t) = \sigma_I(\varepsilon(t), \theta(t), \varepsilon_0, \theta_0, g_0, \bar{\sigma}, \bar{\psi}, \bar{\eta}, \bar{k}), \\ \eta(t) = \eta_I(\varepsilon(t), \theta(t), \varepsilon_0, \theta_0, g_0, \bar{\sigma}, \bar{\psi}, \bar{\eta}, \bar{k}), \\ k(t) = k_I(\varepsilon(t), \theta(t), g(t), \varepsilon_0, \theta_0, g_0, \bar{\sigma}, \bar{\psi}, \bar{\eta}, \bar{k}), \end{cases} \tag{3.9}$$

verifying the equations (3.8) without the terms a_4, b_4, c_4, d_4 (that is the homogeneous system corresponding to the system (3.8)) and for $\bar{\sigma}, \bar{\psi}, \bar{\eta}, \bar{k}$ being constant. Then, we will consider $\bar{\sigma}, \bar{\psi}, \bar{\eta}, \bar{k}$ as functions of t and we will determine them so that equations (3.8) be verified when there is a straight member (i.e. we determine a particular solution to the non homogeneous system (3.8)). Those solutions will be determined as functions of class C^{01}. They are solutions for the next system and initial conditions to it:

$$
\begin{cases}
\dfrac{\partial \psi_I}{\partial \sigma_0}\dot{\bar{\sigma}} + \dfrac{\partial \psi_I}{\partial \psi_0}\dot{\bar{\psi}} + \dfrac{\partial \psi_I}{\partial \eta}\dot{\bar{\eta}} + \dfrac{\partial \psi_I}{\partial k_0}\dot{\bar{k}} = a_4(\varepsilon, \theta, g, \sigma_I, \psi_I, \eta_I, k_i), \\
\dfrac{\partial \sigma_I}{\partial \sigma_0}\dot{\bar{\sigma}} + \dfrac{\partial \sigma_I}{\partial \psi_0}\dot{\bar{\psi}} + \dfrac{\partial \sigma_I}{\partial \eta}\dot{\bar{\eta}} + \dfrac{\partial \sigma_I}{\partial k_0}\dot{\bar{k}} = b_4(\varepsilon, \theta, g, \sigma_I, \psi_I, \eta_I, k_i), \\
\dfrac{\partial \eta_I}{\partial \sigma_0}\dot{\bar{\sigma}} + \dfrac{\partial \eta_I}{\partial \psi_0}\dot{\bar{\psi}} + \dfrac{\partial \eta_I}{\partial \eta}\dot{\bar{\eta}} + \dfrac{\partial \eta_I}{\partial k_0}\dot{\bar{k}} = c_4(\varepsilon, \theta, g, \sigma_I, \psi_I, \eta_I, k_i), \\
\dfrac{\partial k_I}{\partial \sigma_0}\dot{\bar{\sigma}} + \dfrac{\partial k_I}{\partial \psi_0}\dot{\bar{\psi}} + \dfrac{\partial k_I}{\partial \eta}\dot{\bar{\eta}} + \dfrac{\partial k_I}{\partial k_0}\dot{\bar{k}} = d_4(\varepsilon, \theta, g, \sigma_I, \psi_I, \eta_I, k_i), \\
\sigma(0) = \sigma_I(\varepsilon \ldots
\end{cases}
\tag{3.10}
$$

it:
$$\begin{cases} \dfrac{\partial\psi_I}{\partial\sigma_0}\dot{\bar{\sigma}} + \dfrac{\partial\psi_I}{\partial\psi_0}\dot{\bar{\psi}} + \dfrac{\partial\psi_I}{\partial\eta}\dot{\bar{\eta}} + \dfrac{\partial\psi_I}{\partial k_0}\dot{\bar{k}} = a_4(\varepsilon,\theta,g,\sigma_I,\psi_I,\eta_I,k_i), \\ \dfrac{\partial\sigma_I}{\partial\sigma_0}\dot{\bar{\sigma}} + \dfrac{\partial\sigma_I}{\partial\psi_0}\dot{\bar{\psi}} + \dfrac{\partial\sigma_I}{\partial\eta}\dot{\bar{\eta}} + \dfrac{\partial\sigma_I}{\partial k_0}\dot{\bar{k}} = b_4(\varepsilon,\theta,g,\sigma_I,\psi_I,\eta_I,k_i), \\ \dfrac{\partial\eta_I}{\partial\sigma_0}\dot{\bar{\sigma}} + \dfrac{\partial\eta_I}{\partial\psi_0}\dot{\bar{\psi}} + \dfrac{\partial\eta_I}{\partial\eta}\dot{\bar{\eta}} + \dfrac{\partial\eta_I}{\partial k_0}\dot{\bar{k}} = c_4(\varepsilon,\theta,g,\sigma_I,\psi_I,\eta_I,k_i), \\ \dfrac{\partial k_I}{\partial\sigma_0}\dot{\bar{\sigma}} + \dfrac{\partial k_I}{\partial\psi_0}\dot{\bar{\psi}} + \dfrac{\partial k_I}{\partial\eta}\dot{\bar{\eta}} + \dfrac{\partial k_I}{\partial k_0}\dot{\bar{k}} = d_4(\varepsilon,\theta,g,\sigma_I,\psi_I,\eta_I,k_i), \\ \sigma(0) = \sigma_I(\varepsilon^+(0),\theta^+(0),s_0), \\ \psi(0) = \psi_I(\varepsilon^+(0),\theta^+(0),s_0), \\ \eta(0) = \eta_I(\varepsilon^+(0),\theta^+(0),s_0), \\ k(0) = k_I(\varepsilon^+(0),\theta^+(0),g^+(0),s_0). \end{cases} \tag{3.10}$$

As $\tau^+(0) \in B_0$, the matrix of the coefficients of functions $\dot{\bar{\sigma}}, \dot{\bar{\psi}}, \dot{\bar{\eta}}, \dot{\bar{k}})$ has a determinant different from zero, so there is a unique solution of class C^{01} of this Cauchy problem over an interval $[0, T_s]$. with $0 < T_s \leq T$. □

2. Rate Type Constitutive Equations for the Three-Dimensional Case

Definition 3.3. *We will say that the thermodynamic state s of a particle $\omega \in \Omega$ is known at a moment t if the functions $F, \theta, d, \tilde{S}, \eta, \psi, k = \dfrac{1}{\rho_0\theta}, q$, whose physical significance is given by §1.*

We will write
$$s = \left(F, \theta, g, \tilde{S}, \eta, \psi, k\right) \in \mathcal{S}, \tag{3.11}$$
where $\mathcal{S}$ is the space of all posible thermodynamic states (see (1.14), (1.21), (1.25), (1.27)).

To shorten the writing process, we introduce the notations
$$Y = (F, \theta), \quad f = (\frac{1}{\rho_0}\tilde{S}, -\eta) \tag{3.12}$$
and
$$U = \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3) \times \mathbb{R}.$$
Also, let us consider the functions:
$$\begin{cases} a_1, B_3 : \mathcal{S} \to U, \quad a_2 : \mathcal{S} \to \mathbb{R}^3, \quad a_3 : \mathcal{S} \to \mathbb{R}, \\ B_1 : \mathcal{S} \to \mathcal{L}(U, U), \quad B_2 : \mathcal{S} \to \mathcal{L}(\mathbb{R}^3, U), \\ Q_1 : \mathcal{S} \to \mathcal{L}(U, \mathbb{R}^3), \quad Q_2 : \mathcal{S} \to \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3), \quad Q_3 : \mathcal{S} \to \mathbb{R}^3. \end{cases} \tag{3.13}$$
that we admit as smooth as is necessary to make when proceeding with operations.

Definition 3.4. *We call* **thermodynamic process** *for a particle $\omega \in \Omega$ a function $s : [0, T_s] \to \mathcal{S}$.*

We call **process trajectory** *a curve $\tau = (Y, g) : [0, T_s] \to U \times \mathbb{R}^3 \subset \mathcal{S}, \tau(t) = (Y(t), g(t)),\ t \in [0, T_\tau], T_\tau \geq T_s > 0$.*

Any point $(Y,g) \in V$ can by associated to smooth trajectory $(Y,g) : [0,T_\tau] \to U \times \mathbb{R}^3$, $\big(Y(t),g(t)\big) \in V, \forall t \in [0,T_\tau]$ so that $(Y(0),g(0)) = (Y_0,g_0)$ şi $\big(\dot{Y}(t),\dot{g}(t)\big) = (u,v)$, where (u,v) is an arbitrary vector of $U \times \mathbb{R}^3$.

By writing the equation $(3.18)_1$ at the moment t, we will have:

$$< \nabla_Y \hat{\psi}(Y,g,s_0) - \hat{f}(Y,g,s_0), u > + < \nabla_g(Y,g,s_0), v >= 0,\ \forall(u,v) \in U \times \mathbb{R}^3$$

and that will yield:

$$\begin{cases} \nabla_g \hat{\psi}(Y,g,s_0) = 0, \\ \nabla_Y \hat{\psi}(Y,g,s_0) = \hat{f}(Y,g,s_0). \end{cases} \tag{3.20}$$

From the relation $(3.20)_1$ it can be inferred that the function ψ is independent of g, and from $(3.20)_2$ it follows that also the function f is independent of g.

Using an analogue reasoning, from the relations $(3.18)_{2,3}$ we can get:

$$\begin{cases} \hat{B}_2(Y,g,s_0) = B_2(Y,g,\hat{f},\hat{\psi},\hat{k}) = 0, \\ \hat{B}_1(Y,g,s_0) = B_1(Y,g,\hat{f},\hat{\psi},\hat{k}) = \nabla_Y^2 \hat{\psi}(Y,s_0) \end{cases} \tag{3.21}$$

and

$$\begin{cases} \hat{Q}_2(Y,g,s_0) = Q_2(Y,g,\hat{f},\hat{\psi},\hat{k}) = \nabla_Y \hat{k}(Y,g,s_0), \\ \hat{Q}_1(Y,g,s_0) = Q_1(Y,g,\hat{f},\hat{\psi},\hat{k}) = \nabla_g \hat{k}(Y,g,s_0) \end{cases} \tag{3.22}$$

If in the relation $(3.21)_1$ we make $(Y,g) = (Y_0,g_0)$, by taking into account (3.17), we get

$$B_2(s_o) = 0. \tag{3.23}$$

Having in view that s_0 was arbitrarily selected in $\mathcal{S}$, it follows that for any materials with instantaneous response, the equality (3.23) takes place throughout all $\mathcal{S}$.

Also, from $(3.21)_2$ we can notice that B_1 does not depend on g since it is determined along the hypersurface of the instantaneous response by the function $\nabla_Y^2 \hat{\psi}(Y,s_0)$, and also from$(3.21)_2$, by derivation with respect to g and by taking into account (3.17) and $(3.22)_2$ for $(Y,g) = (Y_0,g_0)$,

$$\nabla_g B_1(s_0) + \{\nabla_k B_1(s_0)\}\{\nabla_g Q_2(s_0)\} = 0, \tag{3.24}$$

will result as an equality valid for any s_0. From (3.24) it can be inferred that B_1 is not dependent on g if and only if the second term of the left member of the equality (3.24) is zero.

The fact that the material possesses or does not possesse an instantaneous response can be described by conditions laid direcly on the coefficients of equation (3.18).

To simplify the description, we will introduce the following notations:

$$Z = \begin{pmatrix} Y \\ g \end{pmatrix}, \quad W = \begin{pmatrix} f \\ \psi \\ k \end{pmatrix}, \quad D = \begin{pmatrix} B_1 & B_2 \\ f & 0 \\ Q_1 & Q_2 \end{pmatrix} \tag{3.25}$$

by which the system is written under the form of a matrix:

$$\dot{W} = D(s) * \dot{Z}, \tag{3.26}$$

where $s = \big(Z^t, W^t\big) \in \mathcal{S}$.

To characterize the instantaneous response, we will provide the following theorem:

Theorem 3.2. *If the matrix D is of class C^1 over $\mathcal{S}$, then a necessary and sufficient condition for the material to possess an instantaneous response is that*

$$\frac{\partial D_{ki}}{\partial Z_j}+\sum_{l=1}^{3}\frac{\partial D_{ki}}{\partial W_l}D_{lj}=\frac{\partial D_{kj}}{\partial Z_i}+\sum_{l=1}^{3}\frac{\partial D_{kj}}{\partial W_l}D_{li}. \tag{3.27}$$

For proof, see *Hartman* [8], Chap. VI, part I.

Remark 3.2. *By using the relations* (3.27) *and* (3.23) *for k=5, we get*

$$B_1^{ij}(s)=B_1^{ji}(s), \tag{3.28}$$

that is the matrix B_1 is symmetic.

The same result can be obtained also directly from (3.21) *by taking into account that B_1 is expressed by means of the partial second order derivatives of the function ψ.*

4 Totally Relaxed States

Definition 4.1. *We will say that a state $s^*=\left(Y^*,g^*=0,f^*,\psi^*,k^*\right)\in\mathcal{S}$ is* **totally relaxed** *if*

$$a_3(s^*)=0,\quad B_3(s^*)=0,\quad Q_3(s^*)=0. \tag{4.1}$$

If we take into account the relations $(4.1)_1)$ and (3.16), it follows in a relaxed state in egergy we have:

$$\frac{\partial a_3}{\partial Y}(s^*)=0,\quad \frac{\partial a_3}{\partial f}(s^*)=0,\quad \frac{\partial a_3}{\partial \psi}(s^*)=0,\quad \frac{\partial a_3}{\partial k}(s^*)=0,\quad \frac{\partial a_3}{\partial g}(s^*)=0, \tag{4.2}$$

which point out the fact that for a relaxed state in energy, we have

$$k^*=0\Leftrightarrow\frac{\partial a_3}{\partial g}(s^*)=0.$$

Definition 4.2. *A regular surface Σ, with the equation $\psi=a(Y,f,k)$, is a* **regular surface of total relaxation** *if:*

$$(R_1)\qquad \begin{cases} a_3(Y,0,f,a(Y,f,k),k)=0,\\ B_3(Y,0,f,a(Y,f,k),k)=0,\\ Q_3(Y,0,f,a(Y,f,k),k)=0 \end{cases} \tag{4.3}$$

and

$$a_3(Y,0,f,\psi,k)\neq 0$$

in a neighbourhood of the surface Σ_R for $\psi\neq a(Y,f,k)$;

(R_2) there is a continuous function $g:\mathbb{R}\to\mathbb{R}^+$ such that

$$g(u)=0\Leftrightarrow u=0 \tag{4.4}$$

in a neighbourhood of the surface Σ *and;*

$$\begin{cases} a_3 - \beta < -g(\psi - a(Y,f,k)) & \text{dacă} \quad \psi > a(Y,f,k), \\ g(\psi - a(Y,f,k)) < a_3 - \beta & \text{dacă} \quad \psi < a(Y,f,k). \end{cases} \tag{4.5}$$

where

$$\beta = \nabla_f a(Y,f;k) B_3(Y,o,f,\psi,k) + \nabla_k a(Y,f,k) Q_3(Y,o,f,\psi,k); \tag{4.6}$$

(R_3) *for any set* $\bar{\psi}$, $a(Y,f,k) = \bar{\psi}$ *is a closed surface in* $U \times U \times \mathbb{R}^3$ *and in a neighbourhood of the surface* Σ_R, *the following relation holds*

$$\beta(\psi,f,\psi,k) < 0 \ \text{ if } \ \psi \neq a(Y,f,k). \tag{4.7}$$

Definition 4.3. *We will say that the state* $s_0 = (Y_0, g_0 = 0, f_0, \psi_0, k_0)$ *is totally attracted by the state* $s_1 = (Y_1, g_1 = 0, f_1, \psi_1, k_1)$ *if*

$$Y_0 = Y_1, \ \ g_0 = g_1 = 0 \tag{4.8}$$

and if the admissible process $s(t)$ *start at* s_0 *(that is* $s(0) = s_0$*) and is generated by the trajectory*

$$\tau(t) = \tau_0 = (Y_0, g_0 = 0), \text{pentru } t \geq 0, \tag{4.9}$$

satisfies the conditions:

$$\psi(t) = \psi_1, \ f(t) = f_1, \ k(t) = k_1, \ \ \text{for} \ \ t \geq t_1 \tag{4.10}$$

wirh $t_1 \geq 0$ *(finite or infinite).*

Theorem 4.1. *Let* Σ_R *be a regular surface of total relaxation, with the equation* $\psi = a(Y,f,k)$, *and let* $\mathfrak{U}$ *be the cylinder with generatices parallel with the axis* ψ *passing throught the surface* Σ_R *having the equation* $\psi_1 = a(Y,f,k)$, *and with lower faces, and the upper faces, respectively, contained within the surfaces with the equations:*

$$\Sigma_- : \psi - a(Y,f,k) = \gamma_-, \ \ \Sigma_+ : \psi - a(Y,f,k) = \gamma_+$$

where γ_- *and* γ_+ *are two numbers in a maner that* $\gamma_- < 0 < \gamma_+$ *(see Figure 1).*

Furthermore, we will admit that:

i) a_3, B_3, Q_3 *and* β *are of class* C^1 *over* $\bar{\mathfrak{U}} \setminus \Sigma_R$ *and continuous over* $\bar{\mathfrak{U}}$;

ii) $\dfrac{1}{g(u)}$ *is Lebesgue integrable over* $[0, \gamma_+]$ *and over* $[\gamma_-, 0]$.

Then, for any state $s_0 \in \mathfrak{U} \setminus \Sigma_R$ *there is a unique state* $s_1 \in \Sigma_R$ *by which* s_0 *is totally attracted. If* $s_0 \in \Sigma_R \cap \mathfrak{U}$ *then the state* s_0 *remains there for any* $t \in]0, \infty[$ *and the time* t_1 *necessary for the state* s_0 *to reach the state* s_1 *is finite.*

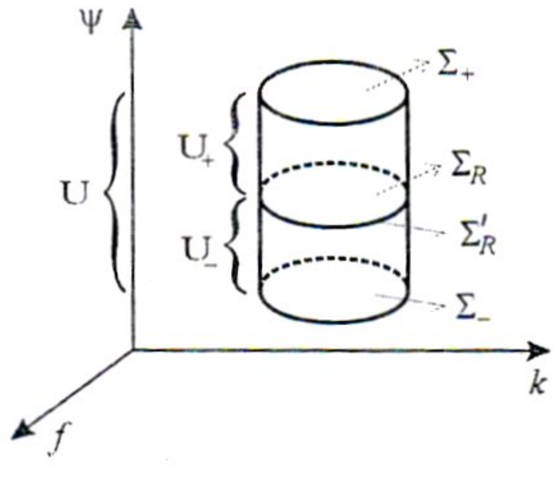

Fig. 1

Proof. Let's admit, first, that $s_0 \in \Sigma \cap \mathcal{U}$. Then the system:

$$\begin{cases} \dot{\psi} = A_3(Y_0, 0, f, \psi, k), & \psi(0) = \psi_0, \\ \dot{f} = B_3(Y_0, 0, f, \psi, k), & f(0) = f_0, \\ \dot{k} = Q_3(Y_0, 0, f, \psi, k), & k(0) = k_0 \end{cases} \tag{4.11}$$

obtained from (3.14), has a unique solution $\psi(t) = \psi_0, f(t) = f_0, k(t) = k_0$ for any $t \geq 0$. Let's suppose that there is another solution $\psi = \psi_1(t), f = f_1(t), k = k_1(t)$ that verifies the initial conditions $\psi_1(0) = \psi_0, f_1(0) = f_0, k_1(0) = k_0$. Having in view that hypothesis (R_2) of defintion 4.2, with $u(t) = \psi_1(t) - a(Y_0, f_1(t), k_1(t))$, we obtain:

$$\dot{u}(t) \leq -g(u(t)) \text{ dacă } u(t) > 0, \;\; \dot{u}(t) \geq -g(u(t)) \text{ dacă } u(t) \leq 0, \;\; u(0) = 0.$$

Let's suppose that there is $t_0 > 0$ in a manner that $u(t_0) > 0$. Then, as $u(t)$ is a continuous function, there is an interval $]t_0 - \varepsilon, t_0 + \varepsilon[$, $\varepsilon > 0$, with the property that $u(t) > 0, \forall t \in]t_0 - \varepsilon, t_0 + \varepsilon[$, while $u(t_0 - \varepsilon) = 0$. But as $u(t) < 0 \Rightarrow 0 < u(t_0) \leq u(t)$, $\forall t \in]t_0 - \varepsilon, t_0[$ so it can be inferred that $u(t_0) = 0$. Thus, the new solution must satisfy the condition $\psi_1(t) = a(Y_0, f_1(t), k_1(t))$. Taking into account the relation (R_1), (4.3), we will get $\dot{\psi}_1(t) = 0, \dot{f}_1(t) = 0, \dot{k}_1(t) = 0$ thus the state s_0 remains there, $\forall t \in]0, \infty[$.

Now, we'll consider the case $s_0 \in \mathcal{U} \setminus \Sigma_R$. Then, the system (4.11) has the unique solution $(f(t), \psi(t), k(t))$ with $f(0) = f_0, \psi(0) = \psi_0, k(0) = k_0$ and the solution remains in $\mathcal{U}_+ \cup \Sigma_R$, $\forall t \in]0, \infty[$. The existence and uniqueness are ensured by the hypothesis i). If the solution was not to remain in $\mathcal{U}_+ \cup \Sigma_R$ that would mean that there is a $t_0 > 0$ so that $(Y_0, 0, f(t_0), \psi(t_0), k(t_0)) \in \partial\mathcal{U}_+ \setminus \Sigma_R$ and $(Y_0, 0, f(t), \psi(t), k(t)) \in \mathcal{U}_+$, $\forall t \in [0, t_0[$. But, in account the condition (R_2) following that $(Y_0, 0, f(t_0), \psi(t_0), k(t_0))$ cannot be on the face Σ_+ because
$\dot{\psi}(t_0) - < \nabla_f a(Y_0, f(t_0), k(t_0)), \dot{f}(t_0) > - < \nabla_k a(Y_0, f(t_0), k(t_0)), \dot{k}(t_0) >= a_3(t_0) - \beta(y_0) < -g(\psi(t_0) - a(Y_0, f(t_0), k(t_0)) < 0$,
which means that the angle between the tangent to the solution curve and the normal to Σ_+ is greater than $\pi/2$. From the condition (R_3) it follows that this point can neither be on the lateral face of $\mathcal{U}_+$ because, there we have:

$<\nabla_f a(Y_0, f(t_0), k(t_0)), \dot{f}(t_0)> + <\nabla_k a(Y_0, f(t_0), k(t_0), \dot{k}(y_0)> = <\nabla_f a(Y_0, f(t_0), k(t_0)), B_3(Y_0, 0, f(t_0), \psi(t_0), k(t_0))> + <\nabla_k a(Y_0, f(t_0), k(t_0)), Q_3(Y_0, 0, f(t_0), \psi(t_0), k(t_0))> = = b(t_0) < 0$. Now our task is to show that $\lim_{t\to\infty} \big(Y_0, 0, f(t), \psi(t), k(t)\big)$ exists as a point on $\Sigma_R \cap \mathfrak{U}$. Having in view the condition (R_2), we can write:

$$(*) \qquad \begin{cases} \dot{u}(t) \le -g(u(t)), \\ u(t) = \psi(t) - a(Y_0, f(t), k(t)), \\ u(0) = u_0 = \psi_0 - a(Y_0, f_0, k_0) > 0, \end{cases}$$

which implies $\lim_{t\to\infty} \dot{u}(t) = 0$ and $\lim_{t\to\infty} u(t) = 0$. From $(3.16)_3$ and (R_1) we have $a_3(Y_0, 0, f, \psi, k) < 0$, so from $(4.11)_1$ it will result that ψ is decreasing and as it is bounded, it follows that $\lim_{t\to\infty} \psi(t) = \psi_1$ with $\psi_1 = \lim_{t\to\infty} a(Y_0, f(t), k(t))$.

In order to prove that $\lim_{t\to\infty} f(t)$ and $\lim_{t\to\infty} k(t)$ do exist, we will first establish that the time necessary for the state s_0 to reach Σ is finite. Indeed, by using the relations $(*)$ and the hypothesis *ii*), we get:

$$(**) \qquad t_1 \le \int_0^{u_0} \frac{1}{g(u)} du < \infty.$$

Therefore, from the uniqueness of the solution, we find:

$$\begin{cases} f(t) = f(t_1) = f_1, \\ k(t) = k(t_1) = k_1, \end{cases} \quad \text{cu } t \ge t_1$$

and

$$\psi_1 = a(Y_0, f_1, k_1).$$

The case when $s_0 \in \mathfrak{U}_-$ is dealt with in an analogue manner. □

Remark 4.1. **1°** *The final state* $\big(Y_0, g_0, f_1, \psi_1, k_1\big)$ *is not necessarily determined by* Y_0*, that is, as a general rule, we don't have* $f_1 = f_R(Y_0), \psi_1 = \psi_R(Y_0), k_1 = k_R(Y_0)$*, which means that the material is not necessarily semielastic to effect of Noll (see* [2]*).*

2° *If the function* g *from the relation* (R_2) *is zero for* $u \le 0$ *and if* $a_3 = 0$, $B_3 = 0$, $Q_3 = 0$ *for* $\psi \le a(Y, f, k)$ *and if the material possesses an instantaneous response, then we obtain, as a particular case, the constitutive equations for the elastc-viscoplastic materials presented by Perzyna and Wojno in* [12] *and by Kestin and Rice in* [13].

5 Numerical Examples

1° The constitutive equation $\sigma = f(\varepsilon)$ will be deemed to have the form:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \chi\Big((f^{-1})'(\sigma) - \frac{1}{E}\Big)\dot{\sigma} = \Big(\frac{1}{E} + \Phi(\sigma)\Big)\dot{\sigma} + \psi(\sigma, \varepsilon), \tag{5.1}$$

where

$$\chi(\sigma_m) = \begin{cases} 0, & \text{dacă} \quad \sigma \le \sigma_Y \ \text{ sau } \ \sigma_Y \le \sigma < \sigma_m, \\ 1, & \text{dacă} \quad \sigma = \sigma_m, \end{cases} \tag{5.2}$$

with $\sigma_m = \sigma_m(X,t) = \max\limits_{0\leq s\leq t} \sigma(X,s)$, while $\sigma_Y > 0$ is the plastic limit at tension and it represent a constant of the material, determinated by standardised experiences (if $\sigma = \sigma_m$ we say that we have a *loading*, while if $\sigma \leq \sigma_Y$ or $\sigma < \sigma_m$ we have a *relief*).

The movement $(X,t) \xrightarrow{u} u(X,t)$ is defined by the relation $x(X,t) = X + u(X,t)$.

We will denote by V_1 the region (set) of the characteristic plane where the quasi-linear equation of the hyperbolic type takes effect:

$$\frac{\partial^2 u}{\partial t^2} = c^2(\varepsilon)\frac{\partial^2 u}{\partial X^2},$$

with $c^2(\varepsilon) = \dfrac{1}{\rho_0}\dfrac{d\sigma}{dX}(X,t)$ (for loading) and we will denote by V_2 that region (set) where the movement is characterised by the equation:

$$\frac{\partial^2 u}{\partial t^2} = c_0^2(\varepsilon)\frac{\partial^2 u}{\partial X^2} + \frac{d\sigma_m}{dX}(X,t) - c_0^2\frac{d\varepsilon_m}{dX}(X,t),$$

(for relief) $(\varepsilon_m(X,t) = \max\limits_{s\in[t_0,t]} \varepsilon(X,s))$.

The frontier (boundary) separating the two regions, notated $V_1|V_2$, will be designated as *loading-unloading (relief) frontier (boundary)*. It depends on the initial limit conditions, but also on the material properties (when tensions begin to decrease for the first time, the *Hooke's* law will be applied: $\sigma = \sigma_m + E(\varepsilon - \varepsilon_m)$ while for loading $\sigma = f(\varepsilon)$.

Since at the time of starting of the relieving process, the deformation reaches its relative maximum $\varepsilon_m(C,t)$, the boundary $V_1|V_2$ can be defined as the curve of the specific plane having the property that along it the deformation reaches its maximum at every point X. In principle, the boundary $V_1|V_2$ is obtained from the condition that the solution from V_1 and from V_2 be compatible with the limit conditions and with the continuity condition for σ, ε and u when crossing the boundary.

Once the boundary $V_1|V_2$ is determinated, the solution for the loading and also for unloading domain is obtained.

When carrying out a tension experiment (bending or torsion, etc.) a typical diagram is given by Figure 2.

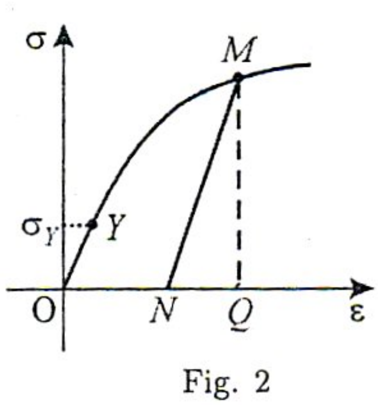

Fig. 2

Torsion is represented on the ordinate, while deformation is represented on the abscissa.

For small deformations the diagram is virtually linear and the point Y marks the end of of this portion.

If the experiment is carried further on and a continuous loading is applied, for $\sigma > \sigma_Y$ we will get, as a general rule, a non-linear diagram $\sigma = f(\varepsilon)$:

- if f is strictly increasing we will say that the material is *cold-hardenable*;

- is starting with $\sigma = \sigma_Y$ over a certain portion of the diagram we have $\sigma = \sigma_Y = constant$, we will say that the material behaves in a *perfectly plastic* manner.

If, during the unloading experiment over a sufficiently large domain of deformation, the behaviour of the material can also be modelled using the same non-linear equation $\sigma = f(\varepsilon)$, we say that the material is *non-linear elastic*.

To conclude, for the loading domains, the non-linear relation $\sigma = f(\varepsilon)$ will be used, while for unloading domain, *Hooke's* relation $\sigma = \sigma_m + E(\varepsilon - \varepsilon_M)$ will be utilised, so upon crossing the boundary $V_1|V_2$, the constitutive law will change.

We can also notice that within the domains V_2, the characteristics appear as straight lines, while within the domains V_1, the characteristics are, as a general rule, represented by curves, which rather tedious the problem of numerical solutions; the problem can be rendered less intricate by using the formula (5.1). Bearing in mind the form (5.1) of the constitutive equation, the equations of the characteristics, both in V_1 and in V_2, will be written under the form:

$$\dot{X} = \pm c = \pm\sqrt{\frac{E}{\rho_0(1+E\Phi)}};\quad dX = 0, \tag{5.3}$$

(where $\dot{X} = \frac{dX}{dt}$), while along them we will have

$$d\sigma = \pm\rho_0 c dv;\quad d\varepsilon^p = \Phi d\sigma;\quad E d\varepsilon^E = d\sigma. \tag{5.4}$$

For $\psi(\sigma,\varepsilon)$ of (5.1) let's admit that:

$$\psi(\sigma,\varepsilon) = \begin{cases} \dfrac{k(\varepsilon)}{E}[\sigma - f(\varepsilon)], & \text{for} \quad \sigma > f(\varepsilon) \text{ şi } \varepsilon \geq \dfrac{\sigma}{E}, \\ 0, & \text{for} \quad \sigma \leq f(\varepsilon), \end{cases} \tag{5.5}$$

and for the relaxation boundaru (frontier) we'll take

$$f(\varepsilon) = \begin{cases} \sigma_Y, & \text{if} \quad \varepsilon \leq \varepsilon_Y, \\ \beta(\varepsilon + \varepsilon_0)^{1/\alpha}, & \text{if} \quad \varepsilon > \varepsilon_Y. \end{cases} \tag{5.6}$$

Sometimes, we can use

$$f(\varepsilon) = \begin{cases} \sigma_Y, & \text{if} \quad \varepsilon < \varepsilon_Y, \\ \sigma_Y + \dfrac{\beta}{2}\varepsilon_z^{-1/2}(\varepsilon - \varepsilon_Y), & \text{if} \quad \varepsilon \in [\varepsilon_Y, \varepsilon_z], \\ \beta\varepsilon^{1/\alpha}, & \text{if} \quad \varepsilon > \varepsilon_z, \end{cases} \tag{5.7}$$

with

$$\varepsilon_z = \left(\frac{\beta\varepsilon_Y}{\sigma_Y - \sqrt{\sigma_Y^2 - \varepsilon_Y\cdot\beta^2}}\right)^2. \tag{5.8}$$

The coefficient $k(\varepsilon) \geq 0$ is taken, by a first approximation, as a constant of the form:

$$k(\varepsilon) = \begin{cases} 0, & \text{if } \ \varepsilon < \varepsilon_0, \\ k_1, & \text{if } \ \varepsilon \in [\varepsilon_0, \varepsilon_1], \\ k_2 + \dfrac{k_2 - k_1}{\varepsilon_2 - \varepsilon_1}(\varepsilon - \varepsilon_2), & \text{if } \ \varepsilon \in]\varepsilon_1, \varepsilon_2], \\ k_2, & \text{if } \ \varepsilon > \varepsilon_2, \end{cases} \tag{5.9}$$

where $0 < k_1 < k_2$, $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2$.

For example, we take $\varepsilon_0 = \varepsilon_Y$, $k_1 = 10^5, \varepsilon_Y = 0,0001471$, $\varepsilon_1 = 0,0005$, $\varepsilon_2 = 0,004$, $k_1 = 10^6$ (see [14]).

The experiment was done for the case of symmetric and longitudinal striking of two identical bars of which one is at rest and the other strikes a speed $v = 14,98 m/s, D = 2,5 cm$ being the bar diameter, and $l = 10D$ being its length.

We will assume that the bar for which the calculations are being mode is, at the initial moment, at rest and not deformed. Thus $t = 0$, $X \in [0, l]$, $\sigma = \varepsilon = v = 0$ are the initial conditions. The end $X = l$ is assumed as being free and thus $X = l$, $t \geq 0, \sigma = 0$ aree the limit conditions.

At the end $X = 0$ the bar receives a blow at a moment $t = 0$ from another identical bar, moving at the initial speed V. For the second bar, the initial conditions are $t = 0$, $-l \leq X \leq 0$, $\sigma = \varepsilon = 0$, $v = V$.

It is admitted that the striking speed V is transmitted to this bar in a very fast and continuous way, so we will select a short time initial $t \in [0, t_m]$ $(t_m = 0, 5\mu s)$ where the unvaried increase of the speed at the end of the bar will take place (for example, a linear increase), from zero to $v_{max} = \dfrac{V}{2}$. So the limit conditions are:

$$X = 0; \begin{cases} 0 \leq t \leq t_m : & v = \dfrac{t}{t_m} v_{max}, \\ t_m < t < T_c : & v_H = v_s, \\ t \geq T_c : & \sigma_s = 0, \end{cases}$$

where T_c is a calculated magnitude designed as *the time of contact* ($T_c \in \{301, 5; 303; 306, 5; 310, 8\}$).

Within the interval $t \in [t_m, T_c]$ the two bars move together. The speed of the particles at $X = 0$ in the two bars being the same. This condition will be applied as long as $\sigma(0, t) > 0$. The minimum time T_c, when for the first time $\sigma(0, t) = 0$, is the contact time. For $t \geq T_c$ the end $X = 0$ becomes free of tension. Since the initial conditions prescribe at $X = 0, t = 0$ distinct values for v this discontinuity will be propagated within the bar under the form of a shock wave.

In the case of constitutive laws (5.1). where for $\sigma \leq \sigma_Y$, the former will be linear $\sigma = E\varepsilon$ an elastic compression shock wave will be propagated within the bar with respect with the speed c_o. At $t = \dfrac{l}{c_o}$ it is being reflected at the end $X = l$ and it becomes a tension shock wave (OA). During propagation this tension shock wave will be gradually absorbed by the plastic compression wave being propagated in the reverse way (see Figure 3). Let X_1 be the bar section where the tension shock wave (and thus the relief wave) will be completely absorbed, that is, this wave will be propagated in the portion $X_A < X \leq l$ of the bar. Hence, on this portion, the straight line

$$(*) \qquad X = -c_0 \left(t - \frac{2l}{c_0} \right)$$

is the front itself of the reflected shock wave.

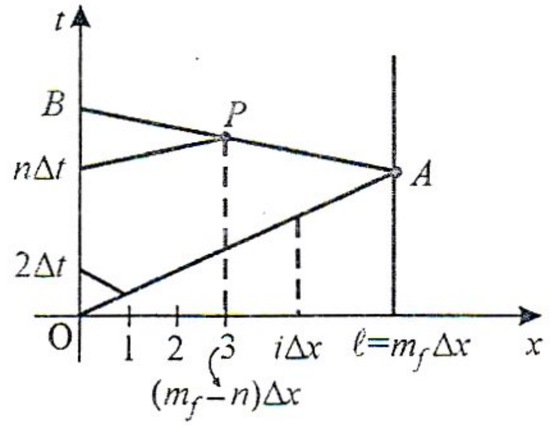

Fig. 3

So, crossing the front of the shock wave (i.e. of the straight line (*)), σ, v and ε will assume a leap. In order to calculate this leap, we first calculate the solution of the problem in the domain beneath the straight line (*). From the equation of the positive slope characteristics $X = c(\sigma)t$ and from $dv = \dfrac{d\sigma}{\rho c(\sigma)}$, we will get

$$v = \int_0^{\sigma} \frac{d\tau}{\rho_0 c(\tau)},$$

while from (*) we can obtain the solution from the domain of loading along the straight line (*). Thus, if a constitutive equation of the type

$$\begin{cases} \sigma = E\varepsilon, & \sigma \leq \sigma_Y, \\ \sigma = \beta(\varepsilon + \varepsilon_0), & \sigma > \sigma_Y, \end{cases} \tag{5.10}$$

with $\varepsilon = 0$ being used, we get:

$$\begin{cases} \sigma_b = \dfrac{\beta^2}{2\rho_0 c_0^2} \cdot \dfrac{t^2}{\left(\dfrac{2l}{c_0} - t \right)^2}, \\ v_b = \dfrac{2}{3\beta} \sqrt{\sigma^3} \sqrt{\dfrac{2}{\rho_0}} + \left[\dfrac{1}{\rho_0 c_0} - \dfrac{2}{3\beta} \sqrt{\sigma_Y} \sqrt{\dfrac{2}{\rho_0}} \right] \cdot \sigma_Y. \end{cases} \tag{5.11}$$

The solution along the straight line (*), but within the unloading domain, is obtained by utilising the leap conditions $(6_1, 6_2, 6_3)$ and (6_4) (§1) where $U = -c_0$. We get (see [14]):

$$\begin{cases} \sigma_a = \sigma_b - \dfrac{1}{2}[2\sigma_Y - (\rho_0 c_0 v_b - \sigma_b)], \\ v_a = v_b - \dfrac{1}{\rho_0 c_0}(\sigma_a - \sigma_b), \\ \varepsilon_a^E = \dfrac{\sigma_a}{E}, \\ \varepsilon_a^P = \varepsilon_b^P. \end{cases} \tag{5.12}$$

Remark 5.1. *The index "a" designates the values of all magnitudes in the boundary domain, while the index "b" those of the loading domain.*

Within these equalities, it is necessary to know also the value of one of the functions σ_a, v_a and ε_a^E at a point of the straight line $(*)$. Since immediately after reflection at $X = l$ and at $t = t_T = \dfrac{l}{c_0}$ we have $\sigma_a = 0$, the equalities (5.12) will provide σ_a, v_a and ε_a^E along $(*)$ as along $\sigma_a \neq \sigma_b$; the calculations are to be made in succesive points along $(*)$, starting from $X = l$ and $t = \dfrac{l}{c_0}$. In this manner, we can also get the point (noted A) on the straight line $(*)$ where the unloading (relieving) wave is completely absorbed by the direct plastic waves.

2. The case of shock wave propagation throught a material satisfying a quasi-linear constitutive equation of the form

$$\dot{\sigma} = \varphi(\sigma, \varepsilon)\dot{\varepsilon} + \psi(\sigma, \varepsilon), \tag{5.13}$$

that can be written in a functional form

$$\sigma(t) = f(\varepsilon(t), \tau(t)), \tag{5.14}$$

where $\tau(t)$ the *history parameter*, that is, it depends on the history of the deformation over $[a, b]$ for a fixed X, while f is being defined by the equation

$$\frac{\partial f}{\partial \varepsilon}(\varepsilon, \tau) = \varphi(f(\varepsilon, \tau), \varepsilon). \tag{5.15}$$

The equation curve $\sigma = f(\varepsilon, 0)$ of the plane (ε, σ) is *the instantaneous curve* with respect to the natural condition of rest, being defined by the equation

$$\begin{cases} \dfrac{\partial f}{\partial \varepsilon}(\varepsilon, 0) = \varphi(f(\varepsilon, 0), \varepsilon), \\ f(0, 0) = 0. \end{cases} \tag{5.16}$$

Thus, the solution $\sigma = f(\varepsilon, 0)$ of the equation (5.16) could stand for the following three situations:

(*i*) coincide with the *Hooke's* straight line, that is $\sigma = E\varepsilon$;
(*ii*) be a convex curve;
(*iii*) be a concave curve (see Figure 4).

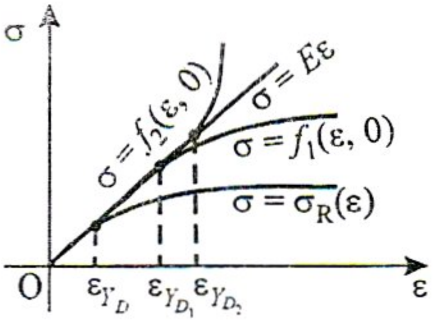

Fig. 4

We will consider as posible any of the three situations and we will suppose that for all those situations there is an interval $[0, \varepsilon_{Y_0}]$ where f is linear, ε_{Y_0} is the deformation corresponding to a certain dynamic limit of non-linearity. Also, in the same figure, there is a representation of the relaxation curve, $\sigma = \sigma_R(\varepsilon)$ (see Figure 4).

In order to describe the propagation of the shock wave, we use the leap relations

$$\begin{cases} \rho_0 U[v] - [\sigma] = 0, \\ U[\varepsilon] - [v] = 0, \\ \sigma = f(\varepsilon, 0), \end{cases} \tag{5.17}$$

where $U(\varepsilon)$ is the propagation speed of the shock wave, which is variable when f has its concavity towards σ-positive.

(For $0 \leq t \leq 2\mu s$ we will take $\Delta t = \frac{1}{200}\mu s$, whille for $2\mu s < t \leq 10\mu s$, Δt will gradually increase to $\Delta t = \frac{1}{4}\mu s$). In this case, OA could be something else than a straight line (see Figure 3)).

The initial conditions $\sigma = \varepsilon = v = 0$ for $t = 0$ and $X \in [0, l]$ and the limit conditions

$$v(0, t) = v_0, \;\; \sigma(l, t) = 0 \;\text{ pentru }\; t > 0 \tag{5.18}$$

and leap conditions (5.17) provide to us, along OA, the equalities

$$\rho_0 U v_a - f(\varepsilon_a, 0) = 0, \;\; v_a - U(\varepsilon_a) = 0, \;\; \rho_0 U^2 = \frac{f(\varepsilon_a, 0)}{\varepsilon_a}. \tag{5.19}$$

By using the movement equations

$$\frac{\partial v}{\partial t} = \frac{1}{\rho_0} \frac{\partial \sigma}{\partial X} \tag{5.20}$$

and the compatibility condition

$$\frac{\partial v}{\partial X} = \frac{\partial \varepsilon}{\partial t}, \tag{5.21}$$

where we take into account (5,14), we get (along OA)

$$\begin{cases} \left(\dfrac{\partial v}{\partial t}\right)_a + c^2 \left(\dfrac{\partial \varepsilon}{\partial X}\right)_a = -\dfrac{1}{\rho_0} \dfrac{\partial f}{\partial \tau}(\varepsilon, 0) \left(\dfrac{\partial \tau}{\partial X}\right)_a, \\ \left(\dfrac{\partial \varepsilon}{\partial t}\right)_a + \left(\dfrac{\partial v}{\partial X}\right)_a = 0. \end{cases} \tag{5.22}$$

Furthermore

$$\begin{cases} \dfrac{dv_a}{dt} = \left(\dfrac{\partial v}{\partial t}\right)_a + U\left(\dfrac{\partial v}{\partial X}\right)_a, \\ \dfrac{d\varepsilon_a}{dt} = \left(\dfrac{\partial \varepsilon}{\partial t}\right)_a + U\left(\dfrac{\partial \varepsilon}{\partial X}\right)_a. \end{cases} \tag{5.23}$$

As τ is continuous upon crossing of the shock wave, then

$$\begin{cases} \dfrac{d[\tau]}{dt} = \left[\dfrac{\partial \tau}{\partial t}\right] + U\left[\dfrac{\partial \tau}{\partial X}\right] = 0, \\ \dfrac{\partial f}{\partial \tau}\dfrac{\partial \tau}{\partial t} = \psi(f(\varepsilon,\tau),\varepsilon). \end{cases} \tag{5.24}$$

As $\left(\dfrac{\partial \tau}{\partial t}\right)_b = \left(\dfrac{\partial \tau}{\partial X}\right)_b = 0$, we will get from $(5.24)_2$

$$\begin{cases} \dfrac{\partial f}{\partial \tau}(\varepsilon_a,0)\left(\dfrac{\partial \tau}{\partial t}\right)_a = \psi(f(\varepsilon_a,0),\varepsilon_a), \\ \dfrac{\partial f}{\partial \tau}(\varepsilon_a,0)\left(\dfrac{\partial \tau}{\partial X}\right)_a = -\dfrac{1}{U}\psi(f(\varepsilon_a,0),\varepsilon_a). \end{cases} \tag{5.25}$$

From (5.22) and (5.23), we will get

$$\begin{cases} \left(\dfrac{\partial v}{\partial X}\right)_a = -\left(\dfrac{\partial \varepsilon}{\partial t}\right)_a = -\dfrac{d\varepsilon_a}{dt} + U\left(\dfrac{\partial \varepsilon}{\partial X}\right)_a, \\ \left(\dfrac{\partial v}{\partial t}\right)_a = \dfrac{dv_a}{dt} + U\dfrac{d\varepsilon_a}{dt} - U^2\left(\dfrac{\partial \varepsilon}{\partial X}\right)_a, \end{cases} \tag{5.26}$$

which, together with $(5.22)_1$ and (5.25) will lead to

$$\frac{dv_a}{dt} + U\frac{d\varepsilon_a}{dt} = \frac{1}{\rho_0 U}\psi(f(\varepsilon_a,0),\varepsilon_a) + (U^2 - c^2)\left(\frac{\partial \varepsilon}{\partial X}\right)_a. \tag{5.27}$$

If we use (5.19) in (5.27), we can eliminate v_a and get

$$\frac{dU}{dt}(\varepsilon_a) + U(\varepsilon_a)\frac{d\varepsilon_a}{dt} = \frac{1}{\rho_0 U(\varepsilon_a)}\psi(f(\varepsilon_a,0)\varepsilon_a) + (U^2 - c^2)\left(\frac{\partial \varepsilon}{\partial X}\right)_a, \tag{5.28}$$

where $U(\varepsilon_a)$ is defined by (5.19). For the equation (5.28) we have the initial condition

$$\begin{cases} v_a(0) = v_0, \\ \varepsilon_a(0) = \varepsilon_0 \end{cases} \tag{5.29}$$

with ε_0 from

$$\rho_0 v_0^2 = f(\varepsilon_0,0)\varepsilon_0. \qquad (5.30).$$

Remark 5.2. *Unlike in the semi-linear case (when $U = c$) where ε along the shock wave is determinated only by knowing the initial condition $\varepsilon_a(0) = \varepsilon_0$, here, in the equation for determining ε_a we will also introduce its derivative with respect to X*

and in principle, it can be only determined only concurrently with the determination of the complete solution above the OA curve. Jet, the equation (5.28) contains both information on the instantaneous response by means of $U(\varepsilon)$, and information on the relaxation properties of the material, owing to the presence of ψ. The function $f(\varepsilon, 0)$ can by considered, in a preliminary approximation, as being determined when knowing two constants of the material (for example E and v). Hence, the equation (5.28) can be used for determining the function ψ along the curve of instantaneous response of the material, if we know sufficient experimental data along the shock wave for ε_a and for $\left(\dfrac{\partial\varepsilon}{\partial X}\right)_a$.

Conclusion If we know $\varepsilon_a = \varepsilon_a(t)$ from $(5.19)_2$ we get $v_a = v_a(t)$, and from $(5.17)_3$ we get $\sigma_a = \sigma_a(t)$. As

$$\begin{cases} \dfrac{dX}{dt} = c(\varepsilon_a(t)), \\ X(0) = 0 \end{cases}$$

we get the equation of the shock wave OA.

Experimental data (see J. F. Bell [14])

Ex. no.	σ_{max} N/cm^2		ε_{max}		T_c μs	V_f m/s	$\bar{\varepsilon}$ x=1D	Ingress of reliering regime
	peak	plateau	peak	plateau				
1	7158	5783	0,02359	0,02248	306,5	22,46	0,0108	4D
2	9284	5763	0,02403	0,02229	303,0	22,48	0,0107	4,25D
3	20999	5738	0,02360	0,02210	301,0	22,41	0,0101	4,25D

- D is the diameter of the bar;
- $\bar{\varepsilon}$ is the deformation at the inflexion point on the time-deformation curve.

The experiments shows that maximum deformation (on the plateau) is constant over a certain portion along the bar (generally, over a few diameters from the stricken end) but, approximately over a distance of half a diameter, maximum deformation is somewhat greater.

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