PARETO MAXIMA AS MAXIMUM POINTS FOR WEIGHTED REAL VALUED FUNCTIONS

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Abstract

We study the relationship between Pareto maximum points and maximum points for some suitable value functions associated to multi-criteria optimization problems. A generalization of the classical framework is given, by introducing Pareto manifolds: these form a quite large class of manifolds, on which Pareto maxima may be covariantly defined and studied.

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1 Introduction

Differential geometric techniques in Optimization theory were considered only recently; most of them belong to Riemannian geometry (see [3], [4] for a review), but there are also affine differential ones ([2]). The main notions and results from classical optimization (of real valued functions on sets in \(\mathbb{R}^n\)) may be generalized on manifolds.

When multicriteria optimization problems are considered, such a generalization might seem an utopia: the preference criteria are (usually) inspired by ordering real numbers, fact which is difficult to translate on manifolds. (See however some recent attempt to define monotony of vector fields, as particular "multivalued" functions on manifolds in [1]). Moreover, the notions of efficiency (Pareto maxima, etc.) do not seem to have covariant character (in order to be transported on manifolds, via charts or parametrizations).

In this paper, we consider differentiable functions \(f : M \to N\), where \(M\) and \(N\) are differentiable manifolds, and we look for an analogue of the (classical) Pareto maximum for \(f\).

First (§1) we have to restrict ourselves to some special class of target manifolds \(N\) (which will be called Pareto manifolds): these manifolds are characterized by the
existence of an adapted atlas, with "order preserving" coordinates changes. Several examples of such manifolds are provided.

In dealing with multicriteria optimization, one often simplify the setting by weakening the requirements: despite of studying the variation of the multi-valued function \( f \), one defines (ad hoc) \( f \)-dependent weighted "linearized" and real valued functions \( F \) on \( M \) and one looks for their maximum points. Along this path, we associate to a differentiable function \( f \) (on manifolds, as above) a family of weighted functions \( F \), in such a way that maximum points of \( F \) offer information concerning the Pareto maximum points of \( f \).

Our main results (Theorems 1 and 2, §3) state that:
- any maximum point of a (previous) function \( F \) is a Pareto maximum point for \( f \).
- for any Pareto maximum point \( x_0 \) of \( f \), there exists a suitable chosen weighted function \( F \), such that \( x_0 \) be a maximum point of \( F \).

2 Pareto manifolds

Convention. A vector \( v \in \mathbb{R}^n \) is non-negative (notation \( v \geq 0 \)) if each of its components is non-negative.

Definition 1. Let \( M \) be an \( n \)-dimensional differentiable manifold. We say \( M \) is a Pareto manifold if there exists a compatible atlas \( A \) on \( M \) such that for any two overlapping charts \((U, \varphi)\) and \((V, \psi)\) from \( A \), the following property holds:

\[
\text{if } v, w \in \varphi(U \cap V) \text{, } v \geq w \text{ then } \psi \circ \varphi^{-1}(v) \geq \psi \circ \varphi^{-1}(w)
\]

Such an atlas will be called an adapted atlas on \( M \).

Examples. (i) The canonical atlas on any open set of \( \mathbb{R}^n \) produces a Pareto manifold structure on this set.
(ii) The 2-dimensional torus admits a 6-charts atlas which determines a Pareto manifold structure on it.
(iii) Consider

\[
M = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid y = 1 \text{ , } x \geq 0\}
\]

Define the charts \((U_1, h_1)\) and \((U_2, h_2)\) by

\[
U_1 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \text{ , } h_1 : U_1 \to \mathbb{R} \text{ , } h_1(x, 0) = x
\]

\[
U_2 = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ , } y = 1\} \cup \{(x, y) \in \mathbb{R}^2 \mid x < 0 \text{ , } y = 0\}
\]
\[
G_i = g_i \circ f \quad , \quad F = \sum_{i=1}^{k} \lambda_i G_i
\]

The function \(F\) is a weighted function associated to \(f\).

**Theorem 1.** Let \(x_0\) be a (local) maximum point for \(F\) in \(M\). Suppose:
(i) the functions \(g_1, ..., g_k\) are monotone increasing around \(f(x_0)\).
(ii) $F$ is injective or there exists at least one injective “term” $\lambda_i g_i$.

Then $x_0$ is a (local) Pareto point for $f$.

**Proof.** Let $(U, h)$ be a chart around $x_0$ in $M$ and $(V, \varphi)$ an adapted chart around $f(x_0)$ such that $f(U) \subseteq V$. By restricting (if necessary) the domains $U$ and $V$, we may suppose that $x_0$ is a maximum point of $F$ on the whole $U$, and the functions $g_i$ are non-decreasing on the whole $V$.

Take an arbitrary $x \in U$ such that $f(x) \geq f(x_0)$. Then $G_i(x) \geq G_i(x_0)$ for every index $i = 1, k$, so $F(x) \geq F(x_0)$ on $U$. Because $x_0$ is a maximum point, we deduce $F(x) = F(x_0)$. Hence, for every index $i$, we have

$$\lambda_i g_i(f(x)) = \lambda_i g_i(f(x_0))$$

From the definition of $F$ and the hypothesis (ii), it follows that $f(x) = f(x_0)$; thus $x_0$ is a (local) Pareto point for $f$. \(\square\)

**Examples.** Consider the particular case of a function $f : U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$, and define

$$F(x) = \frac{1}{n} \{ f^1(x) + ... + f^n(x) \}$$

and

$$\tilde{F}(x) = \text{the volume of the simplex spanned by } f^1(x), ..., f^n(x).$$

Then any maximum point for $F$ and $\tilde{F}$, respectively, is a Pareto point for $f$.

These two examples show that there are a lot of possibilities to construct such “weighted”, real valued auxiliary functions, in order to obtain many Pareto points for a given multi-valued function $f$. But is it possible to obtain all the Pareto points by this method? At least for the following particular but important case, the answer is affirmative.

**Theorem 2.** Let $M$ be a differentiable manifold and a continuous function $f : M \rightarrow \mathbb{R}^n$. Let $x_0$ be a fixed (local) Pareto point for $f$.

Then there exists a function $F$ as above with $x_0$ as a (local) maximum point.

**Proof.** Let $x_0$ be a (local) Pareto point for the function $f$. We define the functions $G_i : M \rightarrow \mathbb{R}, i = 1, n$, by

$$G_i(x) = - | f^i(x) - f^i(x_0) | + f^i(x) - f^i(x_0)$$

(we denoted by $| y |$ the absolute value of $y$).

We define now the weighted function $F : M \rightarrow \mathbb{R}$, by

$$F(x) = \sum_{i=1}^{n} G_i(x)$$

one can easily see that $x_0$ is a (local) maximum point for $F$. \(\square\)
Conclusions. The "duality" between Pareto points of multi-valued functions $f$ and maximum points of associated "weighted", real valued functions $F$ may be further investigated; it is difficult to consider all the associated functions $F$ (an huge infinity), so it would be interesting to find a (finite ?) "minimal" set of such functions which provide all the Pareto points for a given function $f$.

References


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