SEMI-SYMMETRIC METRICAL N - LINEAR CONNECTIONS IN THE K-OSCULATOR BUNDLE

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Abstract

The study of higher order Lagrange spaces based on the notion of k-osculator bundle was made by Radu Miron and Gheorghe Atanasiu in [2] - [5]. The applications of the Lagrange geometry of order k in Physics and Mechanics are quite numerous and important [7].

In this paper we introduce the concept of semi-symmetric metrical N - linear connections on the total space $E = Osc^k M$ as a straightforward extension of that on the 2-osculator bundle [8]. We determine all semi-symmetric metrical N-linear connections in the k-osculator bundle and we study the group of transformations of these connections and its invariants. This paper is a generalization of the same subject in the bundle of accelerations [8]. As to the terminology and notations we use those from [6], which are essentially based on M.Matsumoto's book [1].

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Key words: k - osculator bundle, curvature, torsion, semi-symmetric metrical N-linear connection.

1 Group of transformations of N - linear connections in the k-osculator bundle

Let M be a real n - dimensional C^∞- manifold and $(Osc^k M, \pi, M), k \geq 1$ its k-osculator bundle. The local coordinates on the total space $E = Osc^k M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i}, ..., y^{(k)i})$. If N is a nonlinear connection on E with the coefficients $N_{(1)}{}^i{}_j$, $N_{(2)}{}^i{}_j$,...,$N_{(k)}{}^i{}_j$, then let $D\Gamma(N) = (L^i_{jm}, C_{(1)}{}^i_{jm}, C_{(2)}{}^i_{jm}, ..., C_{(k)}{}^i_{jm})$ be an N-linear connection D on $E = Osc^k M$.

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We consider a metrical d-structure on E, defined by a d-tensor field of the type (0,2), marked as say $g_{ij}(x^i, y^{(1)i}, ..., y^{(k)i})$. This d-tensor field is symmetric and non-degenerate.

Given a metrical d-structure g_{ij} on E, we associate Obata's operators:

$$\text{(1.1)} \quad \Omega^{ir}_{sj} = \tfrac{1}{2}(\delta^i_s \delta^r_j - g_{sj} g^{ir}),\ \Omega^{*ir}_{sj} = \tfrac{1}{2}(\delta^i_s \delta^r_j + g_{sj} g^{ir}),$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

Obata's operators have the same properties as the ones associated with a Finsler space [6].

The elements of the Abelian group $\mathcal{T}_N = \{t(0, 0, B^i_{jm}, D_{(1)}{}^i_{jm}, D_{(2)}{}^i_{jm}, ..., D_{(k)}{}^i_{jm}) \in \mathcal{T}\}$ are transformations $t(0, 0, B^i_{jm}, D_{(1)}{}^i_{jm}, D_{(2)}{}^i_{jm}, ..., D_{(k)}{}^i_{jm}) : D\Gamma(N) = (L^i_{jm}, C_{(1)}{}^i_{jm}, C_{(2)}{}^i_{jm}, ..., C_{(k)}{}^i_{jm}) \to D\overline{\Gamma}(N) = (\overline{L}^i_{jm}, \overline{C}_{(1)}{}^i_{jm}, \overline{C}_{(2)}{}^i_{jm}, ..., \overline{C}_{(k)}{}^i_{jm})$ given by:

$$\text{(1.2)} \quad \overline{N}_{(\alpha)}{}^i{}_j = N_{(\alpha)}{}^i{}_j,\ \overline{L}^i_{jm} = L^i_{jm} - B^i_{jm},\ \overline{C}_{(\alpha)}{}^i_{jm} = C_{(\alpha)}{}^i_{jm} - D_{(\alpha)}{}^i_{jm},$$
$$(\alpha = 1, 2, ..., k).$$

Proposition 1.1 *The transformation of the group $\mathcal{T}_N$, given by (1.2) leads to the transformation of the torsion and curvature d-tensor fields in the following way:*

$$\text{(1.3)} \quad \overline{R}_{(0\alpha)}{}^i{}_{jm} = R_{(0\alpha)}{}^i{}_{jm}, \quad (\alpha = 1, 2, ..., k),$$

$$\text{(1.4)} \quad \overline{T}_{(0)}{}^i{}_{jm} = T_{(0)}{}^i{}_{jm} + (B^i_{mj} - B^i_{jm}),$$

$$\text{(1.5)} \quad \overline{S}_{(\alpha)}{}^i{}_{jm} = S_{(\alpha)}{}^i{}_{jm} + (D_{(\alpha)}{}^i_{mj} - D_{(\alpha)}{}^i_{jm}), (\alpha = 1, 2, ..., k),$$

$$\text{(1.6)} \quad \overline{R}_h{}^i{}_{jm} = R_h{}^i{}_{jm} - \sum_{\gamma=1}^{k} D_{(\gamma)}{}^i_{hs} R_{(0\gamma)}{}^s{}_{jm} - B^i_{hs} T_{(0)}{}^s{}_{jm} +$$
$$+\mathcal{A}_{jm}\{-B^i_{hj|m} + B^s_{hj} B^i_{sm}\},$$

$$\text{(1.7)} \quad \overline{P}_{(\alpha)h}{}^i{}_{jm} = P_{(\alpha)h}{}^i{}_{jm} + L^s_{mj} D_{(\alpha)}{}^i_{hs} - \sum_{\gamma=1}^{k} D_{(\gamma)}{}^i_{hs} B_{(\alpha\gamma)}{}^s{}_{jm} - B^i_{hs} C_{(\alpha)}{}^s_{jm} -$$
$$-B^i_{hj} \mid^{(\alpha)}_m + D_{(\alpha)}{}^i_{hm|j} + B^s_{hj} D_{(\alpha)}{}^i_{sm} - D_{(\alpha)}{}^s_{hm} B^i_{sj}, (\alpha = 1, 2, ..., k),$$

$$\text{(1.8)} \quad \overline{S}_{(\beta\alpha)h}{}^i{}_{jm} = S_{(\beta\alpha)h}{}^i{}_{jm} - C_{(\beta)}{}^s_{jm} D_{(\alpha)}{}^i_{hs} + C_{(\alpha)}{}^s_{mj} D_{(\beta)}{}^i_{hs} - D_{(\alpha)}{}^i_{hj} \mid^{(\beta)}_m +$$
$$+D_{(\beta)}{}^i_{hm} \mid^{(\alpha)}_j + D_{(\alpha)}{}^s_{hj} D_{(\beta)}{}^i_{sm} - D_{(\beta)}{}^s_{hm} D_{(\alpha)}{}^i_{sj} - \sum_{\gamma=1}^{k} D_{(\gamma)}{}^i_{hs} C^{(\gamma)}_{(\alpha\beta)}{}^s{}_{jm},$$

$$(\alpha, \beta = 1, 2, ..., k; \beta \leq \alpha).$$

We shall consider the tensor fields:

$$(1.9) \quad K_{h\ jm}^{\ i} = R_{h\ jm}^{\ i} - \sum_{\gamma=1}^{k} C_{(\gamma)hs}^{\ \ i} R_{(0\gamma)\ jm}^{\ \ \ s},$$

$$(1.10) \quad \mathcal{P}_{(\alpha)h\ jm}^{\ \ \ i} = \mathcal{A}_{jm}\{P_{(\alpha)h\ jm}^{\ \ \ i} - \sum_{\gamma=1}^{k} C_{(\gamma)hs}^{\ \ i} B_{(\alpha\gamma)\ jm}^{\ \ \ \ s}\}, (\alpha = 1, 2, ..., k),$$

$$(1.11) \quad \mathcal{S}_{(\beta\alpha)h\ jm}^{\ \ \ \ i} = \mathcal{A}_{jm}\{S_{(\beta\alpha)h\ jm}^{\ \ \ \ i} - \sum_{\gamma=1}^{k} C_{(\gamma)hs}^{\ \ i} C_{(\alpha\beta)\ jm}^{(\gamma)\ \ s}\}, (\alpha, \beta = 1, 2, ..., k;$$

$$\beta \leq \alpha; C_{(\alpha\alpha)}^{(\alpha)} = 0).$$

Proposition 1.2 *By a transformation (1.2) the tensor fields $K_{h\ jm}^{\ i}$, $\mathcal{P}_{(\alpha)h\ jm}^{\ \ \ i}$, $(\alpha = 1, 2, ..., k)$ $\mathcal{S}_{(\beta\alpha)h\ jm}^{\ \ \ \ i}$, $(\alpha, \beta = 1, 2, ..., k; \beta \leq \alpha)$ are transformed according to the following laws:*

$$(1.12) \quad \overline{K}_{h\ jm}^{\ i} = K_{h\ jm}^{\ i} - B_{hs}^{i} T_{(0)\ jm}^{\ \ s} + \mathcal{A}_{jm}\{-B_{hj|m}^{i} + B_{hj}^{s} B_{sm}^{i}\},$$

$$(1.13) \quad \overline{\mathcal{P}}_{(\alpha)h\ jm}^{\ \ \ i} = \mathcal{P}_{(\alpha)h\ jm}^{\ \ \ i} - D_{(\alpha)hs}^{\ \ i} T_{(0)\ jm}^{\ \ s} - B_{hs}^{i} S_{(\alpha)\ jm}^{\ \ s} + \mathcal{A}_{jm}\{-B_{hj}^{i} \,|_{m}^{(\alpha)} -$$

$$-D_{(\alpha)hj|m}^{\ \ i} + B_{hj}^{s} D_{(\alpha)sm}^{\ \ i} + D_{(\alpha)hj}^{\ \ s} B_{sm}^{i}\}, \quad (\alpha = 1, 2, ..., k),$$

$$(1.14) \; \overline{\mathcal{S}}_{(\beta\alpha)h\ jm}^{\ \ \ \ i} = \mathcal{S}_{(\beta\alpha)h\ jm}^{\ \ \ \ i} - D_{(\alpha)hs}^{\ \ i} S_{(\beta)\ jm}^{\ \ s} - D_{(\beta)hs}^{\ \ i} S_{(\alpha)\ jm}^{\ \ s} + \mathcal{A}_{jm}\{-D_{(\alpha)hj}^{\ \ i} \,|_{m}^{(\beta)}$$

$$-D_{(\beta)hj}^{\ \ i} \,|_{m}^{(\alpha)} + D_{(\alpha)hj}^{\ \ s} D_{(\beta)sm}^{\ \ i} + D_{(\beta)hj}^{\ \ s} D_{(\alpha)sm}^{\ \ i}\}, \quad (\alpha, \beta = 1, 2, ..., k; \beta \leq \alpha).$$

2 Semi-symmetric metrical N- linear connections in the k-osculator bundle

Definition 2.1 *An $N-$ linear connection $D\Gamma(N) = (L_{jm}^{i}, C_{(1)jm}^{\ \ i}, C_{(2)jm}^{\ \ i}, ..., C_{(k)jm}^{\ \ i})$ on $E = Osc^k M$, with the property:*

$$(2.1) \quad g_{ij|m} = 0, \quad g_{ij}|_{m}^{(\alpha)} = 0, \quad (\alpha = 1, 2, ..., k)$$

is said to be a metrical $N-$ linear connection on $E = Osc^k M, k > 2$.

A class of metrical $N-$ linear connections, which have interesting properties is that of semi-symmetric metrical $N-$ linear connections.

Definition 2.2 *An $N-$ linear connection $D\Gamma(N) = (L^i_{jm}, C_{(1)}{}^i_{jm}, C_{(2)}{}^i_{jm}, ..., C_{(k)}{}^i_{jm})$ on E is called semi-symmetric if the torsion d-tensor fields $T_{(0)}{}^i{}_{jm}$, $S_{(\alpha)}{}^i{}_{jm}$, $(\alpha = 1, 2, ..., k)$ have the form:*

$$
\begin{aligned}
&T_{(0)}{}^i{}_{jm} = \tfrac{1}{n-1}(T_{(0)j}\delta^i_m - T_{(0)m}\delta^i_j) = \tfrac{1}{n-1}\mathcal{A}_{jm}\{T_{(0)j}\delta^i_m\},\\
&S_{(\alpha)}{}^i{}_{jm} = \tfrac{1}{n-1}(S_{(\alpha)j}\delta^i_m - S_{(\alpha)m}\delta^i_j) = \tfrac{1}{n-1}\mathcal{A}_{jm}\{S_{(\alpha)j}\delta^i_m\}, (\alpha = 1, 2, ..., k),
\end{aligned}
\tag{2.2}
$$

where $T_{(0)j} = T_{(0)}{}^i{}_{ji}$, $S_{(\alpha)j} = S_{(\alpha)}{}^i{}_{ji}$, $(\alpha = 1, 2, ..., k)$.

Definition 2.3 *An $N-$ linear connection $D\Gamma(N) = (L^i_{jm}, C_{(1)}{}^i_{jm}, C_{(2)}{}^i_{jm}, ..., C_{(k)}{}^i_{jm})$ on E is called a semi-symmetric metrical N-linear connection, if the relations (2.1) and (2.2) are verified.*

If $\sigma_j = \frac{T_{(0)j}}{n-1}$, $\tau_{(\alpha)j} = \frac{S_{(\alpha)j}}{n-1}$, $(\alpha = 1, 2, ..., k)$ and if we apply the Theorem 5.4.3, [7] we obtain:

Theorem 2.1 *The set of all semi-symmetric metrical N-linear connections on E, which preserve the nonlinear connection N, $D\Gamma(N) = (L^i_{jm}, C_{(1)}{}^i_{jm}, C_{(2)}{}^i_{jm}, ..., C_{(k)}{}^i_{jm})$ is given by:*

$$
\begin{aligned}
&L^i_{jm} = \overset{0}{L}{}^i_{jm} + \sigma_j\delta^i_m - g_{jm}g^{is}\sigma_s,\\
&C_{(\alpha)}{}^i_{jm} = \overset{0}{C}{}_{(\alpha)}{}^i_{jm} + \tau_{(\alpha)j}\delta^i_m - g_{jm}g^{is}\tau_{(\alpha)s}, \quad (\alpha = 1, 2, ..., k),
\end{aligned}
\tag{2.3}
$$

where $D\overset{0}{\Gamma}(N) = (\overset{0}{L}{}^i_{jm}, \overset{0}{C}{}_{(1)}{}^i_{jm}, \overset{0}{C}{}_{(2)}{}^i_{jm}, ..., \overset{0}{C}{}_{(k)}{}^i_{jm})$ is an arbitrary fixed semi-symmetric metrical $N-$ linear connection on E.

On notices that (2.3) gives the transformations of the semi-symmetric metrical $N-$ linear connections on E, which preserve the nonlinear connection N.

Let $t(\sigma_j, \tau_{(\alpha)j}) : D\Gamma(N) \to D\overline{\Gamma}(N)$, $(\alpha = 1, 2, ..., k)$ be a transformation of this form. It is given by:

$$
\begin{aligned}
&\overline{L}^i_{jm} = L^i_{jm} + \sigma_j\delta^i_m - g_{jm}g^{is}\sigma_s,\\
&\overline{C}_{(\alpha)}{}^i_{jm} = C_{(\alpha)}{}^i_{jm} + \tau_{(\alpha)j}\delta^i_m - g_{jm}g^{is}\tau_{(\alpha)s}, \quad (\alpha = 1, 2, ..., k).
\end{aligned}
\tag{2.4}
$$

Theorem 2.2 *The set $\overset{s}{\mathcal{T}}_N$ of all transformations $t(\sigma_j, \tau_{(1)j}, \tau_{(2)j}, ..., \tau_{(k)j})$ of the semi-symmetric metrical $N-$ linear connections,on E, given by (2.4), together with the mapping product, is an Abelian group. This group acts effectively on the set of all $N-$ linear connections on E.*

By applying the result from §1, one obtains:

Theorem 2.3 *By means of transformations (2.4) the tensor fields* $K_h{}^i{}_{jm}$, $\mathcal{P}_{(\alpha)h}{}^i{}_{jm}$, $S_{(\alpha\alpha)h}{}^i{}_{jm}$, $\mathcal{S}_{(\alpha\alpha)h}{}^i{}_{jm}$, $(\alpha = 1, 2, ..., k)$ *are changing by the laws:*

$$\overline{K}_h{}^i{}_{jm} = K_h{}^i{}_{jm} + 2\mathcal{A}_{jm}\{\Omega^{ir}_{jh}\sigma_{rm}\}, \tag{2.5}$$

$$\overline{\mathcal{P}}_{(\alpha)h}{}^i{}_{jm} = \mathcal{P}_{(\alpha)h}{}^i{}_{jm} + 2\mathcal{A}_{jm}\{\Omega^{ir}_{jh}\rho_{(\alpha)rm}\}, \quad (\alpha = 1, 2, ..., k), \tag{2.6}$$

$$\overline{S}_{(\alpha\alpha)h}{}^i{}_{jm} = S_{(\alpha\alpha)h}{}^i{}_{jm} + 2\mathcal{A}_{jm}\{\Omega^{ir}_{jh}\tau_{(\alpha\alpha)rm}\} + \sum_{\gamma=1}^{k} 2\Omega^{ir}_{sh}\tau_{(\alpha)r}C^{(\gamma)}_{(\alpha\alpha)}{}^s{}_{jm}, \tag{2.7}$$

$$(\alpha, \gamma = 1, 2, ..., k) \text{ and } C^{(\alpha)}_{(\alpha\alpha)}{}^i{}_{jm} = 0,$$

$$\overline{\mathcal{S}}_{(\alpha\alpha)h}{}^i{}_{jm} = \mathcal{S}_{(\alpha\alpha)h}{}^i{}_{jm} + 4\mathcal{A}_{jm}\{\Omega^{ir}_{jh}\tau_{(\alpha\alpha)rm}\}, (\alpha = 1, 2, ..., k), \tag{2.8}$$

$$\sigma_{rm} = \sigma_{r|m} - \sigma_r\sigma_m + \tfrac{1}{2}g_{rm}\sigma - \tfrac{\sigma_r T_{(0)m}}{n-1}, \ (\sigma = g^{rs}\sigma_r\sigma_s), \tag{2.9}$$

$$\rho_{(\alpha)rm} = \sigma_r|^{(\alpha)}_m + \tau_{(\alpha)r|m} - (\sigma_r\tau_{(\alpha)m} + \sigma_m\tau_{(\alpha)r}) + g_{rm}\rho_{(\alpha)} - \tag{2.10}$$

$$-\tfrac{\tau_{(\alpha)r}T_{(0)m} + \sigma_r S_{(\alpha)m}}{n-1}, \quad (\rho_{(\alpha)} = g^{rs}\tau_{(\alpha)r}\sigma_s), \quad (\alpha = 1, 2, ..., k),$$

$$\tau_{(\alpha\alpha)rm} = \tau_{(\alpha)r}|^{(\alpha)}_m - \tau_{(\alpha)r}\tau_{(\alpha)m} + \tfrac{1}{2}g_{rm}\tau_{(\alpha\alpha)} - \tfrac{\tau_{(\alpha)r}S_{(\alpha)m}}{n-1}, \tag{2.11}$$

$$(\tau_{(\alpha\alpha)} = g^{rs}\tau_{(\alpha)r}\tau_{(\alpha)s}), \quad (\alpha = 1, 2, ..., k).$$

Using these results we can determine some invariants of the group $\overset{s}{\mathcal{T}}_N$. To this aim we shall eliminate σ_{ij}, $\rho_{(\alpha)ij}$, $\tau_{(\alpha\alpha)ij}$ from (2.5), (2.6), (2.8).

Theorem 2.4 *Let* $n > 2$. *The semi-symmetric metrical N-linear connection determines the following tensor fields:*

$$H_h{}^i{}_{jm} = K_h{}^i{}_{jm} + \tfrac{2}{n-2}\mathcal{A}_{jm}\{\Omega^{ir}_{jh}(K_{rm} - \tfrac{K g_{rm}}{2(n-1)})\}, \tag{2.12}$$

(2.13) $N_{(\alpha)h\ jm}^{\ \ \ \ i} = \mathcal{P}_{(\alpha)h\ jm}^{\ \ \ \ i} + \frac{2}{n-2}\mathcal{A}_{jm}\{\Omega_{jh}^{ir}(\mathcal{P}_{(\alpha)rm} - \frac{\mathcal{P}_{(\alpha)}g_{rm}}{2(n-1)})\},\ (\alpha = 1, 2, ..., k),$

(2.14) $M_{(\alpha\alpha)h\ jm}^{\ \ \ \ \ \ i} = \mathcal{S}_{(\alpha\alpha)h\ jm}^{\ \ \ \ \ \ i} + \frac{4}{2n-3}\mathcal{A}_{jm}\{\Omega_{jh}^{ir}(\mathcal{S}_{(\alpha\alpha)rm} - \frac{2\mathcal{S}_{(\alpha\alpha)}g_{rm}}{3(n-1)})\},$

$(\alpha = 1, 2, ..., k),$

where

$K_{hj} = K_{h\ ji}^{\ i},\ \mathcal{P}_{(\alpha)hj} = \mathcal{P}_{(\alpha)h\ ji}^{\ \ \ \ i},\ \mathcal{S}_{(\alpha\alpha)hj} = \mathcal{S}_{(\alpha\alpha)h\ ji}^{\ \ \ \ \ \ i},\ K = g^{hj}K_{hj},$

$\mathcal{P}_{(\alpha)} = g^{hj}\mathcal{P}_{(\alpha)hj},\ \mathcal{S}_{(\alpha\alpha)} = g^{hj}\mathcal{S}_{(\alpha\alpha)hj},\ (\alpha = 1, 2, ..., k).$

These tensor fields are invariants of the group $\overset{s}{\mathcal{T}}_N$.

References

[1] M.Matsumoto, *The Theory of Finsler Connections*, Publ.Study Group Geom 5, Depart.Math.,Okayama Univ., 1970.

[2] R.Miron and Gh.Atanasiu,*Differential geometry of the k- osculator bundle*, Rev.Roumaine Math.Pures Appl., 41, 3/4 (1996) 205-236.

[3] R.Miron and Gh.Atanasiu, *Prolongation of Riemannian, Finslerian and Lagrangian structures,* Rev.Roumaine Math.Pures Appl., 41, 3/4 (1996) 237-249.

[4] R.Miron and Gh.Atanasiu, *Higher-order Lagrange spaces,* Rev.Roumaine Math.Pures Appl., 41, 3/4 (1996) 251-262.

[5] R.Miron and Gh.Atanasiu, *Compendium sur les espaces Lagrange d'ordre supérior*, Seminarul de Mecanica no.40, Universitatea din Timişoara, (1994) 1-27.

[6] R.Miron and M.Hashiguchi, *Metrical Finsler Connections*, Rep.Fac.Sci., Kagoshima Univ (Math., Phys.&Chem.), 12 (1979) 21-35.

[7] R.Miron, *The Geometry of Higher-order Lagrange Spaces. Applications in Mechanics and Physics*, Kluwer Academic Publishers , FTPH82, 1997.

[8] M.Purcaru, *Metric Semi-Symmetric N-Linear Connections in the Bundle of Accelerations*, Balkan Journ.of Geometry and Its Applications, vol.2, 1997 113-118.

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