AN APPROXIMATE SOLUTION OF BURGERS EQUATION USING ADOMIAN'S DECOMPOSITION METHOD

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Abstract

The aim of this paper is to find an approximate solution of Burgers equation using the Decomposition Method, which has been developed by George Adomian [3]. The advantage of this method is to avoid simplifications and restrictions which change the non-linear problem to mathematically tractable one, whose solution is not consistent with physical solution. Theoretical analysis and all calculations have been done and the results are discussed.

AMS Subject Classification: 65C20.
Key words: Burger equation, Adomian Decomposition Method, Adomian polynomials.

1 Introduction

The one dimension non-linear differential equation, which is similar to the one dimension Navier-Stokes equation without the stress term, and was presented for the first time in a paper in 1940 from Burger, is the model for the solution of Navier-Stokes equation and is applied to laminar and turbulence flows as well. The Burger equation was first studied by Cole [12] who gave a theoretical solution, based on Fourier series analysis, using the appropriate initial and boundary conditions. Another theoretical solution was given by Madsen and Sincovec [13], based on the "test and trial" method, using the appropriate initial and boundary conditions. In Benton and Platzman [10], are mentioned almost 35 distinct solutions of Burger equation and Agas [9] tried to get approximate solutions of Burger equation using numerical analysis. He tried the method of Finite Differences and the method of Lines in Finite Elements. The problem he faced was that these methods couldn't give solutions for big values of the Reynolds number. He also found some problems in convergence. In this paper,

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we will find solutions using the Adomian decomposition method. This method gives a computable and accurate solution of the problem for a small number of terms and demands to be parallel to any modern supercomputer. The whole paper contains five paragraphs. Each of them analyzed as follows. The first paragraph is the introduction. The formulation of the problem is studied in the second paragraph. The theoretic approach is given in the third paragraph. The determination of the Adomian's Special Polynomials is studied in the fourth paragraph. The fifth paragraph includes the results, the diagrams and the discussions.

2 Formulation of the Problem

The equation of motion in one dimension has the following form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where the first term is the linear, the second is the non-linear and the third is the highest order term. If we define

$$L_t u = \frac{\partial u}{\partial t} = Ru, \quad L_x u = \frac{\partial^2 u}{\partial x^2} = Lu, \quad Nu = u\frac{\partial u}{\partial x}, \tag{2}$$

where Nu represents the non-linear term, Lu is the highest order term, and Ru is the rest of the equation, equation (1) takes the form

$$Ru + Nu = \nu Lu.$$

The boundary conditions are defined as follows:

$$u(0,t) = u(1,t) \ \text{ for } \ t \succeq 0 \tag{3}$$

and the initial condition:

$$u(x,0) = 4x\left(1-x\right). \tag{4}$$

3 Theoretic approach

We solve equation (1) for $L_t u$ and $L_x u$ separately and we get

$$L_t u = \nu L_x u - Nu, \tag{5}$$

$$L_x u = \nu^{-1}\left(L_t u + Nu\right). \tag{6}$$

Let L_t^{-1} and L_x^{-1} be the inverse operators of $L_t u$ and $L_x u$ respectively, given by the form:

$$L_t^{-1} = \int (\cdot)\, dt \ \text{ and } \ L_x^{-1} = \int\int (\cdot)\, dx dx. \tag{7}$$

Then operating both sides of equations (5) and (6) with the inverse operators (7) we obtain.

$$u = \phi_0 + L_t^{-1}\left(\nu\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right), \tag{8}$$

$$u = \psi_0 + \nu^{-1}L_x^{-1}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right), \tag{9}$$

where ϕ_0 and ψ_0 are the solutions of the equations:

$$\frac{\partial u}{\partial t} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = 0 \tag{10}$$

respectively. The equations (10) can be solved subjected to the corresponding initial condition (4) and boundary conditions (3) and we obtain:

$$\phi_0 = 4x\,(1-x) \quad \text{and} \quad \psi_0 = 0. \tag{11}$$

Now, adding (8) and (9) and dividing by 2, we get the following form:

$$\begin{aligned} u &= \frac{1}{2}\left[(\phi_0+\psi_0) + L_t^{-1}\left(\nu\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right) + \nu^{-1}L_x^{-1}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\right] \\ &= 2x\,(1-x) + \frac{1}{2}\left[L_t^{-1}\left(\nu\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right) + \nu^{-1}L_x^{-1}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)\right], \end{aligned} \tag{12}$$

where

$$u_0 = \frac{1}{2}(\phi_0+\psi_0) = 2x\,(1-x)\,. \tag{13}$$

After that, we write the parametrized form of (12) which is:

$$u = u_0 + \lambda\frac{1}{2}\left[\nu^{-1}L_x^{-1}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + L_t^{-1}\left(\nu\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\right)\right] \tag{14}$$

and the parametrized decomposition forms of u and Nu as

$$u = \sum_{n=0}^{\infty} \lambda^n u_n, \tag{15}$$

$$u\frac{\partial u}{\partial x} = Nu = \sum_{n=0}^{\infty} \lambda^n A_n, \tag{16}$$

where A_n are the Adomian's special polynomials [1,2] to be determined. Here the parameter λ looks like a perturbation parameter; but actually is not a perturbation parameter; it is used only for grouping the terms. Now substitution of (15) and (16) into (14) gives

$$\sum_{n=0}^{\infty} \lambda^n u_n = u_0 + \lambda\frac{1}{2}\left[\nu^{-1}L_x^{-1}\left(\frac{\partial \sum\limits_{n=0}^{\infty}\lambda^n u_n}{\partial t} + \sum_{n=0}^{\infty}\lambda^n A_n\right) + \right.$$

$$L_t^{-1}\left(\nu\frac{\partial^2\sum\limits_{n=0}^{\infty}\lambda^n u_n}{\partial x^2}-\sum_{n=0}^{\infty}\lambda^n A_n\right)\Bigg]. \tag{17}$$

If we compare like-power terms of λ from both sides of equation (17), and taking under consideration that parameter λ is being proved that has the unique value $\lambda = 1$ [7,8], we get

$$\begin{aligned}
u_0 &= 2x\left(1-x\right),\\
u_1 &= \frac{1}{2}\left[\nu^{-1}L_x^{-1}\left(\frac{\partial u_0}{\partial t}+A_0\right)+L_t^{-1}\left(\nu\frac{\partial^2 u_0}{\partial x^2}-A_0\right)\right],\\
u_2 &= \frac{1}{2}\left[\nu^{-1}L_x^{-1}\left(\frac{\partial u_1}{\partial t}+A_1\right)+L_t^{-1}\left(\nu\frac{\partial^2 u_1}{\partial x^2}-A_1\right)\right],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
u_{n+1} &= \frac{1}{2}\left[\nu^{-1}L_x^{-1}\left(\frac{\partial u_n}{\partial t}+A_n\right)+L_t^{-1}\left(\nu\frac{\partial^2 u_n}{\partial x^2}-A_n\right)\right],\ n=0,1,2,\ldots,n
\end{aligned}\tag{18}$$

Next, we proceed to determine Adomian's special polynomials A_n.

4 Determination of Adomian's Special Polynomials

The A_n polynomials are defined in such a way that each A_n depends only on $u_0, u_1, ..., u_n$ for $n = 0, 1, 2, ..., n$, i.e., $A_0 = A\left(u_0\right),\ A_1 = A_1\left(u_0, u_1\right),\ A_2 = A_2\left(u_0, u_1, u_2\right)$, etc. In order to do this we substitute (15) into (16) and we have

$$\begin{aligned}
Nu &= u\frac{\partial u}{\partial x} = \left(u_0+\lambda u_1+\lambda^2 u_2+\lambda^3 u_3+...\right)\left(\frac{\partial u_0}{\partial x}+\lambda\frac{\partial u_1}{\partial x}+\lambda^2\frac{\partial u_2}{\partial x}+\lambda^3\frac{\partial u_3}{\partial x}+...\right)\\
&= u_0\frac{\partial u_0}{\partial x}+\lambda\left(u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x}\right)+\lambda^2\left(u_0\frac{\partial u_2}{\partial x}+u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_0}{\partial x}\right)+\\
&\quad+\lambda^3\left(u_0\frac{\partial u_3}{\partial x}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_1}{\partial x}+u_3\frac{\partial u_0}{\partial x}\right)+\lambda^4\left(...\right).
\end{aligned}\tag{19}$$

From (19) we conclude that the Adomian Polynomials have the following form:

$$\begin{aligned}
A_0 &= u_0\frac{\partial u_0}{\partial x},\\
A_1 &= u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x},\\
A_2 &= u_0\frac{\partial u_2}{\partial x}+u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_0}{\partial x},\\
&\ldots\ldots\ldots\ldots
\end{aligned}\tag{20}$$

Hence, the polynomial A_0 has the following form:

$$A_0 = u_0\frac{\partial u_0}{\partial x} = 4\left(x-3x^2+2x^3\right). \tag{21}$$

Using (13) and A_0 from (21) into the expression of u_1 in (20) and then performing the integrations with respect to t and x respectively, we have

$$u_1 = 2\left[\nu^{-1}\left(\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10}\right) - \left(\nu + x - 3x^2 + 2x^3\right)t\right]. \tag{22}$$

If we suggest as a solution of u an approximation of only two terms then from (13) and (22) we have the solution

$$u = u_0 + u_1. \tag{23}$$

We use Mathematica 3.0 in order to get numerical results and we use the results obtained from Agas [9] in order to compare these results. The program in Mathematica is:

```
u0=2x-2x^2
xu0=D[u0,x]
tu0=D[u0,t]
du0=D[u0,x,x]
a=Integrate[4x-4x^2-8x^2+8x^3,x,x]
b=Integrate[4n-4x+4x^2,-8x^2+8x^2-8x^3,t]
u1=Expand[1/2(b+n^-1a)]
u=Expand[u0-u1]
u/.{x->.75, t->.01, n->1}
Clear[u]
```

5 Results, Diagrams and Discussion

x=25 v=1

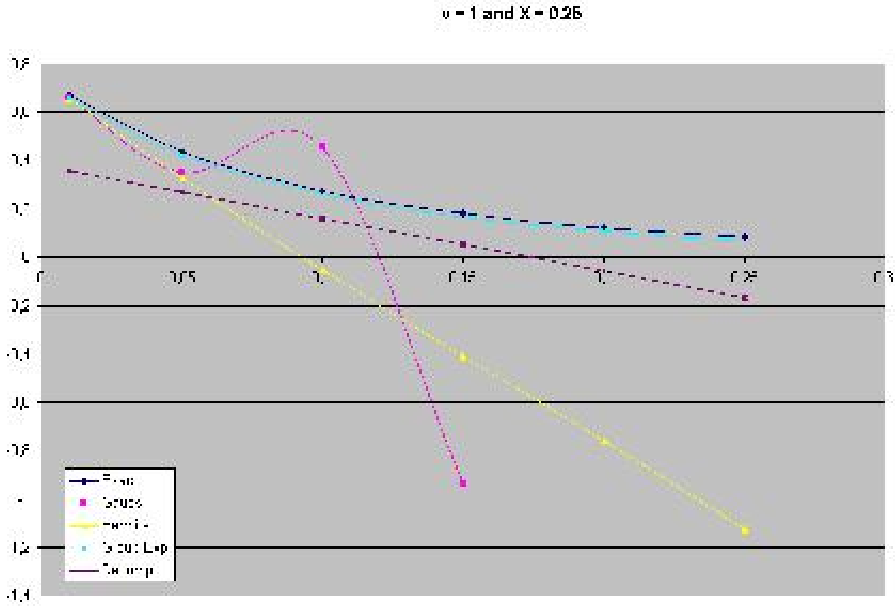
t	Exact	Gauss	Hermite	Group Ex	Decomp
0,01	0,6724	0,6584	0,6583	0,6575	0,35658
0,05	0,4356	0,3503	0,3254	0,4206	0,26908
0,1	0,2751	0,4527	-0,053	0,2601	0,15971
0,15	0,1794	-0,934	-0,408	0,1644	0,05033
0,2	0,1191		-0,7593	0,1041	-0,05905
0,25	0,0807		-1,1234	0,0657	-0,16843

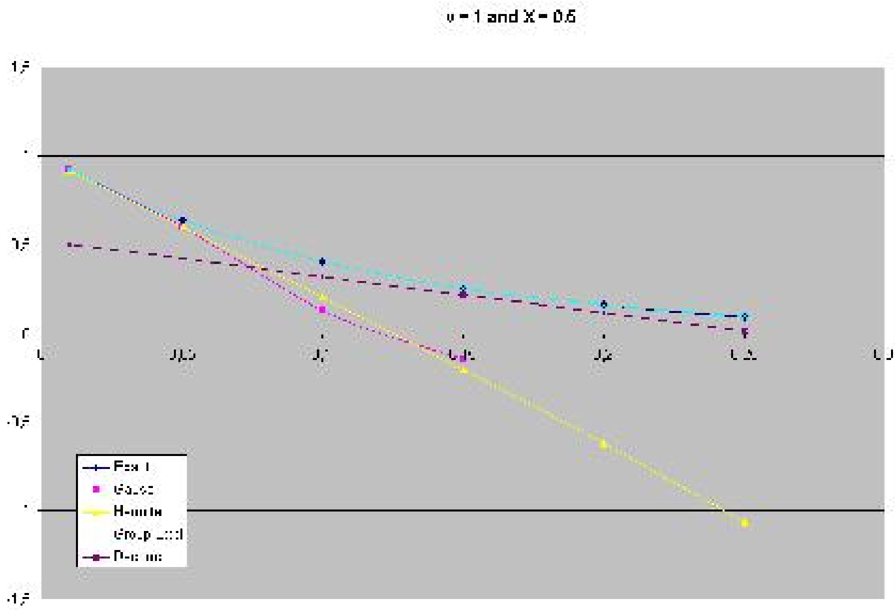
x=0,5

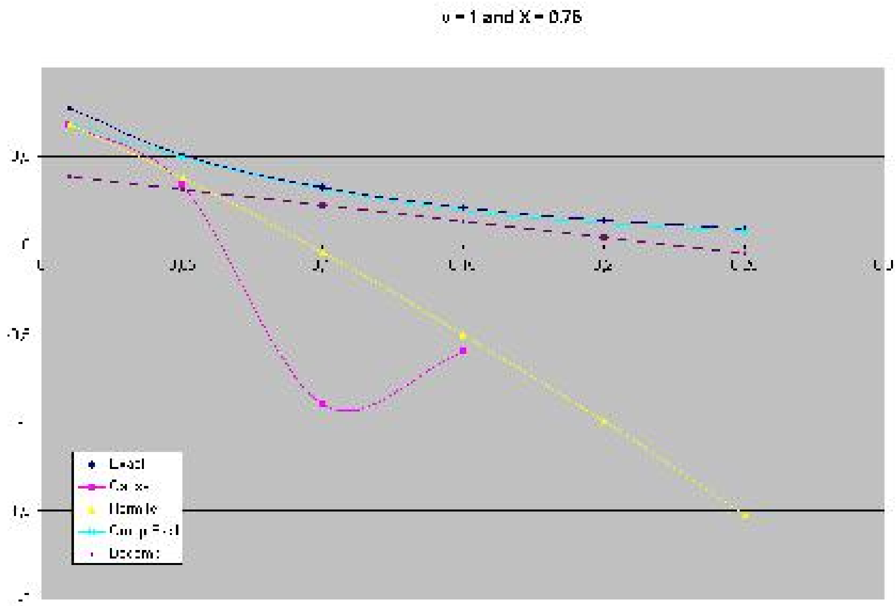
t	Exact	Gauss	Hermite	Group Ex	Decomp
0,01	0,9184	0,9197	0,9204	0,918	0,49667
0,05	0,639	0,5978	0,5983	0,6386	0,41667
0,1	0,4019	0,1275	0,2	0,4015	0,31667
0,15	0,2524	-0,1408	-0,2023	0,252	0,21667
0,2	0,1585		-0,6192	0,1581	0,11667
0,25	0,0914		-1,0642	0,0991	0,01667

x=0,75

t	Exact	Gauss	Hermite	Group Ex	Decomp
0,01	0,7677	0,6818	0,6824	0,6927	0,38676
0,05	0,5065	0,3355	0,3772	0,4915	0,31426
0,1	0,3239	-0,8926	-0,046	0,3089	0,22364
0,15	0,2069	-0,602	-0,5021	0,1919	0,13301
0,2	0,1342		-0,992	0,1192	0,04239
0,25	0,0892		-1,5243	0,0742	-0,04825


v = 1 and X = 0.25


v = 1 and X = 0.5



x=25 v=0.1

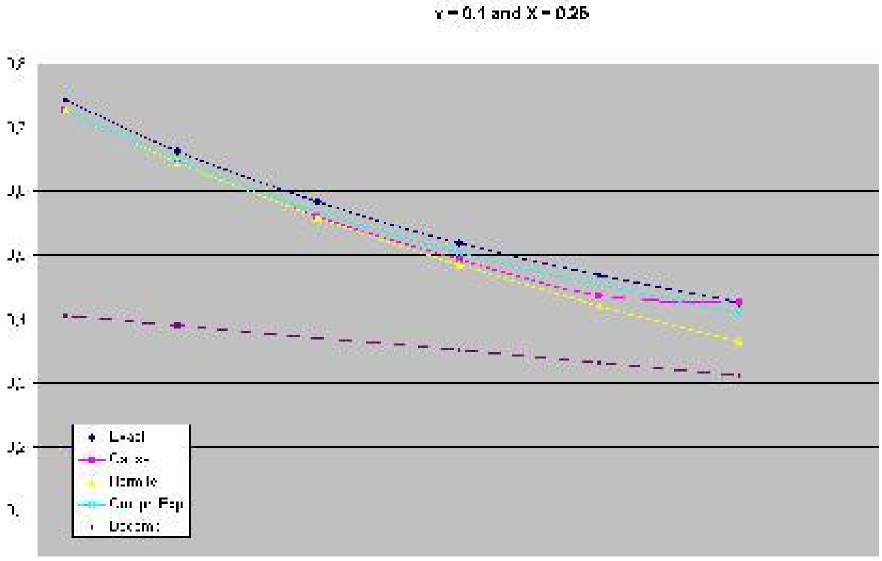
t	Exact	Gauss	Hermite	Group Ex	Decomp
0,01	0,7422	0,7274	0,7273	0,7272	0,40563
0,05	0,6621	0,6453	0,6452	0,6471	0,39013
0,1	0,584	0,5608	0,5592	0,5671	0,37075
0,15	0,5189	0,4931	0,4853	0,5039	0,35138
0,2	0,4681	0,4362	0,4213	0,4531	0,33201
0,25	0,4265	0,4263	0,3652	0,4115	0,31263

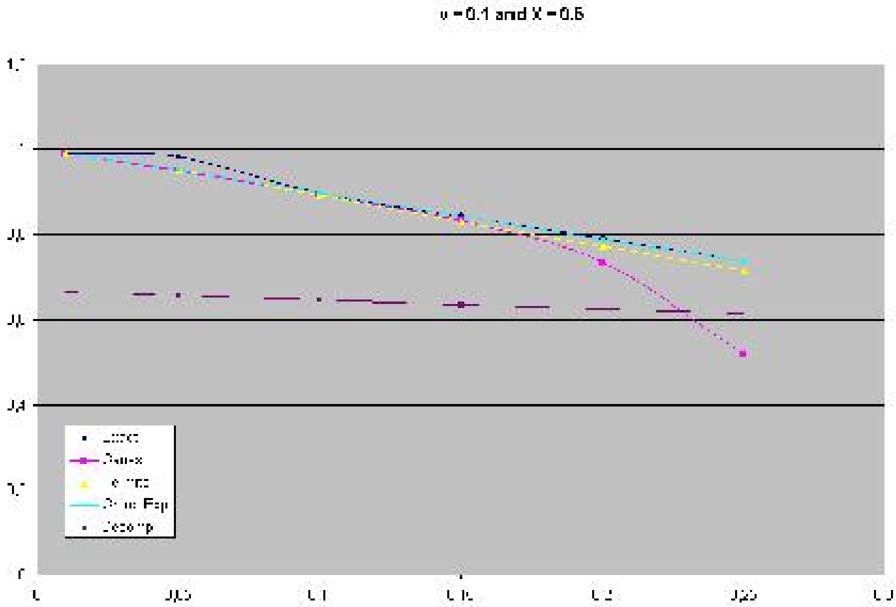
x=0.50

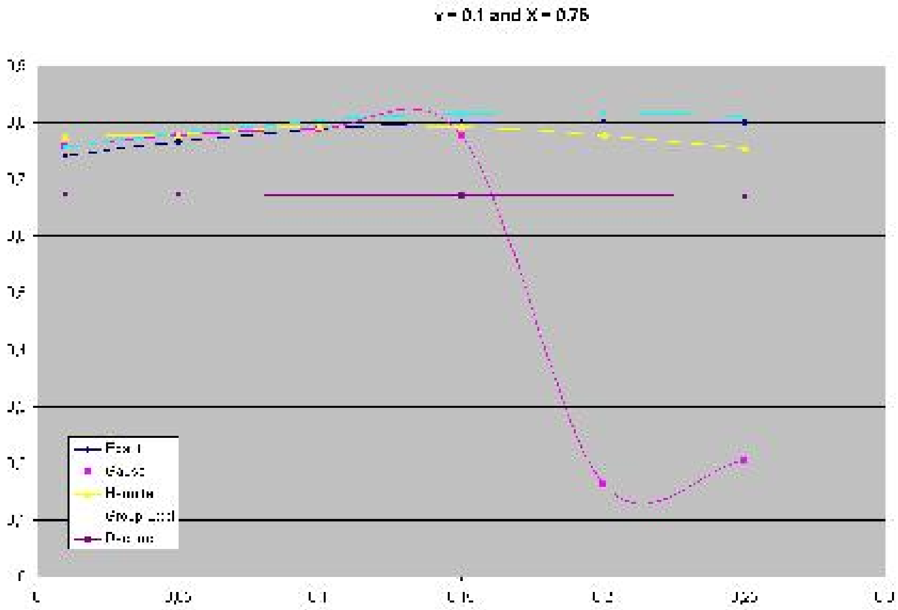
t	Exact	Gauss	Hermite	Group Ex	Decomp
0,01	0,9917	0,9916	0,9923	0,9914	0,66467
0,05	0,9833	0,9516	0,9524	0,953	0,65667
0,1	0,8993	0,893	0,8933	0,899	0,64667
0,15	0,8434	0,8317	0,8313	0,8431	0,63667
0,2	0,7889	0,7352	0,7714	0,7886	0,62667
0,25	0,7375	0,5198	0,7143	0,7372	0,61667

x=0,75

t	Exact	Gauss	Hermite	Group Ex	Decomp
0,01	0,7417	0,7567	0,7752	0,757	0,67371
0,05	0,7663	0,778	0,7793	0,7816	0,67321
0,1	0,7882	0,7892	0,7934	0,8035	0,67258
0,15	0,7999	0,778	0,7923	0,8152	0,67195
0,2	0,802	0,1649	0,7782	0,8173	0,67133
0,25	0,7995	0,206	0,7553	0,8108	0,67071


ν = 0.1 and X = 0.25


ν = 0.1 and X = 0.5



From the above diagrams it is obvious how powerful this method is. Using only two terms we can obtain similar results. Of course, in some cases the present solutions deviate from the solutions given in the table. The decomposition solution can be further improved if more-term approximations of the solution are obtained. As far as accurate results are concerned, computational experience has shown that they can be obtained easily by taking half a dozen terms. In case we do not have sufficiently high precision by using a few of the A_n, then accordingly to Rach R. [14] there are two alternatives. One is to compute additional terms by any of the available procedures. The second approach is to use the Adomian-Malakian "convergence acceleration" procedure [15]. This unique approach conveniently yields the error-damping effect of calculating many more terms of the A_n to determine whether further calculation is justified. The advantage of this global methodology is that it leads to an analytical continuous approximated solution that is very rapidly convergent [2,7,8]. This method does not take any help of linearization or any other simplifications for handling the non-linear terms. Since the decomposition parameter in general is not perturbation parameter, it follows that the non-linearities in the operator equation can be handled easily and accurate solution may be obtained for any physical problem.

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