GENERALIZED FERMI- WALKER PARALLELISM INDUCED BY GENERALIZED SCHOUTEN CONNECTIONS

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Abstract

We define a rich family of induced linear connections along observers on arbitrary spacetimes, by analogy with the extension of Schouten's connection from the non-holonomic geometrization of differential systems (Chevalley distributions).In particular, for accelerating observers we recover the classical Fermi-Walker transport. We illustrate our construction with some exotic gyroscope precessions in the Schwarzschild spacetime. The same topics are resumed for observer fields (alias "reference frames"), by means of specific Lorentzian almost product structures on the spacetime. By these methods, we can define *proper* parallel transports for each (Frenet frame owner) observer, and canonical *proper* connections for each (similar) observer field.

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1 Introduction

The Universe is perceptible through observation. A relativistic observer γ needs reference frames, for measurements of durations and of ("geo")metric quantities: the proper time ("proper clock") is given by its own canonical parameter running on an interval of the real numbers axis; the restspaces are refer ed to "fixed" directions, maintained by gyroscopes focused toward "fixed" celestial bodies.

In Special Relativity, the term "fix" is quite clear. On arbitrary spacetimes, the respective notion must be covariant; if γ is freely falling, its restspaces are transported through Levi-Civita parallelism, so a fix spacelike direction has, by definition, a null covariant derivative. For accelerated observers, the restspaces are not transported by

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the Levi-Civita parallelism, anymore. Then, in order to define "constant" directions, one uses the (classical) Fermi-Walker parallelism ([1], [2], [10], [14], [15]), which is an isometry between the tangent spaces along γ.

Several extensions of the classical Fermi-Walker transport, with various physical motivations, are known (see [3], [4] and references therein). In [8], we enlarged the context by defining a rich class of generalized Fermi-Walker connections (which includes all the previous quoted ones) and we pleaded for the relativity of the Fermi-Walker-like transport. This means that an accelerating observer γ must be able to choose between several parallel transports and not resume itself to the classical Fermi-Walker one. Such "proper" parallel transport may be useful - for example - when an accelerating observer γ cannot find "fixed stars" (i.e. there might be no remarquable directions along γ), parallel with respect to the *classical* Fermi-Walker connection.

Another reason for enlarging the theory is that the classical Fermi-Walker transport is an "absolute" one, forcing different accelerating observers to agree on the same method of "proper frame construction" (and thus disturbing the spirit of the Principle of Relativity).

This paper has two aims:

(i) first, we prove that our generalized Fermi-Walker transport is very flexible: given a spacelike frame along an accelerating observer, there exists a unique generalized Fermi-Walker connection along it which makes the velocity and the three directions parallel (§2). Thus, each accelerating observer may choose its own favourite "fixed" three directions by its own parallel transport.

In particular, if an observer γ admits a Frenet frame, he can choose a *unique* parallel transport $\nabla^{(\gamma)}$, which parallelizates this frame; thus, the Frenet frame may be designed as *fixed* frame for γ, and $\nabla^{(\gamma)}$ may be interpreted as *proper* parallelism for γ.

As an application, we determine (§2) the generalized Fermi-Walker transports and some "proper" frames of Schwarzschid observers.

(ii) For an observer field (alias "reference frame" in [10]), we construct a Lorentzian almost product structure, with canonical complementary orthogonal (timelike and spacelike) distributions. First, we point out that the classical Schouten connection associated to this structure induces exactly the classical Fermi-Walker transport along each particular observer.

But the Schouten connection is by no means the only linear connection able to deal with the almost product geometric properties. On arbitrary affine almost product manifolds, we extended the Schouten connection, obtaining a large family of remarkable linear connections adapted to a couple of complementary distributions ([6], [7]). Particularizing for the previous canonical almost product structure on spacetimes, we prove how these generalized Schouten connections induce the generalized Fermi-Walker transports previously quoted, along each observer (§3).

Corresponding to the classical Vranceanu connection, a specific generalized Fermi-Walker parallelism may be constructed [9]; we recall the main steps and give it a new application in a Robertson- Walker spacetime.

2 Generalized Fermi- Walker parallelism

Consider a spacetime (M, g), with signature convention (-, +, +, +). (It is worth noting that all our constructions and results easily extend for time-oriented Lorentzian manifolds of arbitrary dimension). An *observer* is a proper time parametrized curve $\gamma : I \to M$, with temporal and future oriented velocity vectors. We denote by ∇ the Levi-Civita connection of g and by $\mathcal{X}_\gamma$ the set of vector fields along γ. The restspace of γ in $\gamma(t)$ is the spacelike subspace, orthogonal complement of $\gamma'(t)$ in the tangent space $T_{\gamma(t)}M$.

Each linear connections on M induces a connection along γ; each such connection defines a parallel transport between the tangent spaces of M in every pair of γ-points. A *generalized Fermi- Walker connection* along γ ([8]) is an induced linear connection $\tilde{\nabla}$ along γ whose parallel transport conserves the restspaces of γ. Such a transport is called *spatial isometric* if moreover it preserves the norms of the restspace vectors along γ. The parallel transport associated with $\tilde{\nabla}$ is called generalized Fermi-Walker transport. If a given non-vanishing vector field X along γ, orthogonal to γ, is $\tilde{\nabla}$-parallel, we say it has constant direction (in the restspace of γ). In the opposite case, the direction changes: this is the so-called *generalized Thomas precession*, measured by $\tilde{\nabla}_\gamma X$.

In particular, we recover the classical Fermi- Walker connection ([1], [2], [10], [14], [15]) given by

(1) $$\nabla^0_{\gamma'} X = \nabla_{\gamma'} X + g(\gamma', X)\nabla_{\gamma'}\gamma' - g(\nabla_{\gamma'}\gamma', X)\gamma'$$

for every vector field $X \in \mathcal{X}_\gamma$. It is known that the corresponding parallel transport is a spatial isometry and that γ is autoparallel with respect to ∇^0. The Fermi-Walker connection ∇^0 is relevant for accelerating observers only (∇ and ∇^0 coincide along γ if and only if γ is a freely falling observer, that is if γ is a geodesic). In exchange, the generalized Fermi-Walker connections are relevant for both accelerating and non-accelerating observers.

Consider $\tilde{\nabla}$ be a generalized Fermi-Walker connection along γ. Then there exists ([8]) a unique (1,1)-tensor field A along γ such that for $X \in \mathcal{X}_\gamma$

(2) $$\tilde{\nabla}_{\gamma'} X = \nabla_{\gamma'} X + g(\gamma', X)\nabla_{\gamma'}\gamma' - g(\nabla_{\gamma'}\gamma', X)\gamma' + A(X)$$

and

(3) $$g(A(Z), \gamma') = 0 \quad . \qquad Z \in \mathcal{X}_\gamma^\perp.$$

Thus, formulae (2)- (3) describe all the generalized Fermi-Walker connections.

The curve γ is $\tilde{\nabla}$-autoparallel if and only if $A(\gamma') = 0$.

Moreover, the following statements are equivalent: (a) the transport of $\tilde{\nabla}$ is spatial isometric; (b) $\tilde{\nabla}_\gamma g = 0$ on all restspace directions; (c) the linear operator A is skew-adjoint. (If the parallel transport of $\tilde{\nabla}$ is an isometry, then each of the previous properties are satisfied.)

Remarks. (i) In a special relativistic framework, Hehl, Lemke and Mielke ([3]) also considered extensions of the Fermi-Walker transport; ab initio, they supposed that the transport was spatial isometric.

(ii) S. Manoff also defined some generalizations for the Fermi-Walker transport ([4]). All of them can be recovered from our formula (2), using some suitable skew-symmetric (1,1)-tensor fields restricted along the observer γ.

Proposition. *Consider E_2, E_3, $E_4 \in \mathcal{X}_\gamma^\perp$, linearly independent in each point. Then there exists a unique generalized Fermi-Walker connection $\tilde{\nabla}$ along γ which parallelizes γ and the three directions (i.e. $\tilde{\nabla}_{\gamma'}\gamma' = 0$ and $\tilde{\nabla}_{\gamma'}E_i = 0$, for $i = 2, 3, 4$).*

Proof. It is sufficient to define, in each point of γ, the tensor field A along γ, by $A(E_i) = -\nabla^0_{\gamma'}E_i$.

One sees that the condition (3) is satisfied; moreover, A is completely determined on the restspaces of γ. Thus, formula (2) gives the required connection. □

Remarks. (i) Under the hypothesis of the Proposition, there exists an infinity of generalized Fermi- Walker connections which parallelize E_2, E_3 and E_4, but all of them agree on spacelike directions.

(ii) The Proposition enables each observer to choose *any* affine co-moving frame as "fixed" (i.e. with $\tilde{\nabla}$-parallel coordinate axes). Of course, in particular the directions may be orthonormal, or with any other property. From parallel transport properties, it follows that the observer may change at any time the "fixed stars" to look for, through a new co-moving frame. Moreover, the observer may *simultaneously* use several (theoretically an infinity of) "fixed" co-moving frames (with respect to appropriate generalized Fermi-Walker connections).

(iii) By analogy with the *proper time* and the *proper space (restspace)* of the observer γ, we may establish various criteria for choosing a *proper* generalized Fermi-Walker parallel transport along the observer γ.

For example: suppose γ is "generic", i.e. admits a (unique) Frenet frame $\{e_1, e_2, e_3, e_4\}$, with $e_1 = \gamma'$. (Obviously, the spacelike vector fields of the frame belong to the restspaces of γ). Then we may choose as *proper* generalized Fermi-Walker transport of γ the unique one which parallelizates e_2, e_3 and e_4.

(iv) Moreover, for an observer γ, we may look for various kinds of Frenet-like frames, using any generalized Fermi-Walker parallelism instead of Levi-Civita's one.

Example. The exterior Schwarzschild spacetime is $N = \mathbf{R}\times(2M,\infty)\times S^2$, where S^2 denote the unit sphere and M the mass of the central star; we consider (t, r, θ, φ) the canonical coordinates on N. The line element on N is a warped product:

$$ds^2 = -hdt^2 + h^{-1}dr^2 + r^2(d\theta^2 + cos^2\theta d\varphi^2),$$

where $h(r) = 1 - 2M/r$. The time orientation on N is defined by the vector field $U = (1/\sqrt{h})\partial_t$, whose integral curves are the Schwarzschild observers. These are accelerating ones, due to the fact that

$$\nabla_U U = \frac{M}{r^2}\partial_r.$$

Hence, the Levi- Civita parallel transport along a Schwarzschild observer α does not conserve α-restspaces.

(i) A generalized Fermi-Walker connection along α acts on the canonical basis as ([8])

$$\tilde{\nabla}_{\alpha'}\partial_t = A(\partial_t) \quad , \quad \tilde{\nabla}_{\alpha'}\partial_r = A(\partial_r) \quad , \quad \tilde{\nabla}_{\alpha'}X = A(X),$$

where X is the lift of a vector field from S^2 and A is a (1,1)-tensor field on along α, satisfying $g(A(\partial_r), \partial_t) = g(A(X), \partial_t) = 0$. (All expressions are restricted along α). When A vanishes, we recover the classical Fermi-Walker transport, which -in particular- paralelizates the canonical basis ∂_r, ∂_θ, ∂_φ.

In particular, we may choose A satisfying $A(\partial_t) = A(\partial_\varphi) = 0$, $A(\partial_r) = \partial_\theta$, $A(\partial_\theta) = -hr^2\partial_r$. Then, as we may check also directly, this generalized Fermi-Walker transport is a spatial isometry. The direction ∂_φ remains parallel, but ∂_r and ∂_θ do not.

(ii) Consider now three arbitrary spacelike directions along the observer α

$$E_i = e_{i1}\partial_r + e_{i2}\partial_\theta + e_{i3}\partial_\varphi \quad , \quad i = 1, 2, 3,$$

where e_{ij} are functions of t. Suppose the vector fields are everywhere linearly independent (that is the matrix $(e_{ij})_{ij}$ is inversible, for every value of t). We may choose the tetrad $\{E_1, E_2, E_3\}$ as an affine co-moving frame along α. Then the generalized Fermi-Walker connection which parallelizes it is given by formula (2), where A is determined by

$$A(E_i) = -\sqrt{h}^{-1}(e'_{i1}\partial_r + e'_{i2}\partial_\theta + e'_{i3}\partial_\varphi)$$

(the prime means derivative with respect to t).

(iii) The Schwarzschild observer α is not "generic", because the third covariant derivative of U is not independent of the first two derivatives; so, there does not exist a *unique* Frenet frame along it. However, we may consider a Frenet-like frame: $e_1 = \alpha'$, $e_2 = (\sqrt{h})^{-1}\partial_r$, $e_3 = r^{-1}\partial_\theta$, $e_4 = (rcos\varphi)^{-1}\partial_\varphi$. Then the generalized Fermi-Walker transport which parallelizates it is exactly the classical Fermi-Walker one.

3 Canonical Lorentzian almost product structure associated to an observer field

Consider ξ an observer field ("reference frame" in the terminology of [10]) on M, i.e. a timelike, unitary, future oriented vector field. Its integral curves are observers on M. Denote by $\mathcal{D}$ the distribution (in Chevalley sense) spanned by ξ, and by $\mathcal{D}'$ its orthogonal complementary distribution; then $\mathcal{D}'$ is a spacelike distribution of rank 3 and, in each point, describes the restspaces of ξ-observers. To the pair of complementary distributions $\mathcal{D}$ and $\mathcal{D}'$, with corresponding projectors V, resp. V', we can associate a Lorentzian almost product structure on M; this one is given by a field of endomorphisms $P \in \mathcal{T}_1^1(M)$, defined through the formula $P = 2V - Id$, and obviously satisfies $P^2 = Id$.

Let ∇ be an *arbitrary* linear connection on M (not necessary the Levi-Civita one). The generalized Schouten connections associated to ∇ write ([6], [7])

$$2\nabla^S_X Y = 2\nabla_X Y + P(\nabla_X P)Y + B(X,Y) \quad , \qquad X, Y \in \mathcal{X}(M),$$

where B is a (1,2)-tensor fields satisfying

$$B(X,Y) = PB(X,PY) \quad , \qquad X \in \mathcal{D} \quad , \quad Y \in \mathcal{D}'.$$

(For $B = 0$ we recover the *classical Schouten connection*).

These connections contain much information concerning the distributions $\mathcal{D}$ and $\mathcal{D}'$ and the structure of the manifold, and allow to consider various kinds of exotic parallelism. Their invariants (torsion, curvature) were studied as an attempt to geometrize the non-holonomic systems.

Let γ be an ξ-observer. Then the two complementary distributions restrict along γ to the tangent direction and the restspaces of γ, respectively. The generalized Schouten (induced) connections along γ, associated with ∇, write

$$2\nabla^S_{\gamma'} Y = 2\nabla_{\gamma'} Y + P(\nabla_{\gamma'} P)Y + B(\gamma', Y) \quad , \qquad Y \in \mathcal{X}_\gamma \tag{4}$$

and

$$B(\gamma', Z) = PB(\gamma', PZ) \quad , \qquad Z \in \mathcal{X}^\perp_\gamma \tag{5}$$

Proposition. (i) *If ∇ is the Levi-Civita connection of M, then formulae (4)-(5) give exactly the generalized Fermi- Walker along γ.*

(ii) *Conversely, each generalized Fermi-Walker parallelism can be recovered from a generalized Schouten's connection (as in §2).*

In fact, we recover the relations (2) and (3), with the identifications $\tilde{\nabla} = \nabla^S$ and $A = B(\gamma', .)$.

Remark. The (generalized) Fermi- Walker and the Schouten constructions are quite similar; in fact, they describe the same phenomenon, at the "observer" or at the "observer field" levels. In particular, the classical Schouten connection induces the classical Fermi-Walker transport.

Example. To the Lorentzian almost product structure P we can associate the classical Vranceanu connection (for a generalization, see [6], [7])

$$\overline{\nabla}_X Y = V\nabla_{VX}VY + V'\nabla_{V'X}V'Y + V[V'X, VY] + V'[VX, V'Y],$$

for every $X, Y \in \mathcal{X}(M)$. Due to the symmetry of ∇, we obtain ([9])

$$4\nabla^V_X Y = 4\nabla_X Y + (\nabla_{PY}P)X + P(\nabla_Y P)X + 2P(\nabla_X P)Y.$$

When we restrict along the observer γ we obtain

$$4\nabla^V_{\gamma'} Y = 4\nabla_{\gamma'} Y + (\nabla_{PY}P)\gamma' + P(\nabla_Y P)\gamma' + 2P(\nabla_{\gamma'} P)Y,$$

for every $Y \in \mathcal{X}_\gamma$. We proved ([9]) that this induced Vranceanu connection is a particular case of a generalized Fermi- Walker connection, replacing in relations (2)-(3)

$$A(Y) = -g(\gamma', Y)\gamma' - \nabla_{\gamma'} Y \quad , \qquad Y \in \mathcal{X}_\gamma.$$

Consider now a Robertson- Walker spacetime $M = I \times_f S$, where I is a real interval, S a 3-dimensional spaceform, f a differentiable, positive, real function on I. On M, the metric is the warped product

$$g = dt^2 + f^2 h,$$

where h is the canonical metric on S. The canonical observer field (which gives the time-orientation on M) is $\xi = \partial_t$; its observers are geodesics on M, so their (classical) Fermi- Walker parallelism coincides with the Levi- Civita parallelism.

Using formula (5), we obtain that the (induced) Vranceanu connection parallelizes the restspaces of ξ, i.e. $\overline{\nabla}_\xi Y = 0$, for every $Y \in \mathcal{X}^\perp_\gamma$.

A "practical" consequence: if several intergalatic observers, navigating in a Robertson-Walker Universe, have difficulties in agreeing on common (Levi- Civita) "fixed" frames, they may use with more chances the Vranceanu parallelism (because *all* restspace directions are ∇- "fixed").

4 Conclusions

(i) The first conclusion is that the generalized Fermi-Walker transport offers a better choice of reference systems than the classical one. An observer γ may choose three (remarquable) independent directions ('fixed stars") in his restspace; then, there exists a generalized Fermi-Walker connection $\tilde{\nabla}$ which parallelizes the respective directions. Three giroscopes put on the three directions will describe the invariance or the precession of any other direction. In case an observer considers several generalized Fermi-Walker connections along it (corresponding to several triplet gyroscopes), these may provide data for average measurements.

Consider now another accelerating observer α. The classical setting force him to use the same Fermi-Walker parallelism ∇^0, violating - at least the spirit of - the

Principle of Relativity. Even in case the two observers may agree on the choice of the "stars", it may happen that the three "fixed (i.e. ∇^0 -parallel) stars" for γ remain no longer fixed for α. If generalized Fermi-Walker connections are considered, the observers are allowed to use the "same" frame directions, but with different tensor fields A (i.e. with different kinds of parallelism).

(ii) The Principle of relativity imposed the replacement of absolute time and space with relative ones. For a relativistic observer, *proper time* and *proper space* ("restspace") are the only relevant. In §2 we defined also a *proper parallelism*: the Frenet frame (in case it exists), which is (in general) non-inertial with respect to the metric, becomes inertial with respect to a properly choosed and canonical generalized Fermi-Walker parallel transport.

(iii) For an observer field ξ, we constructed a canonical family of generalized Schouten connections, which give parallel frames associated to ξ. These linear connections are intrinsic, because depend on the spacetime structure only. Moreover, along each ξ- observer, they give rise to the previous generalized Fermi-Walker parallel transports.

So, the third idea was to construct a bridge between the theory of accelerated observers (with their Fermi-Walker transport) and the geometrization of non-holonomic differential systems (Chevalley distributions) through Schouten-like (and, in particular, through Vranceanu-like) connections. We proved that both parallelisms have common roots; the only difference is that the former refers to a single observer, while the latter refers to observer fields.

Special remarkable classes of generalized Schouten connections may lead to new classes of exotic parallelism of accelerated observers; the Vranceanu connection on a Robertson- Walker spacetime is but an example. (For Vranceanu's non-holonomic cosmological model, see [9], [11]- [13]).

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