Towards a Theory of Complex Holomorphic Functions of Several Variables from Non-Standard Point of View

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Abstract

The main goal of this note is to briefly show how the methods of Non-Standard Analysis may produce theorems in the standard context.

We extend from one to n-complex variables parts of a Robinson Callot's Theorem [Fr], see our Theorem 1 and our Theorem 3; further we apply these theorems in order to get non-standard proofs of the Liouville's Theorem (Corollary 2), the Open Mappi ng Theorem (Corollary 3) and the Maximum Modulus Principle (Corollary 4).

These results may indicate how the theory of Complex Functions Theory may be rewritten starting from a Non-Standard point of view (for Real Functions Theory see [DR]).

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In this note we use the known principles, methods and notions belonging to Non-Standard Analysis from [Ro], [Dv], [DR]. See also [P] and [PP].

Let's fix some notations:

If $\mathcal{U}$ is a superstructure and $\mathcal{L}$ a higher order language on $\mathcal{U}$ in which are defined all the usual mathematical notions (particularly the numbers sets $\mathbf{Z}, \mathbf{R}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$, the sequences, the functions, etc) and in which all the theorems of classical mathematics hold, then we denote by $^*\mathcal{U}$ a non-standard enlargement of $\mathcal{U}$. We remark that in particular if $\mathcal{A}$ is the family of all sentences formulated by means of the language $\mathcal{L}$, which are true in $\mathcal{U}$, then $^*\mathcal{U}$ is a model for $\mathcal{A}$.

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If one consider on $\mathcal{U}$ the ZFC axiomatization [Ru], then in ${}^*\mathcal{U}$ one consider the IST axiomatization [Ne]. We suppose that $\mathcal{U} \subset {}^*\mathcal{U}$ and if A is a set in $\mathcal{U}$ we denote by ${}^*A$ the image of A under the previous inclusion. For instance ${}^*\mathbf{R}$ and ${}^*\mathbf{C}$ are the fields of non-standard real and complex numbers, respectively.

If s is a true sentence in $\mathcal{U}$, then its image ${}^*s$ (under the previous inclusion) remains true in ${}^*\mathcal{U}$.

In particular ${}^*z \in {}^*\mathbf{C} \Leftrightarrow {}^*z = {}^*x + i{}^*y$, with ${}^*x, {}^*y \in {}^*\mathbf{R}$, $i^2 = -1$.

In particular, the notions of functions, continuity, holomorphy etc, and the theorems concerning these notions transfer to ${}^*\mathcal{U}$ and remain true there.

Our next Theorems 1 and 3 hold in ${}^*\mathcal{U}$, while their Corollaries 2,4,5 hold in $\mathcal{U}$.

We recall that IST contains ZFC plus three new principles namely: Transfer Principle, Idealisation Principle and Standardisation Principle see [DR], [Dv], [P].

Theorem 1. *Let $f : U \to {}^*\mathbf{C}$, $U \subset {}^*\mathbf{C}^n$ a *-open set (${}^*\mathbf{C}$ is the field of non-standard complex numbers) be a holomorphic function. Suppose that $hal(x_0) \subset U$, $x_0 \in U$, x_0 a standard number. Suppose moreover that f takes only limited values on $hal(x_0)$. Then there exists $V \subset U$ a standard neighbourhood of x_0 on which:*

a) *f is S-continuous, so the shadow of f exists (cf [Ro], Theorem 4.5.10, pg. 116)*;

b) *the shadow of f is holomorphic*;

c) *there exists the shadow of any partial derivative of standard order of f and it coincides with the same order partial derivative of the shadow of f.*

Remark. If we denote by ${}^\circ f$ the shadow of the function f, then c) says that for any $\alpha \in \mathbf{N}^n$, there exists ${}^\circ\left(\frac{\partial^\alpha f}{\partial z^\alpha}\right)$ on V and, moreover, we have

$$ {}^\circ\left(\frac{\partial^\alpha f}{\partial z^\alpha}\right) = \frac{\partial^\alpha ({}^\circ f)}{\partial z^\alpha}. $$

Proof. The proof given by Robinson-Callot see [Fr], easily extends from one to n variables. We only remark that for a) we use the simple fact that f is S-continuous iff f is separately S-continuous in every variable (this follows from the obvious remark that

$$ hal(x_1, \ldots, x_n) = hal(x_1) \times \ldots \times hal(x_n) $$

for any $(x_1, \ldots, x_n)$ standard, see [Sr] pg. 116) while for b) we use the fact that the function f is holomorphic iff f is separately holomorphic and for c) we use either external induction (starting from Robinson-Callot's results) or alternatively the Cauchy formula for holomorphic functions in n variables.

Based on these remarks and the proofs from [Fr] page 28, (which we do not repeat here) the reader may easily arrive to the complete proofs

Corollary 2. **(Liouville's Theorem).** *Let $f : C^n \to C$ be a standard bounded entire function (in the standard superstructure $\mathcal{U}$). Then f is constant.*

Proof. Let's denote by $B = \bar{B}(0, M) \subset \mathbf{C}$, the closed ball of center 0 and radius M where $|f(z)| \leq M$, $(\forall) z \in \mathbf{C}^n$.

Let ${}^*f : {}^*\mathbf{C}^n \to {}^*B$, be the corresponding function in ${}^*\mathcal{U}$. B is a bounded metric space which is equivalent to the fact that all points of ${}^*B$ are finite ([Ro], Theorem 4.3.1, page 100).

So ${}^*f$ is a limited function.

Let's denote by g the function $g : {}^*\mathbf{C}^n \to {}^*B$, given by the formula $g(x) = {}^*f(\omega x)$ for any x where ω is an infinitely large complex number.

g is also analytic and limited so, Theorem 1 applies arround any standard point x_0 from ${}^*\mathbf{C}^n$.

In particular ${}^\circ g$ exists and it is a holomorphic function by b). It follows (${}^\circ g$ being a standard function) that, for any $\alpha \in \mathbf{N}^n$, the derivative $\dfrac{\partial^\alpha({}^\circ g)}{\partial z^\alpha}$ exists.

From c) the shadow ${}^\circ\left(\dfrac{\partial^\alpha g}{\partial z^\alpha}\right)$ also exists being equal to $\dfrac{\partial^\alpha({}^\circ g)}{\partial z^\alpha}$. So the shadow ${}^\circ\left(\dfrac{\partial^\alpha g}{\partial z^\alpha}\right)(z)$ exists for any $z \in {}^*\mathbf{C}^n$, z standard. This means that $\dfrac{\partial^\alpha g}{\partial z^\alpha}(z)$ is nearly standard for any z from ${}^*\mathbf{C}^n$, z standard.

But $\dfrac{\partial^\alpha g}{\partial z^\alpha}(z) = \omega^{|\alpha|}\dfrac{\partial^\alpha\, {}^\circ({}^*f)}{\partial z^\alpha}(z)$ (if $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$), for any z. We deduce that $\dfrac{\partial^\alpha\, {}^\circ({}^*f)}{\partial z^\alpha}(z)$ is infinitezimal for any z standard.

Since f is a standard function, we have $\dfrac{\partial^\alpha({}^*f)}{\partial z^\alpha}(z) = \dfrac{\partial^\alpha f}{\partial z^\alpha}(z)$ which is a standard number for any standard z.

It follows that $\dfrac{\partial^\alpha f}{\partial z^\alpha}(z) = 0$, $(\forall)\alpha \in \mathbf{N}^n$ and $(\forall) z \in \mathbf{C}^n$.

Because the holomorphy is equivalent to the analyticity for standard functions, we deduce that f is constant.

Theorem 3. *Let $f : U \to {}^*C$, $U \subset {}^*C^n$ a *-open set, be a holomorphic non-constant function.*

Suppose that $hal(x_0) \subset U$, $x_0 \in U$, x_0 standard number. Suppose moreover that f takes only limited values on $hal(x_0)$. Then if f is S-continuous in x_0, we have $f(hal(x_0)) = hal(f(x_0))$.

Proof. The case $n = 1$ is part three of the Robinson Callot's Theorem ([Fr]). We remark here that their result easily extends to several variables. Namely, we first remark that performing linear transformations of coordinates, we may suppose that $x_0 = O_n \in {}^*\mathbf{C}^n$ and $f(O_n) = 0$.

Secondly, it is obvious that there is a complex line L through O_n, such that $f|L$ is not identically zero (otherwise moving the line L through O_n, one gets that f is identically zero around O_n, so constant, a contradiction).

Taking the equation of L as z_n-coordinate, one obtains

$$f(0,\ldots,0,z_n) \not\equiv 0.$$

Let's denote by $f_n(z_n) := f(0,\ldots,0,z_n)$. Since f_n is S-continuous in $0 \in {}^*\mathbf{C}$ (because f is S-continuous in $O_n \in {}^*\mathbf{C}^n$) and not constant one gets from the one variables case that $f_n(hal(0)) = hal(f_n(0))$ $(= hal(0);\ f_n(O_n) = 0)$. But f is S-continuous in $O_n \in {}^*\mathbf{C}^n$

$$\Rightarrow f(hal(O_n)) \subseteq hal(f(0)) \quad (= hal(0)) \tag{1}$$

and

$$hal(O_n) = \underbrace{hal(0) \times \ldots \times hal(0)}_{n\ times}.$$

So

$$f(hal(O_n)) \supseteq f(\underbrace{\{0\} \times \{0\} \times \ldots \times \{0\}}_{(n-1)\ times} \times hal(0)) = hal(f_n(0))(= hal(0)) \tag{2}$$

From (1) and (2) we get $f(hal(O_n)) = hal(f_n(0))$ i.e.

$$f(hal(x_0)) = hal(f(x_0)).$$

Corollary 4 (Open Mapping Theorem). *Let $U \subset C^n$ be a domain and $f : U \to C$ be a standard not-constant holomorphic function. Then f is an open map.*

Proof. We must prove that $f(V)$ is open in $\mathbf{C}$ for any open set $V \subset U$. We recall that a set $X \subset T$ (= a metric space) is open iff $hal(x) \subset {}^*X$, $(\forall)x \in X$ ([Ro], Th. 4.1.4, pag. 90). So, take $y \in f(V)$, $y = f(x)$, $x \in V$. Since V is open, $(\forall)z \in V$, $(\exists)B_z = B(z, r_z)$ an open ball such that $B_z \subset V$. Put $C_x := \bar{B}\left(x, \frac{r_x}{2}\right)$ the closed ball, $C_x \subset B_x \subset V$. Since f is continuous, it follows that $f|_{C_x}$ is bounded (since C_x is compact). Put $D_x := \overset{\circ}{C}_x = B\left(x, \frac{r_x}{2}\right)$ the open ball.

So $f|_{D_x} : D_x \to \mathbf{C}$ is holomorphic and bounded and $D_x \subset V$, D_x open, $x \in D_x \Rightarrow hal(x) \subset {}^*D_x$ ([Ro]) Th. 4.1.3). By the identity theorem, $f|D_x$ is not constant and bounded, so ${}^*f|{}^*D_x : {}^*D_x \to {}^*\mathbf{C}$ is not constant and limited. By Theorem 1 a) we deduce that ${}^*f$ is S-continuous in x. Being not constant, from Theorem 3 it follows that ${}^*f(hal(x)) = hal(f(x)) = hal(y)$. But

$$hal(x) \subseteq {}^*D_x \subseteq {}^*V \Rightarrow {}^*f(hal(x)) \subseteq {}^*f({}^*V) = {}^*(f(V))$$

(since f is S-continuous on ${}^*V$), so $hal(y) \subseteq {}^*(f(V))$. So, for any $y \in f(V)$ it follows that $hal(y) \subseteq {}^*(f(V))$, i.e. $f(V)$ is open by [Ro], Th. 4.1.4., pg. 90 again.

Corollary 5 (Maximum Modulus Principle). *Let $U = \overset{\circ}{U} \subset \mathbf{C}^n$ be a domain and $f : U \to \mathbf{C}$ be a standard holomorphic function. If the modulus of f achieves its maximum in a point from U, then f is constant.*

Proof. Suppose that $|f|$ achieves its maximum in $z_0 \in U = \overset{\circ}{U}$ and f is not constant. Then $|f(z)| \leq |f(z_0)|$, $(\forall) z \in U$.

By transfer, we have ${}^*f : {}^*U \to {}^*\mathbf{C}$ and $Im({}^*f) = {}^*(Imf)$. Using Theorem 3, f being non-constant we obtain ${}^*f(hal(z_0)) = hal(f(z_0))$, hence $hal(f(z_0)) \subseteq Im({}^*f) = {}^*(Imf)$.

There exists $\tilde{V}$ a neighbourhood of $f(z_0)$ such that $\tilde{V} \subseteq hal(f(z_0))$ [cf [Sr], Prop 6.1.4., pg. 112]. Therefore there exists $\tilde{V}$ a neighbourhood of $f(z_0)$, such that $\tilde{V} \subseteq {}^*(Imf)$.

By transfer, there exists V a neighbourhood of $f(z_0)$ such that $V \subseteq Imf$.

Let's denote by $B = B(f(z_0), r)$, the open ball of center $f(z_0)$ and radius r, r standard such that $B \subseteq Im(f)$.

Let's take $\omega = f(z_1) \in B$, $z_1 \in U$, $|\omega| > |f(z_0)|$, $\omega \in B$. So we obtain $|f(z_1)| > |f(z_0)|$, $z_0 \in U$, $z_1 \in U$, a contradiction.

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