SEMI-SYMMETRIC METRICAL N - LINEAR CONNECTIONS IN THE K-OSCULATOR BUNDLE

Monica Purcaru and Emil Stoica

Abstract

The study of higher order Lagrange spaces based on the notion of k-osculator bundle was made by Radu Miron and Gheorghe Atanasiu in [2] - [5]. The applications of the Lagrange geometry of order k in Physics and Mechanics are quite numerous and important [7].

In this paper we introduce the concept of semi-symmetric metrical N-linear connections on the total space $E = \text{Osc}^k M$ as a straightforward extension of that on the 2-osculator bundle [8]. We determine all semi-symmetric metrical N-linear connections in the k-osculator bundle and we study the group of transformations of these connections and its invariants. This paper is a generalization of the same subject in the bundle of accelerations [8]. As to the terminology and notations we use those from [6], which are essentially based on M. Matsumoto’s book [1].

AMS Subject Classification: 53C05.
Key words: k - osculator bundle, curvature, torsion, semi-symmetric metrical N-linear connection.

1 Group of transformations of N - linear connections in the k-osculator bundle

Let $M$ be a real $n$ - dimensional $C^\infty$- manifold and $(\text{Osc}^k M, \pi, M), k \geq 1$ its k-osculator bundle. The local coordinates on the total space $E = \text{Osc}^k M$ are denoted by $(x^i, y^{(1)}_i, y^{(2)}_i, ..., y^{(k)}_i)$. If $N$ is a nonlinear connection on $E$ with the coefficients $N^i_{(1) j}, N^{(2)}_i j, ..., N^{(k)}_i j$, then let $D\Gamma(N) = (L^i_{jm}, C_{(1) jm}^i, C_{(2) jm}^i, ..., C_{(k) jm}^i)$ be an N-linear connection $D$ on $E = \text{Osc}^k M$. 
We consider a metrical d-structure on E, defined by a d-tensor field of the type \((0,2)\), marked as say \(g_{ij}(x^i, y^{(1)i}, \ldots, y^{(k)i})\). This d-tensor field is symmetric and non-degenerate.

Given a metrical d-structure \(g_{ij}\) on \(E\), we associate Obata’s operators:

\[
\Omega^{ij}_{s} = \frac{1}{2}(\delta^i_s \delta^r_j - g_{sj} g^{ir}), \quad \Omega^{ij}_{s} = \frac{1}{2}(\delta^i_s \delta^r_j + g_{sj} g^{ir}),
\]

where \((g^{ij})\) is the inverse matrix of \((g_{ij})\).

Obata’s operators have the same properties as the ones associated with a Finsler space [6].

The elements of the Abelian group \(T_N = \{(0,0, B_{jm}^1, D_{(1)jm}^i, D_{(2)jm}^i, \ldots, D_{(k)jm}) \in T\} \) are transformations \(t(0,0, B_{jm}^i, D_{(1)jm}^i, D_{(2)jm}^i, \ldots, D_{(k)jm}) : D\bar{\Gamma}(N) = (L_{jm}^i, C_{(1)jm}^i, C_{(2)jm}^i, \ldots, C_{(k)jm}^i) \rightarrow D\bar{\Gamma}(N) = (\bar{T}_{jm}^i, \bar{C}_{(1)jm}^i, \bar{C}_{(2)jm}^i, \ldots, \bar{C}_{(k)jm}^i) \) given by:

\[
N_{(\alpha)}^i_{j} = N_{(\alpha)}^i_{j}, \quad L_{jm}^i = L_{jm}^i - B_{jm}^i, \quad \bar{C}_{(\alpha)jm}^i = C_{(\alpha)jm}^i - D_{(\alpha)jm}^i, \quad (\alpha = 1, 2, \ldots, k).
\]

**Proposition 1.1** The transformation of the group \(T_N\), given by (1.2) leads to the transformation of the torsion and curvature d-tensor fields in the following way:

\[
\bar{T}_{(0\alpha) jm}^i = R_{(0\alpha) jm}^i, \quad (\alpha = 1, 2, \ldots, k),
\]

\[
\bar{T}_{(0) jm}^i = T_{(0) jm}^i + (B_{jm}^i - B_{jm}^i),
\]

\[
\bar{T}_{(0) jm}^i = S_{(\alpha) jm}^i + (D_{(\alpha)jm}^i - D_{(\alpha)jm}^i), \quad (\alpha = 1, 2, \ldots, k),
\]

\[
\bar{T}_{h jm}^i = R_{h jm}^i - \sum_{\gamma=1}^{k} D_{(\gamma)hs}^i R_{(0\gamma) jm}^s - B_{hs}^i T_{(0) jm}^s + A_{jm}\{ -B_{hjm}^i + B_{hs}^i B_{sm}^i \},
\]

\[
\bar{T}_{h jm}^i = P_{(\alpha) h jm}^i + L_{(\alpha) jm}^i D_{(\alpha) hs}^i - \sum_{\gamma=1}^{k} D_{(\gamma) hs}^i B_{(\alpha\gamma) jm}^i - B_{hs}^i C_{(\alpha) jm}^s - B_{hjm}^i + D_{(\alpha) jm}^i, \quad (\alpha = 1, 2, \ldots, k),
\]

\[
\bar{T}_{(\beta) h jm}^i = S_{(\beta\alpha) h jm}^i - C_{(\beta) jm}^s D_{(\alpha) hs}^i + C_{(\gamma)jm}^s D_{(\beta) hs}^i - D_{(\beta) hs}^i \mid_{(\beta) jm}^i + D_{(\beta) jm}^i \mid_{(\beta) jm}^i + D_{(\beta) hj}^s D_{(\beta) sm}^i - D_{(\beta) hj}^s D_{(\beta) jm}^i s - \sum_{\gamma=1}^{k} D_{(\gamma) hs}^i C_{(\alpha\beta) jm}^s,
\]

\[
\bar{T}_{(\beta) h jm}^i = S_{(\beta\alpha) h jm}^i + C_{(\beta) jm}^s D_{(\alpha) hs}^i + C_{(\gamma)jm}^s D_{(\beta) hs}^i - D_{(\beta) hs}^i \mid_{(\beta) jm}^i + D_{(\beta) hj}^s D_{(\beta) sm}^i + D_{(\beta) hj}^s D_{(\beta) jm}^i s - \sum_{\gamma=1}^{k} D_{(\gamma) hs}^i C_{(\alpha\beta) jm}^s,
\]
We shall consider the tensor fields:

\[(1.9)\quad K_h^i_{jm} = R_h^i_{jm} - \sum_{\gamma=1}^{k} C(\gamma)_h^i R(\gamma)_{jm},\]

\[(1.10)\quad P(\alpha)_h^i_{jm} = A_{jm} \{ P(\alpha)_h^i_{jm} - \sum_{\gamma=1}^{k} C(\gamma)_h^i B(\alpha)_{jm} \}, (\alpha = 1, 2, ..., k),\]

\[(1.11)\quad S(\beta\alpha)_h^i_{jm} = A_{jm} \{ S(\beta\alpha)_h^i_{jm} - \sum_{\gamma=1}^{k} C(\gamma)_h^i C(\beta\alpha)_{jm} \}, (\alpha, \beta = 1, 2, ..., k; \beta \leq \alpha).\]

**Proposition 1.2** By a transformation (1.2) the tensor fields \(K_h^i_{jm}, P(\alpha)_h^i_{jm}, S(\beta\alpha)_h^i_{jm}\), \((\alpha = 1, 2, ..., k; \beta \leq \alpha)\) are transformed according to the following laws:

\[(1.12)\quad \overline{K}_h^i_{jm} = K_h^i_{jm} - B_h^i T(0)_jm + A_{jm} \{-B^i_{hjm} + B^i_{hj} B^i_{sm}\},\]

\[(1.13)\quad \overline{P}(\alpha)_h^i_{jm} = P(\alpha)_h^i_{jm} - D(\alpha)_h^i T(0)_jm - B_h^i S(\alpha)_jm + A_{jm} \{-B^i_{hjm} \}^{(\alpha)} - D(\alpha)_h^i \} - B_h^i D(\alpha)_sm + D(\alpha)_h j \}^{(\alpha)} B^i_{sm}, (\alpha = 1, 2, ..., k),\]

\[(1.14)\quad \overline{S}(\beta\alpha)_h^i_{jm} = S(\beta\alpha)_h^i_{jm} - D(\alpha)_h^i S(\beta)_jm - D(\beta)_h^i S(\alpha)_jm + A_{jm} \{-D(\alpha)_h^i \}^{(\beta)} - D(\beta)_h^i \}^{(\alpha)} + D(\alpha)_h j D(\beta)_h j D(\alpha)_sm + D(\beta)_h j D(\alpha)_sm, (\alpha, \beta = 1, 2, ..., k; \beta \leq \alpha).\]

2 Semi-symmetric metrical \(N\)-linear connections in the k-osculator bundle

**Definition 2.1** An \(N\)-linear connection \(\Gamma(N) = \{ L^i_{jm}, C^i_{(1)}_{jm}, C^i_{(2)}_{jm}, ...\}, C^i_{(k)}_{jm}\) on \(E = \text{Osc}^k M\), with the property:

\[(2.1)\quad g_{ij|m} = 0, \quad g_{ij|^{(\alpha)}_m} = 0, \quad (\alpha = 1, 2, ..., k)\]

is said to be a metrical \(N\)-linear connection on \(E = \text{Osc}^k M, k > 2\).

A class of metrical \(N\)-linear connections, which have interesting properties is that of semi-symmetric metrical \(N\)-linear connections.
Definition 2.2 An $N$-linear connection $D\Gamma(N) = (L^i_{jm}, C_{(1)ijm}, C_{(2)ijm}, \ldots, C_{(k)ijm})$ on $E$ is called semi-symmetric if the torsion $d$-tensor fields $T^i_{(0)jm}$, $S^i_{(\alpha)jm}$, $(\alpha = 1, 2, \ldots, k)$ have the form:

\begin{equation}
T^i_{(0)jm} = \frac{1}{n-1}(T^i_{(0)}\delta^i_m - T^i_{(0)}\delta^i_j) = \frac{1}{n-1}A_{jm}\{T^i_{(0)}\delta^i_m\},
\end{equation}

\begin{equation}
S^i_{(\alpha)jm} = \frac{1}{n-1}(S_{(\alpha)}\delta^i_m - S_{(\alpha)}\delta^i_j) = \frac{1}{n-1}A_{jm}\{S_{(\alpha)}\delta^i_m\}, (\alpha = 1, 2, \ldots, k),
\end{equation}

where $T^i_{(0)j} = T^i_{(0)}j$, $S^i_{(\alpha)j}$, $(\alpha = 1, 2, \ldots, k)$.

Definition 2.3 An $N$-linear connection $D\Gamma(N) = (L^i_{jm}, C_{(1)ijm}, C_{(2)ijm}, \ldots, C_{(k)ijm})$ on $E$ is called a semi-symmetric metrical $N$-linear connection, if the relations (2.1) and (2.2) are verified.

If $\sigma_j = \frac{T^i_{(0)j}}{n-1}$, $(\alpha = 1, 2, \ldots, k)$ and if we apply the Theorem 5.4.3, [7] we obtain:

Theorem 2.1 The set of all semi-symmetric metrical $N$-linear connections on $E$, which preserve the nonlinear connection $N$, $D\Gamma(N) = (L^i_{jm}, C_{(1)ijm}, C_{(2)ijm}, \ldots, C_{(k)ijm})$ is given by:

\begin{equation}
L^i_{jm} = L^i_{jm} + \sigma_j\delta^i_m - g_{jm}g^{i\text{s}}\sigma_s,
\end{equation}

\begin{equation}
C_{(\alpha)jm} = C_{(\alpha)jm} + \tau_{(\alpha)}\delta^i_m - g_{jm}g^{i\text{s}}\tau_{(\alpha)s}, (\alpha = 1, 2, \ldots, k),
\end{equation}

where $D$ $\Gamma(N) = (L^i_{jm}, C_{(1)ijm}, C_{(2)ijm}, \ldots, C_{(k)ijm})$ is an arbitrary fixed semi-symmetric metrical $N$-linear connection on $E$.

On notices that (2.3) gives the transformations of the semi-symmetric metrical $N$-linear connections on $E$, which preserve the nonlinear connection $N$.

Let $t(\sigma_j, \tau_{(\alpha)}) : D\Gamma(N) \rightarrow D\Gamma(N)$, $(\alpha = 1, 2, \ldots, k)$ be a transformation of this form. It is given by:

\begin{equation}
\bar{T}^i_{jm} = L^i_{jm} + \sigma_j\delta^i_m - g_{jm}g^{i\text{s}}\sigma_s,
\end{equation}

\begin{equation}
\bar{C}_{(\alpha)jm} = C_{(\alpha)jm} + \tau_{(\alpha)}\delta^i_m - g_{jm}g^{i\text{s}}\tau_{(\alpha)s}, (\alpha = 1, 2, \ldots, k).
\end{equation}

Theorem 2.2 The set $\hat{\Gamma}_N$ of all transformations $t(\sigma_j, \tau_{(1)j}, \tau_{(2)j}, \ldots, \tau_{(k)j})$ of the semi-symmetric metrical $N$-linear connections, on $E$, given by (2.4), together with the mapping product, is an Abelian group. This group acts effectively on the set of all $N$-linear connections on $E$. 


By applying the result from §1, one obtains:

**Theorem 2.3** By means of transformations (2.4) the tensor fields $K_{h \ j \ m}^i$, $P_{(\alpha)\ h \ j \ m}^i$, $S_{(\alpha\alpha)\ h \ j \ m}^i$, $S_{(\alpha\alpha\alpha)\ h \ j \ m}^i$, $(\alpha = 1, 2, ..., k)$ are changing by the laws:

\begin{align*}
(2.5) \quad & \mathcal{K}_{h \ j \ m}^i = K_{h \ j \ m}^i + 2\mathcal{A}_{j m}^i \{\Omega_{j h}^r \sigma_{r m}\}, \\
(2.6) \quad & \mathcal{P}_{(\alpha)\ h \ j \ m}^i = \mathcal{P}_{(\alpha)\ h \ j \ m}^i + 2\mathcal{A}_{j m}^i \{\Omega_{j h}^r \rho_{(\alpha) r m}\}, \quad (\alpha = 1, 2, ..., k), \\
(2.7) \quad & \mathcal{S}_{(\alpha\alpha)\ h \ j \ m}^i = S_{(\alpha\alpha)\ h \ j \ m}^i + 2\mathcal{A}_{j m}^i \{\Omega_{j h}^r \tau_{(\alpha\alpha) r m}\} + \sum_{\gamma=1}^{k} 2\Omega_{j h}^r \tau_{(\alpha) r m} C_{(\alpha) \ j \ m}^{(\gamma)} s, \\
& \quad (\alpha, \gamma = 1, 2, ..., k) \text{ and } C_{(\alpha) \ j \ m}^{(\alpha)} = 0, \\
(2.8) \quad & \mathcal{S}_{(\alpha\alpha\alpha)\ h \ j \ m}^i = S_{(\alpha\alpha\alpha)\ h \ j \ m}^i + 4\mathcal{A}_{j m}^i \{\Omega_{j h}^r \tau_{(\alpha\alpha\alpha) r m}\}, \quad (\alpha = 1, 2, ..., k),
\end{align*}

\begin{align*}
(2.9) \quad & \sigma_{r m} = \sigma_{r | m} - \sigma_{r} \sigma_{m} + \frac{1}{2} g_{r m} \sigma - \frac{\sigma \tau_{(\alpha) m}}{n-1}, \quad (\sigma = g^{rs} \sigma_{r} \sigma_{s}), \\
(2.10) \quad & \rho_{(\alpha) r m} = \sigma_{r | m}^{(\alpha)} + \tau_{(\alpha) r | m} - (\sigma_{r} \tau_{(\alpha) m} + \sigma_{m} \tau_{(\alpha) r}) + g_{r m} \rho_{(\alpha)} - \\
& \quad - \frac{\tau_{(\alpha) r} \tau_{(\alpha) m} + \sigma_{r} S_{(\alpha) m}}{n-1}, \quad (\rho_{(\alpha)} = g^{rs} \tau_{(\alpha) r} \sigma_{s}), \quad (\alpha = 1, 2, ..., k), \\
(2.11) \quad & \tau_{(\alpha\alpha) r m} = \tau_{(\alpha) r | m}^{(\alpha)} - \tau_{(\alpha) r} \tau_{(\alpha) m} + \frac{1}{2} g_{r m} \tau_{(\alpha\alpha)} - \frac{\tau_{(\alpha) \tau_{(\alpha) m}}}{n-1}, \\
& \quad (\tau_{(\alpha\alpha)} = g^{rs} \tau_{(\alpha) r} \tau_{(\alpha) s}), \quad (\alpha = 1, 2, ..., k).
\end{align*}

Using these results we can determine some invariants of the group $\mathcal{T}_N$. To this aim we shall eliminate $\sigma_{ij}$, $\rho_{(\alpha) ij}$, $\tau_{(\alpha\alpha) ij}$ from (2.5), (2.6), (2.8).

**Theorem 2.4** Let $n > 2$. The semi-symmetric metrical $N$-linear connection determines the following tensor fields:

\begin{align*}
(2.12) \quad & H_{h \ j \ m}^i = K_{h \ j \ m}^i + \frac{2}{n-2} \mathcal{A}_{j m}^i \{\Omega_{j h}^r (K_{r m} - \frac{K_{g_{r m}}}{2(n-1)})\},
\end{align*}
(2.13) \[ N^i_{(\alpha)hjm} = P^i_{(\alpha)hjm} + \frac{2}{n-2} A_{jm} \{ \Omega^{ir}_{jh} (P^i_{(\alpha)rm} - \frac{P^i_{(\alpha)g_{rm}}}{2(n-1)}) \}, \quad (\alpha = 1, 2, \ldots, k), \]

(2.14) \[ M^i_{(\alpha\alpha)hjm} = S^i_{(\alpha\alpha)hjm} + \frac{4}{2n-3} A_{jm} \{ \Omega^{ir}_{jh} (S^i_{(\alpha\alpha)rm} - \frac{2S^i_{(\alpha\alpha)g_{rm}}}{3(n-1)}) \}, \quad (\alpha = 1, 2, \ldots, k), \]

where

\[ K_{hj} = K^i_{hji}, \quad P^i_{(\alpha)hj} = P^i_{(\alpha)hji}, \quad S^i_{(\alpha\alpha)hj} = S^i_{(\alpha\alpha)hji}, \quad K = g^{hi} K_{hj}; \]

\[ P^i_{(\alpha)} = g^{hi} P^i_{(\alpha)hj}, \quad S^i_{(\alpha\alpha)} = g^{hi} S^i_{(\alpha\alpha)hj}, \quad (\alpha = 1, 2, \ldots, k). \]

These tensor fields are invariants of the group \( T^s_N \).

References


Authors’ address:

Monica Purcaru and Emil Stoica

*University Transilvania, Department of Geometry*

*Iuliu Maniu 50, 2200, Brașov, ROMANIA*

e-mail: mpurcaru@unitbv.ro