ON VARIATION OF THE METRIC TENSOR IN THE ACTION OF A PHYSICAL FIELD

L.D. Raigorodski

Abstract

The application of the variations of the metric tensor in action integrals of physical fields is examined. Three fields are considered: an electromagnetic field, an auxiliary field and a gravitational field. Components of the 4-velocity of a dust-like matter play the role of potentials of an auxiliary field.

AMS Subject Classification: 53B50, 53Z05.
Key words: Lie, variation, metric, electromagnetic field, gravitational field, equations of motion.

1 Introduction

The action of a physical system may be written as

$$S = \int \Lambda d\Omega, \tag{1}$$

where $d\Omega = dx^0 dx^1 dx^2 dx^3$ and Λ is the Lagrangian of the system.

In varying the action integral only with respect to the metric tensor components g_{ij} we have

$$\delta_g S = -\frac{1}{2c} \int T^{ik} \delta g_{ik} \sqrt{-g} d\Omega, \tag{2}$$

where g is the determinant of g_{ik} and T^{ik} is identified with the energy-momentum tensor of the physical object under consideration [1].

Quantities δg_{ik} in (2) play the same part as the variations of electromagnetic potentials δA_i play in the integral describing the interaction between an electromagnetic field and charged particles :

$$\delta_A S_{(i)} = a \int j^{\,i} \delta A_i \sqrt{-g} d\Omega, \tag{3}$$

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2000,* 132-137

where a is a constant, j^i is the 4-vector of the current density that is the source of the field.

In the action integral components of the metric tensor appear (obviously or not obviously) in the role of gravitational potentials. Variation of them leads to the finding of quantities characterizing the system as a gravitational source and the ability of the system to interact with a gravitational field (likewise the variation of electromagnetic potentials δA_i in (3) leads to the finding of an electromagnetic field source j^i).

Let us consider the application of variational principles to physical fields.

To form a field action integral one has to construct a geometrical object L_{ilkm} from the tensor of the field or from first derivatives of potentials. The contraction of the geometrical object L_{ilkm} with the term $\sqrt{-g}g^{ik}g^{lm}$ gives the lagrangian of the field

$$\Lambda = \sqrt{-g}g^{ik}g^{lm}L_{ilkm}. \tag{4}$$

And the action of the field integral is

$$S_{(f)} = \int \sqrt{-g}g^{ik}g^{lm}L_{ilkm}d\Omega. \tag{5}$$

The equations of the field can be derived from an action principle when potentials of the fields are varied.

For an electromagnetic field, for example, we have

$$L_{ilkm} = F_{il}F_{km}, \tag{6}$$

where F_{il} is the field tensor

$$F_{il} = A_{l;i} - A_{i;l} = A_{l,i} - A_{i,l}, \tag{7}$$

δ_A-variation of the summary action

$$S = S_{(f)} + S_i,$$

gives the field equations

$$F^{ik}_{;k} = \frac{4\pi}{c}j^i. \tag{8}$$

If we perform δ_g-variation of the action (5) (when only g_{ik} are varied) we find the energy-momentum tensor of the electromagnetic field

$$T^{ik}_{(em)} = \frac{1}{4\pi}\left(F^{il}F^{k\cdot}_{\cdot l} - \frac{1}{4}g^{ik}F^{lm}F_{lm}\right). \tag{9}$$

While considering a variational field the meaning of the variational procedure is not so obvious as in the case of an electromagnetic field.

The δ_g-variation of the gravitational action

$$S_{(g)} = \int \sqrt{-g} R d\Omega = \int R_{ik} \sqrt{-g} g^{ik} d\Omega \tag{10}$$

gives the expression

$$R_{ik} - \frac{1}{2} g_{ik} R, \tag{11}$$

where R is the scalar curvature of the space-time and R_{ik} is the Ricci Tensor.

The expression (9) is assumed to be interpreted as the field part of the gravitational equations.

At the same time, in accordance with (2) one would consider (with accuracy to a coefficient) the expression (9) to be the energy-momentum tensor of the gravitational field. And this was proposed at one time by Lorentz [2] and by Levi-Civita [3] and was rejected by Einstein [4].

Such ambiguity of the physical interpretation of the results of the use of variational principles to the gravitational action is caused by two circumstances. First, the integral (8) is the action of the field properly constructed by the mode of (5). Secondly, it is the action corresponding to the interaction of the field represented by the potentials g^{ik} with itself, considered as a source by the analogy with (3).

Some indications giving the possibility to interpret reasonably the results obtained from variational principles applied to the gravitational action, can be received if we consider a vector field and let the variations of the associated potentials be induced by variations of metric tensor components only.

It is expedient to compare the results of the usage of variational principles for the action integral of this vector field with well understood equations and quantities of electromagnetic fields and to try to extend the obtained results and conclusions to gravitational fields.

Here it is necessary to take into account the common properties of a gravitational field and this auxiliary vector field — these fields are both possessed of the self-action, that leads to nonlinear field equations.

As an example of the auxiliary field we can use the field of 4-velocities of particles of a dust like matter moving freely in the curved space-time in a region which does not contain material gravitational sources :

$$u^i = \frac{dx^i}{ds},$$

where $ds = \left(-g_{lm} dx^l dx^m\right)^{1/2}$.

Variations of "potentials" u^i are generated by variations of metric tensor components only [5] :

$$\delta u^i = \delta_g u^i = -\frac{1}{2} u^i u_l u_m \delta g^{lm} \tag{12}$$

and

$$\delta u_k = \delta_g u_k = u^i \delta g_{ik} + \frac{1}{2} u_k u^i u^l \delta g_{il}. \tag{13}$$

Let us define the tensor of this vector field by

$$f_{ik} = u_{k,i} - u_{i,k},$$

form the field function

$$L_{ilkm} = f_{il} f_{km}$$

and put it in the integral of the field action (5).

Using the variational principles and taking into account (**??**) and (1) we obtain equations [5],[6] :

$$-f^{il}_{;l} u^k - f^{kl}_{;l} u^i - f^{ml}_{;l} u_m u^i u^k + \left[f^{il} f^{k\cdot}_{\cdot l} - \frac{1}{4} g^{ik} f^{lm} f_{lm} \right] = 0. \tag{14}$$

The bracketed expression on the left-hand side of the equations is fully analogous to the energy-momentum tensor of the electromagnetic field (7). This expression is the result of the variation of the term $\sqrt{-g} g^{ik} g^{lm}$ only.

The rest of the terms of equations (10) result from the δ_g-variation of the field function L_{ilkm}.

On multiplying (10) by u_k and using the equality

$$u_k u^k = -1$$

we get the proper equations of the auxiliary vector field :

$$f^{il}_{;l} = - \left[f^{il} f^{k\cdot}_{\cdot l} - \frac{1}{4} g^{ik} f^{lm} f_{lm} \right] u_k \tag{15}$$

the left hand side of which is the obvious analog of the field part of the Maxwell equations (6) and is the result of the variation of the field function L_{ilkm} only. The right-hand side of (11) is the “current” which determines the contribution of the field as its source.

The example of the vector field just considered gives the possibility to suggest that the energy-momentum tensor of a physical system results from the δ_g-variation not of the whole Lagrangian, but of the multiplier

$$\sqrt{-g} g^{ik} g^{lm}$$

of the geometrical object L_{ilkm}. And the variation if the geometrical object L_{ilkm} gives the set of equations of the system (of the field equations).

Let us now come back to the gravitational action and determine the field function as

$$L_{ilkm} = R_{ilkm}. \tag{16}$$

Substitution of L_{ilkm} (12) in (5) gives the gravitational action

$$S_{(g)} = \frac{c^3}{16\pi k} \int \sqrt{-g} g^{ik} g^{lm} R_{ilkm} d\Omega, \tag{17}$$

where k is Newton's constant of gravity and R_{ilkm} are covariant components of the Riemann tensor (it is well known, that all tensors which can be constructed from the metric tensor g_{ik} and its first and second derivatives are functions of g_{ik} and R_{ilkm}). R_{ilkm} is the simplest tensor formed from the "intensities" of the gravitational field—Christoffel symbols, and is the nearest analog of the electromagnetic field function (**??**). The existence of a gravitational field manifests itself in the existence of non-zero components of the Riemann tensor only, but not the Ricci tensor. Accordingly we suppose the field function must identified with R_{ilkm}.

If we examine the gravitational field in the empty space-time we use the action $S = S_{(g)}$ (13).

The variational procedure leads to the Einstein equations of the gravitational field in the following form:

$$-R_{ik} = \frac{8\pi k}{c^4} T_{ik_{(g)}}, \tag{18}$$

where $T_{ik_{(g)}}$ is

$$T_{ik_{(g)}} = -\frac{c^4}{4\pi k} \left(R_{ik} - \frac{1}{4} g_{ik} R \right), \tag{19}$$

$T_{ik_{(g)}}$ is the quantity resulting from a δ_g-variation the term $\sqrt{-g} g^{ik} g^{lm}$ only and therefore it can be formally interpreted as the energy-momentum tensor of the gravitational field in accordance with concept developed above.

The left hand side (field part) of the equations (**??**) is the result of the δ_g-variation of the field function $L_{ilkm} = R_{ilkm}$ only.

A gravitational field does not exist without its energy and its energy generates the field so it is not possible to separate the right-hand side of the equations (**??**) from the left-hand side.

For the calculation of gravitational energy pseudo-tensors are usually used but the prestige of the variational formalism makes us offer the above-considered results.

References

[1] S. Weinberg, *Gravitation and Cosmology. Principles and Applications of the General Theory of Relativity*, John Wiley and Sons, Inc., New-York–London–Sydney–Toronto, 1972, 384.

[2] H.A. Lorentz, Amst. Versl., 25 (1916) 468.

[3] T. Levi-Civita, Rend. Acc. Linc. (5), 26 (1917) 381.

[4] A. Einstein, Phys. Z., 19 (1918) 115.

[5] L.D. Raigorodski, Acta Phys. Acad. Sci. Hung., 37, 1–2 (1974) 133.

[6] L.D. Raigorodski, Acta Phys. Acad. Sci. Hung., 41, 3 (1976) 153.

Author's address:

L.D. Raigorodski
Department of Physics
Marine Technical University
St. Petersburg, Russia