FIELD OF CONES ON COTANGENT BUNDLE OF SYMPLECTIC MANIFOLDS

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Abstract

Section 1 defines and studies a natural field of cones on the cotangent bundle.
Section 2 analysis natural morphisms between two cotangent bundles.

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1 Field of cones on the cotangent bundle

A symplectic manifold is a couple \((M, \omega)\), where \(M\) is a smooth manifold and \(\omega\) is a symplectic form, i.e., a nondegenerate closed 2-form on \(M\). If \((M, \omega)\) is a symplectic manifold, each pair \((T_xM, \omega_x)\) is a symplectic vector space and the manifold \(M\) is necessarily of even dimension.

Let \((M, \omega)\) be a symplectic manifold. Using the symplectic tangent space \((T_xM, \omega_x)\), \(x \in M\) we define the map

\[
h_x : T_xM \rightarrow T_x^*M, \quad X_x \rightarrow h_x(X_x) = i_{X_x}\omega_x = \omega_x(X_x).
\]

Since \(\omega_x\) is nondegenerate, this map is an isomorphism between the tangent space \(T_xM\) and cotangent space \(T_x^*M\). The map \(h : TM \rightarrow T^*M, h_x/T_xM = h_x, \forall x \in M\) is an isomorphism between tangent fiber bundle \(TM\) and cotangent fiber bundle \(T^*M\).

**Proposition 1.** Let \(N\) be an \(n\) - dimensional smooth manifold, and let \((T^*N, \pi_N, N)\) be the cotangent bundle of \(N\), where \(\pi_N : T^*N \rightarrow N\) is the natural projection. There is a natural symplectic structure on the \(2n\) - dimensional manifold \(T^*N\).

**Proof.** Let be \(q = (x, \theta) \in T^*N\), where \(x = \pi_N(q) \in N\) and \(\theta \in T^*_x N\).

Let \(T_q\pi_N : T_q(T^*N) \rightarrow T_xN\) be the tangent map of \(\pi_N\).

We can define a 1-form \(\lambda\) on manifold \(T^*N\), \(\lambda_q(X_q) = \theta(T_q\pi_N(X_q)), \forall X_q \in T_q(T^*N)\). This 1-form is the Liouville form on \(T^*N\). Then \(\omega = -d\lambda\) is a symplectic form on \(T^*N\).

If \((U, u)\) is a local coordinate chart on \(N\),

\[
u : x \in U \rightarrow u(x) = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,
\]
then on the manifold $T^*N$ we have a local coordinate chart $(T^*U = \pi^{-1}_N(U), u^\#)$, where:

$$u^\# : q = (x, \theta) \in T^*U \rightarrow (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_n) \in \mathbb{R}^{2n},$$

such that $u(x) = (x_1, x_2, \ldots, x_n)$ and $\theta = \sum_{i=1}^{n} y_i \text{d}x_i / z$.

In these coordinates the Liouville form has the expression $\lambda = \sum_{i=1}^{n} y_i \text{d}x_i$ and the symplectic forms $\omega = \sum_{i=1}^{n} \text{d}x_i \wedge \text{d}y_i$.

Recall that if $V$ is a real vector space, a subset $K$ of $V$ is a cone if $\forall v \in K$, $s \in \mathbb{R}$, $s \geq 0$ imply $sv \in K$.

If moreover $v \in K$ and $-v \in K$ imply $v = 0$, then $K$ is called pointed cone.

A field of cones in a vector bundle is a map $K : x \in M \rightarrow K(x) \subset E_x = p^{-1}(x)$ such that the following two condition are satisfied:

i) For each $x \in M$, the set $K(x)$ is a closed convex pointed cone of $E_x$ with interior points;

ii) The sets $\bigcup_{x \in M} \text{Int} K(x)$ and $\bigcup_{x \in M} (E_x - K(x))$ are open in $E$.

Proposition 2. Let $N$ be an $n$-dimensional smooth manifold. There is a field of cones in the vector bundle $(T(T^*N), \pi_{T^*N}, T^*N)$.

Proof. The smooth manifold $T^*N$ has a natural symplectic structure given by the symplectic form $\omega = -d\lambda$.

Let $J$ be an almost complex structure on the manifold $T^*N$ (a section of $\text{End}(T(T^*N))$ such as $J^2 = -\text{Id}$), tamed by the symplectic form $\omega$, i.e., $\omega(X, JX) > 0$, $\forall X \in T(T^*N) - \{0\}$. If moreover $\omega$ is $J$-invariant, $J$ is said to be calibrated. The space of almost complex structures on a given symplectic manifold $(M, \omega)$ which are tamed (resp. calibrated) by $\omega$ is nonempty and contractible (particularly these spaces are connected).

Based on bilinearity of $\omega$ and linearity of $J$, we obtain the bilinearity of the map $g(X, Y) = \omega(X, JY) - \omega(JX, Y), \forall X, Y \in T(T^*N)$.

However:

$$g(X, X) = \omega(X, JX) - \omega(JX, X) = 2\omega(X, JX) > 0, \forall X \in T(T^*N) - \{0\}$$

$$g(JX, JY) = \omega(JX, J^2Y) - \omega(J^2X, JY) = \omega(JX, -Y) - \omega(-X, JY) =$$

$$= -\omega(JX, Y) + \omega(X, JY) = g(X, Y), \forall X, Y \in T(T^*N)$$

$$g(Y, X) = g(JY, JX) = \omega(JY, J^2X) - \omega(J^2Y, JX) = \omega(JY, -X) -$$

$$-\omega(-Y, JX) = \omega(X, JY) - \omega(JX, Y) = g(X, Y), \forall X, Y \in T(T^*N).$$

Then, $g$ is a $J$-invariant Riemannian metric on $T(T^*N)$.

Let

$$h : X_q \in T(T^*N) \rightarrow h_q(X_q) = i_{X_q} \omega_q = \omega_q(X_q, \cdot) \in T^*(T^*N), \quad q \in T^*N$$
oe the isomorphism between tangent fiber bundle $T(T^*N)$ and cotangent fiber bundle $T^*(T^*N)$ of symplectic manifold $T^*N$ and $\lambda X = h^{-1} \circ \lambda$ be a vector field on $T^*N$ (a section of vector bundles $(T(T^*N), \pi_{T^*N}, T^*N)$).

For each $q \in T^*N$ we define a cone:

$$K_\lambda(q) = \{ Y_q \in T_q(T^*N) / \frac{1}{2} (g(\lambda X_q, \lambda X_q) g(Y_q, Y_q))^{\frac{1}{2}} \leq g(Y_q, Y_q) \leq (g(\lambda X_q, \lambda X_q) g(Y_q, Y_q))^{\frac{1}{2}} \} \forall Y_q \in K(x), s \in \mathbb{R}, s \geq 0 \Rightarrow$$

$$\Rightarrow \frac{1}{2} (g(\lambda X_q, \lambda X_q) g(sY_q, sY_q))^{\frac{1}{2}} \leq g(\lambda X_q, \lambda X_q) g(sY_q, \lambda Y_q))^{\frac{1}{2}}$$

implies $sY_q \in K_\lambda(q)$, then. If $Y_q \in K(q)$ and $-Y_q \in K_\lambda(q)$ imply $Y_q = 0$, then $K_\lambda(q)$ is pointed cone.

Then, $K_\lambda(q)$ is a closed convex pointed cone, with $T_q(T^*M)$ interior points, and the sets $\bigcup_{q \in M} \text{Int}K(q)$ and $\bigcup_{q \in M} (T_q(T^*N) - K(q))$ are open in $T(T^*N)$.

2. Morphisms between two cotangent bundles

Proposition 3. Let $P$ and $Q$ be two manifolds. Any diffeomorphism $\varphi : N \rightarrow M$ lift to a symplectic diffeomorphism (symplectomorphism) $\phi : T^*N \rightarrow T^*M$.

Proof. Let $(T^*N, \pi_{T^*N}, N)$, $(T^*M, \pi_{T^*M}, M)$ be the cotangent bundles of $n$-dimensional manifolds $N$ resp. $M$; $\lambda_N$, $\lambda_M$ the corresponding Liouville 1-forms.

One defines $\phi$ by the formula: $\phi(q) = (\varphi(x), (T_x\varphi)^{-1}(\theta))$ for any $q = (x, \theta) \in T^*N$.

Because $(T_x\varphi)^{-1} : T_x(x)N \rightarrow T_xN$, and $\theta \in T_x^*N$, we have $(T_x\varphi)^{-1}(\theta) \in T_{\varphi(x)}^*M$, and the map $\phi$ is well defined. The maps involved in definition of $\phi$ are diffeomorphisms and consequently $\phi$ is also a diffeomorphism. Also

$$\varphi^*\lambda_M = \lambda_N$$

The diffeomorphism $\phi$ satisfy the equality $\phi^*\lambda_M = \lambda_N$. Then $\phi^*\omega_M = \omega_N$ and hence $\phi$ is a symplectomorphism.

Remark. Let $(E, p, M)$ be a regular vector bundle endowed with a field of cones denoted by $[(E, p, M); K]$. It is not difficult to see that the structures $[(E, p, M); K]$ are the objects of a category. A morphism from $[(E, p, M); K]$ to $[(E', p', M'); K']$ in this category is a morphisms $f : E \rightarrow E'$ of vector bundles such that $f(K(x)) \subset K(f(x))$, $\forall x \in M$. The map $\bar{f} : M \rightarrow M$ is defined such that the diagram

$$\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
p & \downarrow & \downarrow p' \\
M & \xrightarrow{\bar{f}} & M
\end{array}$$

commutes.

Let $N$ and $M$ be two smooth manifolds, and $\varphi : N \rightarrow M$ a diffeomorphism. The symplectic diffeomorphism $\phi : T^*N \rightarrow T^*M$ is the lift of $\varphi$ defined in Proposition 3. The diagram:
$T(T^*N) \xrightarrow{T\phi} T(T^*M)$

$\pi_{T^*N} \downarrow \downarrow \pi_{T^*M}$ commutes. Then $T\phi : T(T^*N) \rightarrow T(T^*M)$ (the tangent map of $\phi$) is an isomorphism of vector bundles $(T(T^*N), \pi_{T^*N}, T^*N)$ and $(T(T^*M), \pi_{T^*M}, T^*M)$.

**Proposition 4.** Let $\lambda^N X$ be the vector field corresponding to Liouville form $\lambda^N$ on the manifold $T^*N$ and $\lambda^M X$ is the vector field corresponding to Liouville form $\lambda^M$ on the manifold $T^*M$. If $\varphi : N \rightarrow M$ is a diffeomorphism and $\phi : T^*N \rightarrow T^*M$ is the natural lift of $\varphi$ defined in Proposition 3, then $T\phi(\lambda^N X) = \lambda^M X$.

**Proof.** Let $\omega_N$ and $\omega_M$ be the symplectic forms determined by $\lambda^N$ resp. $\lambda^M$ on the manifold $T^*N$ resp. $T^*M$. Then, $\lambda^N X$ and $\lambda^M X$ are the vector fields determined by equalities:

$$i_{\lambda^N X} \omega_N = \omega_N(\lambda^N X, \cdot) = \lambda^N, \quad i_{\lambda^M X} \omega_M = \omega_M(\lambda^M X, \cdot) = \lambda^M.$$

The equality $T\phi(\lambda^N X) = \lambda^M X$ is equivalent to the equality

$$\omega_M(T\phi(\lambda^N X), \cdot) = \lambda^M.$$

But,

$$\omega_M(T\phi(\lambda^N X), \cdot) = \lambda^M \iff \phi^*(\omega_M(T\phi(\lambda^N X), \cdot)) = \phi^*(\lambda^M) \iff$$

$$[\phi^*(\omega_M(T\phi(\lambda^N X), \cdot))](Y) = [\phi^*(\lambda^M)](Y), \forall Y \in \chi(T^*N) \iff$$

$$\omega_M(T\phi(\lambda^N X), T\phi Y) = \lambda^M(T\phi Y), \forall Y \in \chi(T^*N) \iff (\phi^*\omega_M)(\lambda^N X, Y) =$$

$$= (\phi^*\lambda^M)(Y), \forall Y \in \chi(T^*N) \iff \omega_N(\lambda^N X, Y) = \lambda^N(Y), \forall Y \in \chi(T^*N)$$

(because the diffeomorphism $\phi$ satisfies the equality $\phi^*\lambda_M = \lambda_N$, $\phi^*\omega_M = \omega_N$) $\iff$

$$\omega_N(\lambda^N X, \cdot) = \lambda^N.$$

If $J_N$ is an almost complex structure on the manifold $T^*N$, then $J_M = T\phi \circ J \circ (T\phi)^{-1}$ is an almost complex structure on the manifold $T^*M$.

**Remark.** Let $\varphi : N \rightarrow M$ be a diffeomorphism and $\phi : T^*N \rightarrow T^*M$ be the lift of $\varphi$ defined in Proposition 3. $J_N$ is an almost complex structure on the manifold $T^*N$, then $J_M = T\phi \circ J \circ (T\phi)^{-1}$ is the corresponding almost complex structure on the manifold $T^*M$.

Let $K_{\lambda^N}$ and $K_{\lambda^M}$ be the field of cones determined (Proposition 2) using the Riemannian metrics corresponding to natural symplectic forms and almost complex structures $J_N$ and $J_M = T\phi \circ J_N \circ (T\phi)^{-1}$.

By Proposition 4 we have $T\phi(K_{\lambda^N}) = K_{\lambda^M}$ and then, $T\phi$ is an isomorphism of structures $[(T(T^*N), \pi_{T^*N}, T^*N); K_{\lambda^N}]$ and $[(T(T^*M), \pi_{T^*M}, T^*M); K_{\lambda^M}]$.

Let $Man(n)$ be the category of $n$-dimensional manifolds. The morphisms of this category are diffeomorphisms.

To every manifold $N \in Ob(Man(n))$ and to an almost structure we can associate the structure $[(T(T^*N), \pi_{T^*N}, T^*N); K_{\lambda^N}]$. Similarly, to every diffeomorphism $\varphi : N \rightarrow M$ we associate the isomorphism $T\phi$ of structures $[(T(T^*N), \pi_{T^*N}, T^*N); K_{\lambda^N}]$ and $[(T(T^*M), \pi_{T^*M}, T^*M); K_{\lambda^M}]$. 


References


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