PARETO MAXIMA AS MAXIMUM POINTS FOR WEIGHTED REAL VALUED FUNCTIONS

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Abstract

We study the relationship between Pareto maximum points and maximum points for some suitable value functions associated to multi-criteria optimization problems. A generalization of the classical framework is given, by introducing Pareto manifolds: these form a quite large class of manifolds, on which Pareto maxima may be *covariantly* defined and studied.

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1 Introduction

Differential geometric techniques in Optimization theory were considered only recently; most of them belong to Riemannian geometry (see [3], [4] for a review), but there are also affine differential ones ([2]). The main notions and results from classical optimization (of real valued functions on sets in $\mathbf{R}^n$) may be generalized on manifolds.

When multicriteria optimization problems are considered, such a generalization might seem an utopia: the preference criteria are (usually) inspired by ordering real numbers, fact which is difficult to translate on manifolds.(See however some recent attempt to define monotony of vector fields, as particular "multivalued" functions on manifolds in [1]). Moreover, the notions of efficiency (Pareto maxima, etc.) do not seem to have covariant character (in order to be transported on manifolds, via charts or parametrizations).

In this paper, we consider differentiable functions $f : M \to N$, where M and N are differentiable manifolds, and we look for an analogue of the (classical) Pareto maximum for f.

First (§1) we have to restrict ourselves to some special class of target manifolds N (which will be called *Pareto manifolds*): these manifolds are characterized by the

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existence of an adapted atlas, with "order preserving" coordinates changes. Several examples of such manifolds are provided.

In dealing with multicriteria optimization, one often simplify the setting by weakening the requirements: despite of studying the variation of the multi-valued function f, one defines (ad hoc) f-dependent weighted "linearized" and real valued functions F on M and one looks for their maximum points. Along this path, we associate to a differentiable function f (on manifolds, as above) a family of weighted functions F, in such a way that maximum points of F offer information concerning the Pareto maximum points of f.

Our main results (Theorems 1 and 2, §3) state that:

- any maximum point of a (previous) function F is a Pareto maximum point for f.
- for any Pareto maximum point x_0 of f, there exists a suitable chosen weighted function F, such that x_0 be a maximum point of F.

2 Pareto manifolds

Convention. A vector $v \in \mathbf{R}^n$ is non-negative (notation $v \geq 0$) if each of its components is non-negative.

Definition 1. Let M be an n-dimensional differentiable manifold. We say M is a *Pareto manifold* if there exists a compatible atlas $\mathcal{A}$ on M such that for any two overlapping charts (U, φ) and (V, ψ) from $\mathcal{A}$, the following property holds:

$$if \quad v, w \in \varphi(U \bigcap V) \quad , \quad v \geq w \quad then \quad \psi \circ \varphi^{-1}(v) \geq \psi \circ \varphi^{-1}(w)$$

Such an atlas will be called an *adapted atlas* on M.

Examples. (i) The canonical atlas on any open set of $\mathbf{R}^n$ produces a Pareto manifold structure on this set.

(ii) The 2-dimensional torus admits a 6-charts atlas which determines a Pareto manifold structure on it.

(iii) Consider

$$M = \{(x, y) \in \mathbf{R}^2 \mid y = 0\} \bigcup \{(x, y) \in \mathbf{R}^2 \mid y = 1 \ , \ x \geq 0\}$$

Define the charts (U_1, h_1) and (U_2, h_2) by

$$U_1 = \{(x, y) \in \mathbf{R}^2 \mid y = 0\} \quad , \quad h_1 : U_1 \to \mathbf{R} \ , \ h_1(x, 0) = x$$

$$U_2 = \{(x, y) \in \mathbf{R}^2 \mid x \geq 0 \ , \ y = 1\} \bigcup \{(x, y) \in \mathbf{R}^2 \mid x < 0 \ , \ y = 0\}$$

$$h_2 : U_2 \to \mathbf{R} \quad , \quad h_2(x, y) = x$$

We see that $h_2 \circ h_1^{-1}$ is the identity. This provides an example of a 1-dimensional, non- Hausdorff Pareto manifold.

iv) The product of two Pareto manifolds is again a Pareto manifold.

Definition 2. Consider M a Pareto manifold, (U, h) an adapted chart and $x, y \in U$. We say $x \geq y$ if $h(x) \geq h(y)$.

Due to the property of adapted atlases, the relationship " $\geq$ " is independent of the choice of the adapted chart around x and y.

Definition 3. Consider M a differentiable manifold, N a Pareto manifold and $f : M \to N$ a *continuous* function. We say $x_0 \in M$ is a (local) *Pareto point* for f if there exist a chart (V, φ) of M around x_0 and an adapted chart (U, h) of N around $f(x_0)$, an open set $V_0 \subset V$ around x_0 such that $f(V) \subseteq U$ and for every $x \in V_0$ with $f(x) \geq f(x_0)$ we have $f(x) = f(x_0)$.

Proposition 1. *The notion of Pareto points is covariant.*

Proof. Consider x_0 a Pareto point, the sets V, V_0 and U like in the Definition 3. Suppose that x_0 is contained also in the chart $(\tilde{V}, \tilde{\varphi})$, $f(x_0)$ is contained in the adapted chart $(\tilde{U}, \tilde{h})$ such that $f(\tilde{V}) \subset \tilde{U}$. Denote by $\tilde{V_0} = \tilde{V} \cap V_0$. Let $x \in \tilde{V_0}$ such that $f(x) \geq f(x_0)$. The manifold N is Pareto, so this property do not depend on the chart. Then $f(x) = f(x_0)$.

We proved that x_0 is a Pareto point with respect to the "tilde" charts also; this means the respective notion is covariant.□

3 A useful class of value functions

Consider a differentiable manifold M and a Pareto manifold N. Let $f : M \to N$ be a continuous function. We look for the Pareto points of f. Consider g_1, ..., g_k some real valued functions on $f(M)$ and $\lambda_1, ..., \lambda_k$ non-negative real numbers; construct

$$G_i = g_i \circ f \quad , \quad F = \sum_{i=1}^{k} \lambda_i G_i$$

The function F is a *weighted function* associated to f.

Theorem 1. *Let x_0 be a (local) maximum point for F in M. Suppose:*
(i) the functions g_1, ..., g_k are monotone increasing around $f(x_0)$.

(ii) F is injective or there exists at least one injective "term" $\lambda_i g_i$.
Then x_0 is a (local) Pareto point for f.

Proof. Let (U, h) be a chart around x_0 in M and (V, φ) an adapted chart around $f(x_0)$ such that $f(U) \subseteq V$. By restricting (if necessary) the domains U and V, we may suppose that x_0 is a maximum point of F on the whole U, and the functions g_i are non-decreasing on the whole V .

Take an arbitrary $x \in U$ such that $f(x) \geq f(x_0)$. Then $G_i(x) \geq G_i(x_0)$ for every index $i = \overline{1, k}$, so $F(x) \geq F(x_0)$ on U. Because x_0 is a maximum point, we deduce $F(x) = F(x_0)$. Hence, for every indice i, we have

$$\lambda_i g_i(f(x)) = \lambda_i g_i(f(x_0))$$

From the definition of F and the hypothesis (ii), it follows that $f(x) = f(x_0)$; thus x_0 is a (local) Pareto point for f. $\Box$

Examples. Consider the particular case of a function $f : U \subseteq \mathbf{R}^m \to {\mathbf{R}_+}^n$, and define

$$F(x) = \frac{1}{n}\{f^1(x) + ... + f^n(x)\}$$

and

$\tilde{F}(x) =$ the volume of the simplex spanned by $f^1(x)$, ..., $f^n(x)$.

Then any maximum point for F and $\tilde{F}$, respectively, is a Pareto point for f.

These two examples show that there are a lot of possibilities to construct such "weighted", real valued auxiliary functions, in order to obtain many Pareto points for a given multi-valued function f. But is it possible to obtain *all* the Pareto points by this method ? At least for the following particular but important case, the answer is affirmative.

Theorem 2. *Let M be a differentiable manifold and a continuous function $f : M \to \mathbf{R}^n$. Let x_0 be a fixed (local) Pareto point for f.*
Then there exists a function F as above with x_0 as a (local) maximum point.

Proof. Let x_0 be a (local) Pareto point for the function f. We define the functions $G_i : M \to \mathbf{R}$, $i = \overline{1, n}$, by

$$G_i(x) = - \mid f^i(x) - f^i(x_0) \mid + f^i(x) - f^i(x_0)$$

(we denoted by $\mid y \mid$ the absolute value of y).

We define now the weighted function $F : M \to \mathbf{R}$, by

$$F(x) = \sum_{i=1}^{n} G_i(x)$$

one can easily see that x_0 is a (local) maximum point for F. $\Box$

Conclusions. The "duality" between Pareto points of multi-valued functions f and maximum points of associated "weighted", real valued functions F may be further investigated; it is difficult to consider all the associated functions F (an huge infinity), so it would be interesting to find a (finite ?) "minimal" set of such functions which provide all the Pareto points for a given function f.

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