FIELD OF CONES ON COTANGENT BUNDLE OF SYMPLECTIC MANIFOLDS

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Abstract

Section 1 defines and studies a natural field of cones on the cotangent bundle. Section 2 analysis natural morphisms between two cotangent bundles.

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1 Field of cones on the cotangent bundle

A symplectic manifold is a couple (M, ω), where M is a smooth manifold and ω is a symplectic form, i.e., a nondegenerate closed 2-form on M. If (M, ω) is a symplectic manifold, each pair (T_xM, ω_x) is a symplectic vector space and the manifold M is necessarily of even dimension.

Let (M, ω) be a symplectic manifold. Using the symplectic tangent space (T_xM, ω_x), $x \in M$ we define the map

$$h_x : T_xM \longrightarrow T_x^*M, \quad X_x \longrightarrow h_x(X_x) = i_{X_x}\omega_x = \omega_x(X_x).$$

Since ω_x is nondegenerate, this map is an isomorphism between the tangent space T_xM and cotangent space T_x^*M. The map $h : TM \longrightarrow T^*M$, $h/_{T_xM} = h_x, \forall x \in M$ is an isomorphism between tangent fiber bundle TM and cotangent fiber bundle T^*M.

Proposition 1. *Let N be an n - dimensional smooth manifold, and let (T^*N, π_N, N) be the cotangent bundle of N, where $\pi_N : T^*N \longrightarrow N$ is the natural projection. There is a natural symplectic structure on the $2n$ -dimensional manifold T^*N.*

Proof. Let be $q = (x, \theta) \in T^*N$, where $x = \pi_N(q) \in N$ and $\theta \in T_x^*N$.

Let $T_q\pi_N : T_q(T^*N) \longrightarrow T_xN$ be the tangent map of π_N.

We can define a 1-form λ on manifold T^*N , $\lambda_q(X_q) = \theta(T_q\pi_N(X_q)), \forall X_q \in T_q(T^*N)$. This 1-form is the Liouville form on T^*N. Then $\omega = -d\lambda$ is a symplectic form on T^*N.

If (U, u) is a local coordinate chart on N,

$$u : x \in U \longrightarrow u(x) = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n,$$

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then on the manifold T^*N we have a local coordinate chart $(T^*U = \pi_N^{-1}(U), u^{\#})$, where:

$$u^{\#} : q = (x, \theta) \in T^*U \longrightarrow (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_n) \in \mathbf{R}^{2n},$$

such that $u(x) = (x_1, x_2, \ldots, x_n)$ and $\theta = \sum_{i=1}^{n} y_i dx_{i/x}$.

In these coordinates the Liuoville form has the expression $\lambda = \sum_{i=1}^{n} y_i dx_i$ and the symplectic forms $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

Recall that if V is a real vector space, a subset K of V is a *cone* if $\forall v \in K$, $s \in \mathbf{R}$, $s \geq 0$ imply $sv \in K$.

If moreover $v \in K$ and $-v \in K$ imply $v = 0$, then K is called *pointed cone.*

A field of cones in a vector bundle is a map $K : x \in M \longrightarrow K(x) \subset E_x = p^{-1}(x)$ such that the following two condition are satisfied:

i) For each $x \in M$, the set $K(x)$ is a closed convex pointed cone of E_x with interior points;

ii) The sets $\bigcup_{x \in M} \mathrm{Int} K(x)$ and $\bigcup_{x \in M} (E_x - K(x))$ are open in E.

Proposition 2. *Let N be an n - dimensional smooth manifold. There is a field of cones in the vector bundle $(T(T^*N), \pi_{T^*N}, T^*N)$.*

Proof. The smooth manifold T^*N has a natural symplectic structure given by the symplectic form $\omega = -d\lambda$.

Let J be an almost complex structure on the manifold T^*N (a section of $End(T(T^*N))$ such as $J^2 = -Id$), tamed by the symplectic form ω, i.e., $\omega(X, JX) > 0, \forall X \in T(T^*N) - \{0\}$.If moreover ω is J-invariant, J is said to be *calibrated.* The space of almost complex structures on a given symplectic manifold (M, ω) which are tamed (resp. calibrated) by ω is nonempty and contractible (particularly these spaces are connected).

Based on bilinearity of ω and linearity of J, we obtain the bilinearity of the map $g(X, Y) = \omega(X, JY) - \omega(JX, Y), \forall X, Y \in T(T^*N)$.

However:

$$g(X, X) = \omega(X, JX) - \omega(JX, X) = 2\omega(X, JX) > 0, \forall X \in T(T^*N) - \{0\}$$

$$g(JX, JY) = \omega(JX, J^2Y) - \omega(J^2X, JY) = \omega(JX, -Y) - \omega(-X, JY) =$$

$$= -\omega(JX, Y) + \omega(X, JY) = g(X, Y), \forall X, Y \in T(T^*N)$$

$$g(Y, X) = g(JY, JX) = \omega(JY, J^2X) - \omega(J^2Y, JX) = \omega(JY, -X) -$$

$$-\omega(-Y, JX) = \omega(X, JY) - \omega(JX, Y) = g(X, Y), \forall X, Y \in T(T^*N).$$

Then, g is a J-invariant Riemannian metric on $T(T^*N)$.

Let

$$h : X_q \in T(T^*N) \longrightarrow h_q(X_q) = i_{X_q}\omega_q = \omega_q(X_q, .) \in T^*(T^*N), \quad q \in T^*N$$

be the isomorphism between tangent fiber bundle $T(T^*N)$ and cotangent fiber bundle $T^*(T^*N)$ of symplectic manifold T^*N and ${}^\lambda X = h^{-1} \circ \lambda$ be a vector field on T^*N (a section of vector bundles $(T(T^*N), \pi_{T^*N}, T^*N)$).

For each $q \in T^*N$ we define a cone:

$$K_\lambda(q) = \{Y_q \in T_q(T^*N) / \frac{1}{2}(g({}^\lambda X_q, {}^\lambda X_q) g(Y_q, Y_q))^{\frac{1}{2}} \leq$$

$$\leq g(X_q, Y_q) \leq (g({}^\lambda X_q, {}^\lambda X_q) g(Y_q, Y_q))^{\frac{1}{2}}\} \forall Y_q \in K(x), s \in \mathrm{R}, s \geq 0 \Longrightarrow$$

$$\Longrightarrow \frac{1}{2}(g({}^\lambda X_q, {}^\lambda X_q) g(sY_q, sY_q))^{\frac{1}{2}} \leq g({}^\lambda X_q, sY_q) \leq (g({}^\lambda X_q, {}^\lambda X_q) g(sY_q, \lambda Y_q))^{\frac{1}{2}}$$

implies $sY_q \in K_\lambda(q)$, then. If $Y_q \in K(q)$ and $-Y_q \in K_\lambda(q)$ imply $Y_q = 0$, then $K_\lambda(q)$ is pointed cone.

Then, $K_\lambda(q)$ is a closed convex pointed cone, with $T_q(T^*M)$ interior points, and the sets $\bigcup_{q \in M} \mathrm{Int} K(q)$ and $\bigcup_{q \in M} (T_q(T^*N) - K(q))$ are open in $T(T^*N)$.

2 Morphisms between two cotangent bundles

Proposition 3. *Let P and Q be two manifolds. Any diffeomorphism $\varphi : N \longrightarrow M$ lift to a symplectic diffeomorphism (symplectomorphism) $\phi : T^*N \longrightarrow T^*M$.*

Proof. Let (T^*N, π_N, N) , (T^*M, π_M, M) be the cotangent bundles of n-dimensional manifolds N resp. M; λ_N, λ_M the corresponding Liuoville 1-forms.

One defines ϕ by the formula: $\phi(q) = (\varphi(x), (T_x\varphi)^{-1*}(\theta))$ for any $q = (x, \theta) \in T^*N$.

Because $(T_x\varphi)^{-1} : T_{\varphi(x)}M \longrightarrow T_xN$, and $\theta \in T_x^*N$, we have $(T_x\varphi)^{-1*}(\theta) \in T^*_{\varphi(x)}M$, and the map ϕ is well defined. The maps involved in definition of ϕ are diffeomorphisms and consequently ϕ is also a diffeomorphism. Also

$$\varphi^* \lambda_{M/q}(X_q) = \lambda_{M/\varphi(q)}(T_q\varphi(X_q)) = (T_x\varphi)^{-1*}(\theta)(T_{\varphi(q)}\pi_M T_q(X_q)) =$$

$$= \theta(T_q(\phi \circ \pi_M \circ \varphi)(X_q)) = \theta(T_q \pi_N(X_q)) = \lambda_{N/q}(X_q)$$

The diffeomorphism ϕ satisfy the equality $\phi^*\lambda_M = \lambda_N$. Then $\phi^*\omega_M = \omega_N$ and hence ϕ is a symplectomorphism.

Remark. Let (E, p, M) be a regular vector bundle endowed with a field of cones denoted by $[(E, p, M); K]$. It is not difficult to see that the structures $[(E, p, M); K]$ are the objects of a category. A morphism from $[(E, p, M); K]$ to $[(E', p', M'); K']$ in this category is a morphisms $f : E \longrightarrow E'$ of vector bundles such that $f(K(x)) \subset K(\underline{f}(x))$, $\forall x \in M$. (The map $\underline{f} : M \longrightarrow M$ is defined such that the diagram

$$\begin{array}{ccc} E & \stackrel{\mathrm{f}}{\rightarrow} & E \\ p \downarrow & & \downarrow p' \\ M & \stackrel{\underline{\mathrm{f}}}{\rightarrow} & M \end{array}$$

commutes).

Let N and M be two smooth manifolds, and $\varphi : N \longrightarrow M$ a diffeomorphism. The symplectic diffeomorphism $\phi : T^*N \longrightarrow T^*M$ is the lift of φ defined in Proposition 3. The diagram:

$$\begin{array}{ccc} T(T^*N) & \overset{T\phi}{\to} & T(T^*M) \\ \pi_{T^*N}\downarrow & & \downarrow \pi_{T^*M} \\ T^*N & \overset{\phi}{\to} & T^*M \end{array}$$

commutes. Then $T\phi : T(T^*N) \longrightarrow T(T^*M)$ (the tangent map of ϕ) is an isomorphism of vector bundles $(T(T^*N), \pi_{T^*N}, T^*N)$ and $(T(T^*M), \pi_{T^*M}, T^*M)$.

Proposition 4. *Let ${}^{\lambda_N}X$ be the vector field corresponding to Liuoville form λ_N on the manifold T^*N and ${}^{\lambda_M}X$ is the vector field corresponding to Liuoville form λ_M on the manifold T^*M. If $\varphi : N \longrightarrow M$ is a diffeomorphism and $\phi : T^*N \longrightarrow T^*M$ is the natural lift of φ defined in Proposition 3, then $T\phi({}^{\lambda_N}X) = {}^{\lambda_M}X$.*

Proof. Let ω_N and ω_M be the symplectic forms determined by λ_N resp. λ_M on the manifold T^*N resp. T^*M. Then, ${}^{\lambda_N}X$ and ${}^{\lambda_M}X$ are the vector fields determined by equalities:

$$i_{{}^{\lambda_N}X}\omega_N = \omega_N({}^{\lambda_N}X, .) = \lambda_N, \quad i_{{}^{\lambda_M}X}\omega_M = \omega_M({}^{\lambda_M}X, .) = \lambda_M.$$

The equality $T\phi({}^{\lambda_N}X) = {}^{\lambda_M}X$ is equivalent to the equality

$$\omega_M(T\phi({}^{\lambda_N}X), .) = \lambda_M.$$

But,

$$\omega_M(T\phi({}^{\lambda_N}X), .) = \lambda_M \iff \phi^*(\omega_M(T\phi({}^{\lambda_N}X), .)) = \phi^*(\lambda_M) \iff$$

$$[\phi^*(\omega_M(T\phi({}^{\lambda_N}X), .))](Y) = [\phi^*(\lambda_M)](Y), \forall Y \in \chi(T^*N) \iff$$

$$\omega_M(T\phi({}^{\lambda_N}X), T\phi Y) = \lambda_M(T\phi Y), \forall Y \in \chi(T^*N) \iff (\phi^*\omega_M)({}^{\lambda_N}X, Y) =$$

$$= (\phi^*\lambda_M)(Y), \forall Y \in \chi(T^*N) \iff \omega_N({}^{\lambda_N}X, Y) = \lambda_N(Y), \forall Y \in \chi(T^*N)$$

(because the diffeomorphism ϕ satisfies the equality $\phi^*\lambda_M = \lambda_N$, $\phi^*\omega_M = \omega_N$) $\iff$ $\omega_N({}^{\lambda_N}X, .) = \lambda_N$ Q.E.D.

If J_N is an almost complex structure on the manifold T^*N , then $J_M = T\phi \circ J \circ (T\phi)^{-1}$is an almost complex structure on the manifold T^*M.

Remark. Let $\varphi : N \longrightarrow M$ be a diffeomorphism and $\phi : T^*N \longrightarrow T^*M$ be the lift of φ defined in Proposition 3. J_N is an almost complex structure on the manifold T^*N, then $J_M = T\phi \circ J \circ (T\phi)^{-1}$ is the corresponding almost complex structure on the manifold T^*M.

Let K_{λ_N} and K_{λ_M}be the field of cones determined (Proposition 2) using the Riemannian metrics corresponding to natural symplectic forms and almost complex structures J_N and $J_M = T\phi \circ J_N \circ (T\phi)^{-1}$.

By Proposition 4 we have $T\phi(K_{\lambda_N}) = K_{\lambda_M}$ and then, $T\phi$ is an isomorphism of structures $[(T(T^*N), \pi_{T^*N}, T^*N); K_{\lambda_N}]$ and $[(T(T^*M), \pi_{T^*M}, T^*M); K_{\lambda_M}]$.

Let $Man(n)$ be the category of n-dimensional manifolds. The morphisms of this category are diffeomorphisms.

To every manifold $N \in Ob(Man(n))$ and to an almost structure we can associate the structure $[(T(T^*N), \pi_{T^*N}, T^*N); K_{\lambda_N}]$. Similarly, to every diffeomorphism $\varphi : N \longrightarrow M$ we associate the isomorphism $T\phi$ of structures $[(T(T^*N), \pi_{T^*N}, T^*N); K_{\lambda_N}]$ and $[(T(T^*M), \pi_{T^*M}, T^*M); K_{\lambda_M}]$.

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