Variational Principles in the Linear Thermoelasticity of Solids with Microstructure Having a Symmetric Stress Tensor

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Abstract

In this paper the dynamical linear thermoelasticity theory of an anisotropic and nonhomogeneous solid with microstructure having a symmetric stress tensor (LTSMSST) is considered. A consistent set of field variables is employed, and boundary–initial–value problems of the LTSMSST are defined. An alternative tensorial formulation of a boundary–initial–value problem of the LTSMSST is given. The main aim of this paper is to give some variational principles for the LTSMSST.

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1 Introduction

As C. I. Borş noticed in [1] the asymmetry of the stress tensor in a continuum theory is not a consequence of the presence of microstructure in the body; it is rather a constitutive supposition. Consequently, we may conceive mathematical models of deformable bodies with microstructure having a symmetric stress tensor **T**. Such models have been constructed by C. I. Borş in [1], [2], [3]. In the present paper the mathematical model built by C. I. Borş in [1] is considered. The basic equations, boundary conditions, and initial values of some field variables of the LTSMSST are given.

The solution of a boundary–initial–value problem of the LTSMSST can often also be characterized as the function which yields a minimum value to a related functional. The minimum of such functional is to be sought not among all functions but only within a fairly general class of functions known as *admissible processes*. The class of

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admissible processes may of course change from one problem to another but usually will include all functions satisfying some or all boundary and initial conditions of the boundary–initial–value problem and certain continuity conditions.

One of the important mathematical applications of a variational principle is to prove the existence of a solution of a boundary–initial–value problem of the theory discussed in this paper, but we shall not be concerned with this aspect.

If we assume or prove by an independent method that the boundary–initial–value problem has a solution then we can use the variational principles to construct an approximation of this solution. The principal method for constructing such an approximation is the *Ritz–Rayleigh procedure*, which consists of finding a constrained minimum of specific functional, that is, finding a minimum not among the admissible processes but only within a particularly simple subset of admissible processes.

There is another feature of a variational principle, which is of great importance in applications. The numerical quantity expressing the value of functional in an admissible process is in itself of considerable interest, because it often represents a kind of average of the solution; in fact the functional is proportional to the energy of the physical system under investigation and we might be satisfied with finding a reliable approximation to the value of the functional.

2 Basic Definitions and Governing Equations of the LTSMSST

The notation and format used, for example, by Borş [1] will largely used here. In the following we shall denote by I the time interval $[0, \infty)$, where $t = 0$ is the *initial moment*. Let B be a bounded domain of the three–dimensional Euclidean space $\mathcal{E}_3$ and occupied by an undeformed continuum at the initial moment. Suppose that ∂B, the boundary of B, is a union of a finite number of nonintersecting regular surfaces and let $\mathbf{n}$ be the outward unit normal vector to ∂B. Here the term regular surface is used in the sense of Kellogg [4]. The closure of B will be denoted by $\overline{B}$ and $\boldsymbol{\tau}_k$ will be the linear space of $k-$order tensor functions defined on the set $\overline{B} \times I$ or one of its subsets. A point $\mathbf{x} \in \partial B$ (or a point $(\mathbf{x}, t) \in \partial B \times I$) at which $\mathbf{n}$ is continuous will be called a *regular point*. Let Σ be a subset of ∂B. A function $\mathbf{f}$ will be said to be *piecewise regular* on $\Sigma \times I$ if $\mathbf{f}$ is piecewise continuous on $\Sigma \times I$, and in every regular point $\mathbf{f}$ is continuous. The reference frame is Cartesian, repeated subscripts imply summation and superposed dots indicate the order of time differentiation.

Definition 2.1 By an *admissible process in B* of the LTSMSST we understand an ordered array of functions

$$\mathbf{p} = \{\mathbf{u}, \boldsymbol{\varphi}, \theta, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}, \mathbf{g}, \mathbf{T}, \mathbf{C}, S, \mathbf{q}\} \tag{1}$$

satisfying the following smoothness conditions:

$$\begin{cases} \mathbf{u}, \boldsymbol{\varphi} : \overline{B} \times I \to \boldsymbol{\tau}_1;\ \mathbf{u}, \boldsymbol{\varphi} \in C^{2,2}(B \times (0, +\infty)); \\ \mathbf{u},\ \nabla \mathbf{u},\ \dot{\mathbf{u}},\ \nabla \dot{\mathbf{u}},\ \ddot{\mathbf{u}},\ \boldsymbol{\varphi},\ \nabla \boldsymbol{\varphi},\ \dot{\boldsymbol{\varphi}},\ \nabla \dot{\boldsymbol{\varphi}},\ \ddot{\boldsymbol{\varphi}} \in C^{0,0}(\overline{B} \times I); \end{cases} \tag{2}$$

$$\begin{cases} \theta,\ S : \overline{B} \times I \to \mathbb{R};\ \theta \in C^{2,1}(B \times (0, +\infty)), S \in C^{0,1}(B \times (0, +\infty)); \\ \theta, \nabla \theta, \dot{\theta}, S, \dot{S} \in C^{0,0}(\overline{B} \times I); \end{cases} \tag{3}$$

$$\begin{cases} \boldsymbol{\varepsilon}, \boldsymbol{\kappa} : \overline{B} \times I \to \boldsymbol{\tau}_2;\ \boldsymbol{\varepsilon}, \boldsymbol{\kappa} \in C^{1,1}(B \times (0, +\infty)); \\ \boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, \boldsymbol{\kappa}, \dot{\boldsymbol{\kappa}} \in C^{0,0}(\overline{B} \times I); \\ \mathbf{a} \cdot \boldsymbol{\varepsilon}[\mathbf{b}] = \mathbf{b} \cdot \boldsymbol{\varepsilon}[\mathbf{a}],\ (\forall)\ \mathbf{a}, \mathbf{b} \in \boldsymbol{\tau_1}; \end{cases} \tag{4}$$

$$\begin{cases} \mathbf{g} : \overline{B} \times I \to \boldsymbol{\tau}_1, \\ \mathbf{g} \in C^{1,0}(B \times (0, +\infty)) \cap C^{0,0}(\overline{B} \times I); \end{cases} \tag{5}$$

$$\begin{cases} \mathbf{T}, \mathbf{C} : \overline{B} \times I \to \boldsymbol{\tau}_2;\ \mathbf{T}, \mathbf{C} \in C^{1,0}(B \times (0, +\infty)); \\ \mathbf{T},\ \nabla \cdot \mathbf{T},\ \mathbf{C},\ \nabla \cdot \mathbf{C} \in C^{0,0}(\overline{B} \times I); \\ \mathbf{a} \cdot \mathbf{T}[\mathbf{b}] = \mathbf{b} \cdot \mathbf{T}[\mathbf{a}],\ (\forall)\ \mathbf{a}, \mathbf{b} \in \boldsymbol{\tau_1}; \end{cases} \tag{6}$$

$$\begin{cases} \mathbf{q} : \overline{B} \times I \to \boldsymbol{\tau}_1,\ \mathbf{q} \in C^{1,0}(B \times (0, +\infty)); \\ \mathbf{q},\ \nabla \mathbf{q} \in C^{0,0}(\overline{B} \times I), \end{cases} \tag{7}$$

where: $\mathbf{u}$ is the *displacement vector*, $\boldsymbol{\varphi}$ is the *microrotation field*; $\theta = T - T_0$ is the *temperature difference* (T_0 is the absolute temperature of the natural state), $\boldsymbol{\varepsilon}$ is the *strain tensor*, $\boldsymbol{\kappa}$ is the *torsion tensor*, $\mathbf{g}$ is the *temperature gradient*, $\mathbf{C}$ is the *couple–stress tensor*, S is the *entropy* per unit mass, $\mathbf{q}$ is the *heat flux vector* and ∇ is the *Hamilton operator*.

If we define as usually the addition of two admissible processes and the multiplication of an admissible process by a scalar, the set $\mathcal{A}$ of admissible processes in B becomes a linear space.

Definition 2.2 An *external system of causes in B* of the LTSMSST is the ordered array

$$\mathcal{F} = \{\mathbf{f}, \mathbf{t}, \mathbf{M}, \mathbf{c}, r, h\} \tag{8}$$

whose components satisfy the regularity conditions:

$$\mathbf{f}, \mathbf{M} : \overline{B} \times I \to \boldsymbol{\tau}_1,\ \mathbf{f}, \mathbf{M} \in C^{0,0}(\overline{B} \times I); \tag{9}$$

$$\mathbf{t},\mathbf{c} : \partial B \times I \to \boldsymbol{\tau}_1,\ \mathbf{t},\mathbf{c} \in C^{0,0}(\partial B \times I); \tag{10}$$

$$r : \overline{B} \times I \to \mathbb{R},\ r \in C^{0,0}(\overline{B} \times I); \tag{11}$$

$$h : \partial B \times I \to \mathbb{R},\ h \in C^{0,0}(\partial B \times I), \tag{12}$$

and $\mathbf{t}$ and $\mathbf{c}$ are piecewise regular functions on ∂B, where: $\mathbf{f}$ is the *specific mass force*, $\mathbf{t}$ is the *stress vector*, $\mathbf{M}$ is the *external body–couple*, $\mathbf{c}$ is the *couple stress vector*, r is the *external flow of heat source* and h is the *heat flux vector* per unit time and per unit area of $\overline{B}$.

Definition 2.3 By a *thermoelastic process in* B of the LTSMSST, corresponding to the external system of causes $\mathcal{F}$, we understand an admissible process $\mathbf{p}$ whose elements satisfy:
— the *geometrical equations*

$$\boldsymbol{\varepsilon} = \frac{1}{2}\Big(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\Big), \qquad \boldsymbol{\kappa} = (\nabla\boldsymbol{\varphi})^T; \tag{13}$$

— the *temperature–gradient relation*

$$\mathbf{g} = \nabla\theta; \tag{14}$$

— the *constitutive equations*

$$\begin{cases} \mathbf{T} &= \mathbf{A}[\boldsymbol{\varepsilon}] + \mathbf{B}[\boldsymbol{\kappa}] - \theta\,\boldsymbol{\alpha}; \\ \mathbf{C} &= \boldsymbol{\varepsilon}\,\mathbf{B} + \mathbf{H}[\boldsymbol{\kappa}] - \theta\,\boldsymbol{\beta}; \\ \rho S &= \boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon} + \boldsymbol{\beta}\cdot\boldsymbol{\kappa} + a\,\theta; \end{cases} \tag{15}$$

— the *Fourier law*

$$\mathbf{q} = -\,\mathbf{K}[\mathbf{g}]; \tag{16}$$

— the *equations of motion*

$$\begin{cases} \nabla\cdot\mathbf{T}^T + \rho\,\mathbf{f} = \rho\,\ddot{\mathbf{u}}, \\ \nabla\cdot\mathbf{C}^T + \rho\,\mathbf{M} = \rho\,\mathbf{J}[\ddot{\boldsymbol{\varphi}}]; \end{cases} \tag{17}$$

— the *energy equation*

$$\rho\,T_0\,\dot{S} + \nabla\cdot\mathbf{q} - \rho\,r = 0; \tag{18}$$

— the *Cauchy type relations*

$$\mathbf{t}(\mathbf{n},\mathbf{x}) = \mathbf{T}^T(\mathbf{x})[\mathbf{n}] = \mathbf{T}(\mathbf{x})[\mathbf{n}], \qquad \mathbf{c}(\mathbf{n},\mathbf{x}) = \mathbf{C}^T(\mathbf{x})[\mathbf{n}]; \tag{19}$$

— the *Fourier–Stokes heat flux principle*

$$h(\mathbf{n},\mathbf{x},t)=\mathbf{q}(\mathbf{x},t)\cdot\mathbf{n}; \tag{20}$$

— the *symmetry relation*

$$\mathbf{a}\cdot\mathbf{J}[\mathbf{b}] \quad = \quad \mathbf{b}\cdot\mathbf{J}[\mathbf{a}], \qquad (\forall)\ \mathbf{a},\mathbf{b}\in\boldsymbol{\tau}_1; \tag{21}$$

— the *definite positive conditions*

$$\mathbf{a}\cdot\mathbf{J}[\mathbf{a}]>0, \qquad \mathbf{a}\cdot\mathbf{K}[\mathbf{a}]>0, \qquad (\forall)\ \mathbf{a}\in\mathbb{R}^3\backslash\{\mathbf{0}\}, \tag{22}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{K}$ are continuous differentiable functions on $\overline{B}$ while ρ, $\mathbf{J}$ and c are continuous functions on $\overline{B}$.

In the last definition $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$ represent the *elastic moduli*, $\boldsymbol{\alpha}$ is the *temperature–stress tensor*, $\boldsymbol{\beta}$ is the *temperature–couple–stress tensor*, ρ is the *mass density*, c is the *specific heat*, $a=c/T_0>0$, $\mathbf{J}$ is the *microrotation tensor*, and $\mathbf{K}$ is the *heat conduction tensor*.

The symmetry condition of the stress tensor $\mathbf{T}$, specified in the last equation of (6), implies the following symmetries of the coefficients in the constitutive equations (15) :

$$\mathbf{u}\cdot\Big(\mathbf{A}[\mathbf{U}]\Big)[\mathbf{v}] \quad = \quad \mathbf{v}\cdot\Big(\mathbf{A}[\mathbf{U}]\Big)[\mathbf{u}]; \tag{23}$$

$$\mathbf{U}\cdot\mathbf{A}[\mathbf{V}] \quad = \quad \mathbf{V}\cdot\mathbf{A}[\mathbf{U}]; \tag{24}$$

$$\mathbf{U}\cdot\mathbf{H}[\mathbf{V}] \quad = \quad \mathbf{V}\cdot\mathbf{H}[\mathbf{U}]; \tag{25}$$

$$\mathbf{u}\cdot\Big(\mathbf{B}[\mathbf{U}]\Big)[\mathbf{v}] \quad = \quad \mathbf{v}\cdot\Big(\mathbf{B}[\mathbf{U}]\Big)[\mathbf{u}]; \tag{26}$$

$$\mathbf{u}\cdot\boldsymbol{\alpha}[\mathbf{v}] \quad = \quad \mathbf{v}\cdot\boldsymbol{\alpha}[\mathbf{u}]. \tag{27}$$

By the symbol $\mathbf{A}[\mathbf{V}]$, for example, we understand the two–order tensor

$$\mathbf{A}[\mathbf{V}]=A_{ijkl}V_{kl}\,\mathbf{e}_i\otimes\mathbf{e}_j \tag{28}$$

while $\boldsymbol{\varepsilon}\,\mathbf{B}$ means the two–order tensor

$$\boldsymbol{\varepsilon}\,\mathbf{B} \quad = \quad B_{ijkl}\,\varepsilon_{ij}\,\mathbf{e}_k\otimes\mathbf{e}_l \tag{29}$$

where $\mathbf{e}_k$ is an unit vector of the frame $Ox_1x_2x_3$, $\otimes$ is the notation for the tensorial product, and a centered dot between two tensors designates their inner product.

3 Boundary–Initial–Value Problems in B of the LTSMSST

Let us denote by $\Sigma_1,\ \Sigma_2, \cdots, \Sigma_6$ subsets of ∂B having the properties:

$$\begin{cases} \overline{\Sigma}_1 \cup \Sigma_2 = \overline{\Sigma}_3 \cup \Sigma_4 = \overline{\Sigma}_5 \cup \Sigma_6 = \partial B; \\ \Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Sigma_5 \cap \Sigma_6 = \emptyset, \end{cases} \tag{30}$$

where $\overline{\Sigma}_j$, $j \in \{1,3,5\}$, stands for the closure of Σ_j.

We assume that the following *data* are specified:

1. $\mathcal{F}$, an external system of causes;
2. the functions: *initial displacement* $\mathbf{u}_0 : \overline{B} \to \boldsymbol{\tau}_1$; *initial rotation* $\boldsymbol{\varphi}_0 : \overline{B} \to \boldsymbol{\tau}_1$; *initial velocity vector* $\mathbf{v}_0 : \overline{B} \to \boldsymbol{\tau}_1$; *angular initial velocity vector* $\boldsymbol{\nu}_0 : \overline{B} \to \boldsymbol{\tau}_1$; and *initial entropy* $S_0 : \overline{B} \to \mathbb{R}$, all continuous on $\overline{B}$;
3. *the surface displacement* $\widehat{\mathbf{u}} : \overline{\Sigma}_1 \times I \to \boldsymbol{\tau}_1$, a continuous function;
4. *the surface traction* $\widehat{\mathbf{t}} : \Sigma_2 \times I \to \boldsymbol{\tau}_1$, a piecewise regular function;
5. *the surface rotation* $\widehat{\boldsymbol{\varphi}} : \overline{\Sigma}_3 \times I \to \boldsymbol{\tau}_1$, a continuous function;
6. *the surface couple–stress vector* $\widehat{\mathbf{c}} : \Sigma_4 \times I \to \boldsymbol{\tau}_1$, a piecewise regular function;
7. *the surface temperature* $\widehat{\theta} : \overline{\Sigma}_5 \times I \to \mathbb{R}$, a continuous function;
8. *the surface heat flux* $\widehat{h} : \Sigma_6 \times I \to \mathbb{R}$, a piecewise regular function.

Definition 3.1 A *boundary–initial–value problem in* B (or a *mixed problem in* B) of the LTSMSST, corresponding to the extarnal system of causes $\mathcal{F}$, consists in the problem of finding all thermoelastic processes in B that satisfy:
— the *initial conditions:*

$$\begin{cases} \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}); \quad \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}); \quad \boldsymbol{\varphi}(\mathbf{x},0) = \boldsymbol{\varphi}_0(\mathbf{x}); \\ \dot{\boldsymbol{\varphi}}(\mathbf{x},0) = \boldsymbol{\nu}_0(\mathbf{x}); \quad S(\mathbf{x},0) = S_0(\mathbf{x}), \quad (\forall)\ \mathbf{x} \in \overline{B}; \end{cases} \tag{31}$$

— the *boundary conditions*:

$$\begin{cases} \mathbf{u}(\mathbf{x},t) &= \widehat{\mathbf{u}}(\mathbf{x},t), & (\forall)\ (\mathbf{x},t) \in \overline{\Sigma}_1 \times I; \\ \mathbf{t}(\mathbf{x},t) &= \widehat{\mathbf{t}}(\mathbf{x},t), & (\forall)\ (\mathbf{x},t) \in \Sigma_2 \times I; \\ \boldsymbol{\varphi}(\mathbf{x},t) &= \widehat{\boldsymbol{\varphi}}(\mathbf{x},t), & (\forall)\ (\mathbf{x},t) \in \overline{\Sigma}_3 \times I; \\ \mathbf{c}(\mathbf{x},t) &= \widehat{\mathbf{c}}(\mathbf{x},t), & (\forall)\ (\mathbf{x},t) \in \Sigma_4 \times I; \\ \theta(\mathbf{x},t) &= \widehat{\theta}(\mathbf{x},t), & (\forall)\ (\mathbf{x},t) \in \overline{\Sigma}_5 \times I; \\ h(\mathbf{x},t) &= \widehat{h}(\mathbf{x},t), & (\forall)\ (\mathbf{x},t) \in \Sigma_6 \times I. \end{cases} \tag{32}$$

Note that many different mixed problems in B of the LTSMSST exist, since one or more of the surfaces $\Sigma_1,\ \Sigma_2, \cdots,\ \Sigma_6$ may be the empty set.

Definition 3.2 By a *solution of a mixed problem in* B of the LTSMSST we mean the thermoelastic process $\mathbf{p}$ satisfying the initial conditions (31) and the boundary conditions (32).

Observation 3.1 More complicated boundary–initial–value problems in B of this theory can be introduced, possibly following the work in [5].

Definition 3.3 A *kinematical, thermal and admissible process in* B of the LTSMSST is an admissible process $\mathbf{p} \in \mathcal{A}$ that satisfies (13), (14), (15), (16) and the boundary conditions $(32)_1$, $(32)_3$, $(32)_5$ for the case when all surfaces Σ_1, Σ_3 and Σ_5 are nonempty sets.

The set $\mathcal{K}$ of kinematical, thermal, and admissible processes in B of the LTSMSST is a subset of the admissible processes $\mathcal{A}$. Note that $\mathcal{K}$ is not a linear subspace of the linear space $\mathcal{A}$.

Observation 3.2 The last condition in (31) contains the initial value of entropy, but there is the possibility to give the initial value of temperature difference $\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x})$. On account of $(15)_3$, this initial values must satisfy

$$\rho\, S_0(\mathbf{x}) \quad = \quad \boldsymbol{\alpha}(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_0(\mathbf{x}) + \boldsymbol{\beta}(\mathbf{x}) \cdot \boldsymbol{\kappa}_0(\mathbf{x}) + a(\mathbf{x})\, \theta_0(\mathbf{x}), \tag{33}$$

where:

$$\boldsymbol{\varepsilon}_0(\mathbf{x}) \quad = \quad \frac{1}{2}\Big(\nabla \mathbf{u}_0(\mathbf{x}) + (\nabla \mathbf{u}_0(\mathbf{x}))^T\Big); \qquad \boldsymbol{\kappa}_0(\mathbf{x}) \quad = \quad \Big(\nabla \boldsymbol{\varphi}_0(\mathbf{x})\Big)^T.$$

4 Alternative Formulation of a Mixed Problem in B of the LTSMSST

Let $\mathbf{f}, \mathbf{g} : \overline{B} \times I \to \boldsymbol{\tau}_p$ be continuous tensorial functions. The tensors $\mathbf{f}(\mathbf{x}, t)$ and $\mathbf{g}(\mathbf{x}, t)$ have the following analitical expressions:

$$\begin{cases} \mathbf{f}(\mathbf{x}, t) & = & f_{i_1 i_2 \cdots i_p}(\mathbf{x}, t) \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_p}, \\ \mathbf{g}(\mathbf{x}, t) & = & g_{j_1 j_2 \cdots j_p}(\mathbf{x}, t) \mathbf{e}_{j_1} \otimes \mathbf{e}_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_p}. \end{cases} \tag{34}$$

The inner product of the tensors in (34) is the function defined by $\mathbf{f} \cdot \mathbf{g} : \overline{B} \times I \to \mathrm{I\!R}$ where

$$(\mathbf{f} \cdot \mathbf{g})(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, t) = f_{i_1 i_2 \cdots i_p}(\mathbf{x}, t)\, g_{i_1 i_2 \cdots i_p}(\mathbf{x}, t). \tag{35}$$

The convolution of the tensors $\mathbf{f}$ and $\mathbf{g}$ in (34) is the function:

$$\mathbf{f} * \mathbf{g} : \overline{B} \times I \to \mathrm{I\!R}; \qquad (\mathbf{f} * \mathbf{g})(\mathbf{x}, t) = \int_0^t f(\mathbf{x}, t - \tau) \cdot g(\mathbf{x}, \tau)\, d\tau. \tag{36}$$

We may formally replace the tensor $\mathbf{f}$ with one of the scalar functions i and 1 defined by

$$i(\mathbf{x}, t) = t, \quad 1(\mathbf{x}, t) = 1, \quad (\forall)\ (\mathbf{x}, t) \in \overline{B} \times I. \tag{37}$$

The convolution of the functions 1 and $\mathbf{g}$ will be denoted by $\overline{\mathbf{g}}$:

$$\overline{\mathbf{g}}(\mathbf{x},t) = (1 * \mathbf{g})(\mathbf{x},t) = \int_0^t \mathbf{g}(\mathbf{x},\tau)\,d\tau,\ (\forall)\ (\mathbf{x},t) \in \overline{B} \times I. \tag{38}$$

Obviously, (38) implies

$$\dot{\overline{\mathbf{g}}}\,(\mathbf{x},t) = \mathbf{g}(\mathbf{x},t);\ (1 * \dot{\mathbf{g}})(\mathbf{x},t) = \mathbf{g}(\mathbf{x},t) - \mathbf{g}(\mathbf{x},0),\ (\forall)\ (\mathbf{x},t) \in \overline{B} \times I. \tag{39}$$

The properties of convolution are mentioned, for example, in Gurtin's work [6].

In the proofs of some theorems we shall use the following identities [7]:

$$\mathbf{f} \cdot (\nabla \mathbf{g})^T = \nabla \cdot (\mathbf{f}[\mathbf{g}]) - (\nabla \cdot \mathbf{f}^T) \cdot \mathbf{g}; \tag{40}$$

$$(\nabla \cdot \mathbf{h})\,q = \nabla \cdot (q\,\mathbf{h}) - \mathbf{h} \cdot (\nabla q), \tag{41}$$

where $\mathbf{f}$ is a second order tensor, $\mathbf{g}$ and $\mathbf{h}$ are vectorial functions and q is a scalar one. The divergence theorem aplied both to the second order tensor $\mathbf{f}$ and the vector $\mathbf{g}$ gives

$$\int_B \nabla \cdot (\mathbf{f}[\mathbf{g}])\,dv = \int_{\partial B} \mathbf{f}^T[\mathbf{n}] \cdot \mathbf{g}\,da. \tag{42}$$

If h is a scalar function and $\mathbf{g}$ is a vectorial one, then the same theorem leads to

$$\int_B \nabla \cdot (h\,\mathbf{g})\,dv = \int_{\partial B} \mathbf{g}[\mathbf{n}]\,h\,da. \tag{43}$$

Definition 4.1 An *external system of data in* B of the LTSMSST is the ordered array of functions

$$\mathcal{L} = \{\mathbf{F}, \widehat{\mathbf{u}}, \widehat{\mathbf{t}}, \widehat{\varphi}, \widehat{\mathbf{c}}, \widehat{\theta}, \widehat{h}, \mathbf{u}_0,\ \boldsymbol{\varphi}_0,\ \mathbf{v}_0,\ \boldsymbol{\nu}_0, S_0\}, \tag{44}$$

where

$$\mathbf{F} = (\rho\mathbf{f}, \rho\mathbf{M}, -\rho\,W), \quad W = \frac{1}{T_0}\overline{r} + S_0. \tag{45}$$

Theorem 4.1. *An element* $\mathbf{p} \in \mathcal{K}$ *satisfying the boundary conditions* $(32)_2$, $(32)_4$ *and* $(32)_6$ *is a solution of the mixed problem in* B *of the LTSMSST, corresponding to the external system of data* $\mathcal{L}$ *in* (44), *if and only if the following equations hold:*

$$i * \nabla \cdot \mathbf{T}^T + \widetilde{\mathbf{f}} = \rho\,\mathbf{u}; \tag{46}$$

$$i * \nabla \cdot \mathbf{C}^T + \widetilde{\mathbf{M}} = \mathbf{J}[\boldsymbol{\varphi}]; \tag{47}$$

$$S + \frac{1}{\rho\,T_0}\nabla \cdot \overline{\mathbf{q}} = W, \tag{48}$$

where

$$\begin{cases} \widetilde{\mathbf{f}}(\mathbf{x},t) = \rho(\mathbf{x})\Big((i * \mathbf{f})(\mathbf{x},t) + \mathbf{u}_0(\mathbf{x}) + t\,\mathbf{v}_0(\mathbf{x})\Big), \\ \widetilde{\mathbf{M}}(\mathbf{x},t) = \rho(\mathbf{x})\Big((i * \mathbf{M})(\mathbf{x},t) + \mathbf{J}(\mathbf{x})[\boldsymbol{\varphi}_0(\mathbf{x}) + t\,\boldsymbol{\nu}_0(\mathbf{x})]\Big). \end{cases} \tag{49}$$

Proof. Suppose that $\mathbf{p} \in \mathcal{K}$ is a solution of the mixed problem in B of the LTSMSST. Then:

$$i * \Big(\nabla \mathbf{T}^T + \rho \mathbf{f}\Big)(\mathbf{x}, t) \quad = \quad \rho(\mathbf{x}) \int_0^t (t-\tau)\ \ddot{\mathbf{u}}\ (\mathbf{x}, \tau)\, d\tau; \tag{50}$$

$$i * \Big(\nabla \mathbf{C}^T + \rho \mathbf{M}\Big)(\mathbf{x}, t) \quad = \quad \rho(\mathbf{x}) \int_0^t (t-\tau) \mathbf{J}(\mathbf{x})[\ddot{\boldsymbol{\varphi}}\ (\mathbf{x}, \tau)]\, d\tau. \tag{51}$$

Twice integrating by parts the right hand sides of equation (50) and equation (51), and use of notations (49) leads to (46) and (47).

From the energy equation (18), we get

$$\dot{S} + \frac{1}{\rho T_0} \nabla \cdot \mathbf{q} = \frac{1}{T_0} r. \tag{52}$$

The convolution of (52) with the constant function 1 in (37) along with (38), (39) and the expression of W in (45), leads to (48).

The initial conditions (31) result from (46)−(48) by the differentiation with respect to t of relations (46) and (47), (keeping in mind the notations (45), (49)) followed by their evaluation at $t = 0$.

Conversly, let us assume that $\mathbf{p} \in \mathcal{K}$ satisfying the hypoteses of the Theorem 4.1 is given such that (46) − (48) are fulfilled and let us prove that $\mathbf{p}$ is a solution of the mixed problem in B of the LTSMSST. We need to prove that $\mathbf{p} \in \mathcal{K}$ verifies the initial conditions (31), the motion equations (17) and the energy equation (18).

The initial conditions follow in the same way as in the first part of the theorem. The energy equation results by the differentiation with respect to time of equation (48) and then use of property $(39)_1$, while the equations of motion are deduced from (46) and (47) by the twice differentiating with respect to time and using Leibniz's differentiation formula for an integral depending on a parameter. □

Theorem 4.2. *The admissible process* $\mathbf{p} \in \mathcal{A}$ *is a solution of the mixed problem in* B *of the LTSMSST, corresponding to the external system of data in* $\mathcal{L}$ *in* (44), *if and only if equations* (46) − (48) *as well as the equations*

$$\begin{cases} i \ * \ \Big(\dfrac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2} - \boldsymbol{\varepsilon}\Big) = \mathbf{0}, \\ i \ * \ \Big((\nabla \boldsymbol{\varphi})^T - \boldsymbol{\kappa}\Big) = \mathbf{0}, \\ i \ * \ (\nabla \overline{\theta} - \overline{\mathbf{g}}) = \mathbf{0}; \end{cases} \tag{53}$$

$$\begin{cases} i \ * \ \Big(\mathbf{A}[\boldsymbol{\varepsilon}] + \mathbf{B}[\boldsymbol{\kappa}] - \dfrac{T_0}{c}(\rho\, S - \boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon} - \boldsymbol{\beta} \cdot \boldsymbol{\kappa})\, \boldsymbol{\alpha} - \mathbf{T}\Big) = \mathbf{0}, \\ i \ * \ \Big(\boldsymbol{\varepsilon}\, \mathbf{B} + \mathbf{H}[\boldsymbol{\kappa}] - \dfrac{T_0}{c}(\rho\, S - \boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon} - \boldsymbol{\beta} \cdot \boldsymbol{\kappa})\, \boldsymbol{\beta} - \mathbf{C}\Big) = \mathbf{0}, \\ i \ * \ \Big(\dfrac{T_0}{c}(\rho\, S - \boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon} - \boldsymbol{\beta} \cdot \boldsymbol{\kappa}) - \theta\Big) = 0; \end{cases} \tag{54}$$

$$i \ * \ \Big(\mathbf{K}[\overline{\mathbf{g}}] + \overline{\mathbf{q}}\Big) = \mathbf{0}; \tag{55}$$

$$\begin{cases} i & * & (\widehat{\mathbf{u}}-\mathbf{u})=\mathbf{0}, & on & \overline{\Sigma}_1\times I,\\ i & * & (\mathbf{t}-\widehat{\mathbf{t}})=\mathbf{0}, & on & \Sigma_2\times I,\\ i & * & (\widehat{\boldsymbol{\varphi}}-\boldsymbol{\varphi})=\mathbf{0}, & on & \overline{\Sigma}_3\times I,\\ i & * & (\mathbf{c}-\widehat{\mathbf{c}})=\mathbf{0}, & on & \Sigma_4\times I,\\ i & * & (\widehat{\overline{\theta}}-\overline{\theta})=0, & on & \overline{\Sigma}_5\times I,\\ i & * & (\overline{h}-\widehat{\overline{h}})=\mathbf{0}, & on & \Sigma_6\times I, \end{cases} \tag{56}$$

are satisfied.

Proof. If $\mathbf{p}\in\mathcal{A}$ is a solution of the mixed problem in B of the LTSMSST corresponding to the external system of data in $\mathcal{L}$ in (44), then $\mathbf{p}$ is a thermoelastic process satisfying the boundary conditions (32) and the initial conditions (31). From Theorem 4.1 it results that equations $(46)-(48)$ are satisfied. The conditions $(53)-(56)$ are obviously satisfied because their corresponding second factor in the convolution product is equal to zero.

Conversely, let us suppose that $\mathbf{p}\in\mathcal{A}$ is such that both equations $(46)-(48)$ and $(53)-(56)$ are satisfied. First, from $(56)_2$, $(56)_4$, $(56)_6$ and the properties of the convolution product, it results that the boundary conditions $(32)_2$, $(32)_4$ and $(32)_6$ are fulfilled. Then, from Theorem 4.1 we deduce that $\mathbf{p}$ satisfies the motion equations, the energy equation and the initial conditions. Finally, from equations (53), (54), $(56)_1$, $(56)_3$, $(56)_5$ and properties of the convolution product, we easily see that the remaining conditions that need to be fulfilled by a solution of the mixed problem in B of the LTSMSST are also satisfied and in this way the theorem is completely proved. □

5 Variational Principles of the LTSMSST

Variational principles for the LTSMSST are naturally suggested by the works of Gurtin [6], [7], Nickell and Sackman [8], Ieşan [9] and a recent one by Crăciun [10]. In these treatments of various theories of continuum, the quoted authors, using some operational methods, explicitly introduce the initial conditions appropriate to the problem into the field equations and governing functionals and derive alternative characterizations of the problems. The results in this section represent an extension of these concepts to the LTSMSST.

We shall establish a general variational principle of the LTSMSST which generates all conditions that must be satisfied by an admissible process $\mathbf{p}\in\mathcal{A}$ in order to be a solution of a mixed problem in B of the LTSMSST. Several special variational principles of the LTSMSST can then be derived in the same way as in [6]-[10], depending on the extent to which certain of the requirements in the definition of a thermoelastic process are taken to be identically satisfied. From these special cases we shall take that of a kinematical, thermal and admissible process in B. The obtained results essentially use three lemmas of the variational calculus presented, for example, in [7][p.224].

In this section we shall consider Frechet differentiable functionals defined on the linear space $\mathcal{A}$ or on the set $\mathcal{K}$. If Ω is a such functional, then we formally define $\delta_{\widetilde{\mathbf{p}}}\Omega\{\mathbf{p}\}$ on $\mathcal{A}$, or on the set $\mathcal{K}$, by

$$\delta_{\widetilde{\mathbf{p}}}\Omega\{\mathbf{p}\} \quad = \quad \frac{d}{d\lambda}\Omega\{\mathbf{p}+\lambda\widetilde{\mathbf{p}}\}\Big|_{\lambda=0}\ , \tag{57}$$

where $\mathbf{p}+\lambda\widetilde{\mathbf{p}} \in \mathcal{A}$ for every real number λ, or $\mathbf{p}+\lambda\widetilde{\mathbf{p}} \in \mathcal{K}$, with certain constraints for $\widetilde{\mathbf{p}} \neq \mathbf{0}$. If $\delta_{\widetilde{\mathbf{p}}}\Omega\{\mathbf{p}\}$ exists and it is equal to zero for any $\widetilde{\mathbf{p}}$ satisfying the above requirements, then we write

$$\delta\Omega\{\mathbf{p}\} \quad = \quad 0. \tag{58}$$

For any $t \in I$ and for any external system of data $\mathcal{L}$ in (44) we introduce the functional $\Lambda_t\{\cdot\}$ defined on $\mathcal{A}$ by (for sake of brevity, we shall leave out the arguments $\mathbf{x}$ and t of the integrands)

$$\begin{aligned}
\Lambda_t\{\mathbf{p}\} \quad = \quad & \int_B i * \Big(\frac{1}{2}\boldsymbol{\varepsilon} * \mathbf{A}[\boldsymbol{\varepsilon}] + \boldsymbol{\varepsilon} * \mathbf{B}[\boldsymbol{\kappa}] + \frac{1}{2}\boldsymbol{\kappa} * \mathbf{H}[\boldsymbol{\kappa}] + \\
+ \quad & \frac{1}{2a}(\rho S - \boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon} - \boldsymbol{\beta}\cdot\boldsymbol{\kappa}) * (\rho S - \boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon} - \boldsymbol{\beta}\cdot\boldsymbol{\kappa}) - \\
- \quad & \frac{1}{2T_0}\overline{\mathbf{g}} * \mathbf{K}[\mathbf{g}] - \frac{1}{T_0}\overline{\mathbf{g}} * \mathbf{q} - \mathbf{T} * \boldsymbol{\varepsilon} - \mathbf{C} * \boldsymbol{\kappa}\Big)\, dv + \\
+ \quad & \frac{1}{2}\int_B \rho\Big(\mathbf{u} * \mathbf{u} + \mathbf{J}[\boldsymbol{\varphi}] * \boldsymbol{\varphi}\Big)\, dv - \\
- \quad & \int_B \Big(\Big(i * \nabla\cdot\mathbf{T}^T - \widetilde{\mathbf{f}}\Big) * \mathbf{u} + \Big(i * \nabla\cdot\mathbf{C}^T - \widetilde{\mathbf{M}}\Big) * \boldsymbol{\varphi}\Big)\, dv - \\
- \quad & \int_B i * \rho\Big(S + \frac{1}{\rho T_0}\nabla\cdot\overline{\mathbf{q}} - W\Big) * \theta\, dv + \\
+ \quad & \int_{\overline{\Sigma}_1} i * \mathbf{t} * \widehat{\mathbf{u}}\, da + \int_{\Sigma_2} i * (\mathbf{t} - \widehat{\mathbf{t}}) * \mathbf{u}\, da + \\
+ \quad & \int_{\overline{\Sigma}_3} i * \mathbf{c} * \widehat{\boldsymbol{\varphi}}\, da + \int_{\Sigma_4} i * (\mathbf{c} - \widehat{\mathbf{c}}) * \boldsymbol{\varphi}\, da + \\
+ \quad & \frac{1}{T_0}\Big(\int_{\overline{\Sigma}_5} i * \overline{h} * \widehat{\theta}\, da + \int_{\Sigma_6} i * (\overline{h} - \overline{\widehat{h}}) * \theta\, da\Big).
\end{aligned} \tag{59}$$

Theorem 5.1. *If the heat conduction tensor* $\mathbf{K}$ *is symmetric, then the admissible process* $\mathbf{p} \in \mathcal{A}$ *is a solution of the mixed problem in* B *of the LTSMSST, corresponding to the external data system* $\mathcal{L}$, *if and only if*

$$\delta\Lambda_t\{\mathbf{p}\} \quad = \quad 0, \qquad \textit{for any} \quad t \in I. \tag{60}$$

Proof. We have to calculate $\delta_{\widetilde{\mathbf{p}}}\Omega\{\mathbf{p}\}$ by using (57). In all these calculi we use the definition and the properties of convolution taking into account the identities:

$$\begin{cases}
\widetilde{\boldsymbol{\varepsilon}} * \mathbf{A}[\boldsymbol{\varepsilon}] = \mathbf{A}[\widetilde{\boldsymbol{\varepsilon}}] * \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} * \mathbf{A}[\widetilde{\boldsymbol{\varepsilon}}], \quad \widetilde{\boldsymbol{\kappa}} * \mathbf{H}[\boldsymbol{\kappa}] = \mathbf{H}[\widetilde{\boldsymbol{\kappa}}] * \boldsymbol{\kappa} = \boldsymbol{\kappa} * \mathbf{H}[\widetilde{\boldsymbol{\kappa}}], \\
\widetilde{\boldsymbol{\varphi}} * \mathbf{J}[\boldsymbol{\varphi}] = \mathbf{J}[\widetilde{\boldsymbol{\varphi}}] * \boldsymbol{\varphi} = \boldsymbol{\varphi} * \mathbf{J}[\widetilde{\boldsymbol{\varphi}}], \quad \overline{\widetilde{\mathbf{g}}} * \mathbf{K}[\mathbf{g}] = \overline{\mathbf{g}} * \mathbf{K}[\widetilde{\mathbf{g}}] = \mathbf{K}[\overline{\mathbf{g}}] * \widetilde{\mathbf{g}};
\end{cases} \tag{61}$$

$$\widetilde{\mathbf{t}} = \widetilde{\mathbf{T}}^T[\mathbf{n}] = \widetilde{\mathbf{T}}[\mathbf{n}], \qquad \widetilde{\mathbf{c}} = \widetilde{\mathbf{C}}^T[\mathbf{n}]; \qquad \widetilde{h} = \widetilde{\mathbf{q}}\cdot\mathbf{n}; \tag{62}$$

$$\begin{cases} \int_B \left(\nabla\cdot\widetilde{\mathbf{T}}^T\right) * \mathbf{u}\,dv = \int_{\partial B}\widetilde{\mathbf{t}} * \mathbf{u}\,da - \int_B \widetilde{\mathbf{T}} * (\nabla\mathbf{u})^T\,dv, \\ \int_B \left(\nabla\cdot\widetilde{\mathbf{C}}^T\right) * \boldsymbol{\varphi}\,dv = \int_{\partial B}\widetilde{\mathbf{c}} * \boldsymbol{\varphi}\,da - \int_B \widetilde{\mathbf{C}} * (\nabla\boldsymbol{\varphi})^T\,dv, \\ \int_B (\nabla\cdot\overline{\widetilde{\mathbf{q}}}) * \theta\,dv = \int_{\partial B}\overline{\widetilde{h}} * \theta\,da - \int_B \overline{\widetilde{\mathbf{q}}} * (\nabla\theta)\,dv. \end{cases} \tag{63}$$

By using (57), (59) and (61) – (63), after a straightforward calculation one obtains

$$\begin{aligned} \delta_{\widetilde{\mathbf{p}}}\Lambda\{\mathbf{p}\} = & \int_B i * \Big(\mathbf{A}[\boldsymbol{\varepsilon}] + \mathbf{B}[\boldsymbol{\kappa}] - \frac{1}{a}(\rho S - \boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon} - \boldsymbol{\beta}\cdot\boldsymbol{\kappa})\,\boldsymbol{\alpha} - \mathbf{T}\Big) * \widetilde{\boldsymbol{\varepsilon}}\,dv + \\ + & \int_B i * \Big(\boldsymbol{\varepsilon}\mathbf{B} + \mathbf{H}[\boldsymbol{\kappa}] - \frac{1}{a}(\rho S - \boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon} - \boldsymbol{\beta}\cdot\boldsymbol{\kappa})\,\boldsymbol{\beta} - \mathbf{C}\Big) * \widetilde{\boldsymbol{\kappa}}\,dv + \\ + & \int_B i * \rho\left(\frac{1}{a}(\rho S - \boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon} - \boldsymbol{\beta}\cdot\boldsymbol{\kappa}) - \theta\right) * \widetilde{S}\,dv - \\ - & \int_B i * \frac{1}{T_0}\Big(\mathbf{K}[\overline{\mathbf{g}}] + \overline{\mathbf{q}}\Big) * \widetilde{\mathbf{g}}\,dv - \int_B \Big(i * \nabla\cdot\mathbf{T}^T + \widetilde{\mathbf{f}} - \rho\,\mathbf{u}\Big) * \widetilde{\mathbf{u}}\,dv - \\ - & \int_B \Big(i * \nabla\cdot\mathbf{C}^T + \widetilde{\mathbf{M}} - \rho\,\mathbf{J}[\boldsymbol{\varphi}]\Big) * \widetilde{\boldsymbol{\varphi}}\,dv - \\ - & \int_B i * \rho\left(S + \frac{1}{\rho T_0}\nabla\cdot\overline{\mathbf{q}} - W\right) * \widetilde{\theta}\,dv + \\ + & \int_B \left(\frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^T}{2} - \boldsymbol{\varepsilon}\right) * \widetilde{\mathbf{T}}\,dv + \\ + & \int_B ((\nabla\boldsymbol{\varphi})^T - \boldsymbol{\kappa}) * \widetilde{\mathbf{C}}\,dv + \frac{1}{T_0}\int_B (\nabla\overline{\theta} - \overline{\mathbf{g}}) * \widetilde{\mathbf{q}}\,dv + \\ + & \int_{\Sigma_1} i * (\widehat{\mathbf{u}} - \mathbf{u}) * \widetilde{\mathbf{t}}\,da + \int_{\Sigma_2} i * (\mathbf{t} - \widehat{\mathbf{t}}) * \widetilde{\mathbf{u}}\,da + \\ + & \int_{\Sigma_3} i * (\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) * \widetilde{\mathbf{c}}\,da + \int_{\Sigma_4} i * (\mathbf{c} - \widehat{\mathbf{c}}) * \widetilde{\boldsymbol{\varphi}}\,da + \\ + & \frac{1}{T_0}\Big(\int_{\Sigma_5} i * (\overline{\widehat{\theta}} - \overline{\theta}) * \widetilde{h}\,da + \int_{\Sigma_6} i * (\overline{h} - \overline{\widehat{h}}) * \widetilde{\theta}\,da\Big). \end{aligned} \tag{64}$$

If $\mathbf{p} \in \mathcal{A}$ is a solution of the mixed problem in B of the LTSMSST from (64) and Theorem 4.2 it results that

$$\delta_{\widetilde{\mathbf{p}}}\Lambda_t\{\mathbf{p}\} = 0, \quad (\forall)\ t \in [0, +\infty) \quad \text{i} \quad (\forall)\ \widetilde{\mathbf{p}} \in \mathcal{A}, \tag{65}$$

therefore (60) is satisfied.

Conversly, let us suppose that (60) is fulfilled. This means that (65) is also true. We choose

$$\widetilde{\mathbf{p}} = \{\widetilde{\mathbf{u}}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, \mathbf{0}\} \tag{66}$$

so that $\widetilde{\mathbf{u}} \in C^{\infty,\infty}(\overline{B} \times I)$ vanishes on the set $(N \cap \overline{B}) \times [0, +\infty)$ together with all its spatial derivatives, where N is a neighbourhood of the boundary ∂B. With this choice of $\widetilde{\mathbf{p}}$, (60) becomes

$$\int_B \Big(i * \nabla \cdot \mathbf{T}^T + \widetilde{\mathbf{f}} - \rho\, \mathbf{u} \Big) * \widetilde{\mathbf{u}}\, dv \quad = \quad 0 \qquad 0 \le t < +\infty. \tag{67}$$

Because (67) must be true for any $\widetilde{\mathbf{u}}$ in the class $C^{\infty,\infty}(\overline{B} \times [0, +\infty))$ which vanishes near the boundary of the region B, by using the first Lemma in [7] we deduce (46).

Analogous, by choosing in succession $\widetilde{\mathbf{p}}$ in the forms:

$$\begin{array}{lllllll}
\widetilde{\mathbf{p}} & = & \{\mathbf{0}, \widetilde{\boldsymbol{\varphi}}, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, \mathbf{0}\}; & \quad & \widetilde{\mathbf{p}} & = & \{\mathbf{0}, \mathbf{0}, \widetilde{\theta}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, \mathbf{0}\}; \\
\widetilde{\mathbf{p}} & = & \{\mathbf{0}, \mathbf{0}, 0, \widetilde{\boldsymbol{\varepsilon}}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, \mathbf{0}\}; & & \widetilde{\mathbf{p}} & = & \{\mathbf{0}, \mathbf{0}, 0, \mathbf{0}, \widetilde{\boldsymbol{\kappa}}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, \mathbf{0}\}; \\
\widetilde{\mathbf{p}} & = & \{\mathbf{0}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0}, \widetilde{\mathbf{g}}, \mathbf{0}, \mathbf{0}, 0, \mathbf{0}\}; & & \widetilde{\mathbf{p}} & = & \{\mathbf{0}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0}, \widetilde{\mathbf{T}}, \mathbf{0}, \mathbf{0}, 0, \mathbf{0}\}; \\
\widetilde{\mathbf{p}} & = & \{\mathbf{0}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \widetilde{\mathbf{C}}, 0, \mathbf{0}\}; & & \widetilde{\mathbf{p}} & = & \{\mathbf{0}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \widetilde{S}, \mathbf{0}\}; \\
\widetilde{\mathbf{p}} & = & \{\mathbf{0}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, \widetilde{\mathbf{q}}\}, & & & &
\end{array} \tag{68}$$

where the functions $\widetilde{\boldsymbol{\varphi}}$, $\widetilde{\theta}$, $\widetilde{\boldsymbol{\varepsilon}}$, $\widetilde{\boldsymbol{\kappa}}$, $\widetilde{\mathbf{g}}$, $\widetilde{\mathbf{T}}$, $\widetilde{\mathbf{C}}$, $\widetilde{S}$, $\widetilde{\mathbf{q}}$ (in the class $C^{\infty,\infty}(\overline{B} \times I)$) are such that they vanish on the set $(N \cap \overline{B}) \times [0, +\infty)$ along with their spatial derivatives (where N is a neighbourhood of the boundary ∂B) we respectively obtain the relations: (47); (48); $(54)_1$; $(54)_2$; (55); $(53)_1$; $(53)_2$; $(54)_3$; $(53)_3$.

In tis way, (65) reduces to

$$\begin{aligned}
& \int_{\Sigma_1} i * (\widehat{\mathbf{u}} - \mathbf{u}) * \widetilde{\mathbf{t}}\, da + \int_{\Sigma_2} i * (\mathbf{t} - \widehat{\mathbf{t}}) * \widetilde{\mathbf{u}}\, da + \\
+ \; & \int_{\Sigma_3} i * (\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) * \widetilde{\mathbf{c}}\, da + \int_{\Sigma_4} i * (\mathbf{c} - \widehat{\mathbf{c}}) * \widetilde{\boldsymbol{\varphi}}\, da + \\
+ \; & \frac{1}{T_0} \Big(\int_{\Sigma_5} i * (\overline{\overline{\theta}} - \overline{\theta}) * \widetilde{h}\, da + \int_{\Sigma_6} i * (\overline{h} - \overline{\overline{h}}) * \widetilde{\theta}\, da \Big) = 0,
\end{aligned} \tag{69}$$

for any $t \in [0, +\infty)$ and for any $\widetilde{\mathbf{p}} \in \mathcal{A}$, where the functions $\widetilde{\mathbf{t}}$, $\widetilde{\mathbf{c}}$ and $\widetilde{h}$ are given by (62). By successively choosing the admissible process $\widetilde{\mathbf{p}}$ in (69) in the forms (66), $(68)_1$, $(68)_2$, $(68)_6$, $(68)_7$, $(68)_9$, taking into account (62) and using the other two lemmas in [7], we see that $\mathbf{p}$ also satisfies relations (56). These results, together with Theorem 4.2, prove that the admissible process $\mathbf{p} \in \mathcal{A}$ with property (65) satisfies the equations of motion, the constitutive equations, the geometrical equations, the Fourier law, the boundary conditions and the initial one. This means that the admissible process $\mathbf{p}$ satisfying equation (60) is a solution of the mixed boundary in B of the LTSMSST and thus the theorem is proved. □

Theorem 5.2. *Let $\mathcal{K}$ be the set of kinematical, thermal and admissible process in B*

of the LTSMSST. For any $t \in [0, +\infty)$, *we define the functional* $\Phi_t\{\cdot\} : \mathcal{K} \to \mathrm{I\!R}$,

$$\begin{aligned} \Phi_t\{\mathbf{p}\} &= \frac{1}{2}\int_B i * \Big(\mathbf{T} * \boldsymbol{\varepsilon} + \mathbf{C} * \boldsymbol{\kappa} - \rho\, S * \theta + \frac{1}{T_0}\overline{\mathbf{q}} * \mathbf{g}\Big)\, dv + \\ &+ \int_B \Big(\frac{1}{2}\rho(\mathbf{u} * \mathbf{u} + \mathbf{J}[\boldsymbol{\varphi}] * \boldsymbol{\varphi}) - \widetilde{\mathbf{F}} * \mathbf{U}\Big)\, dv - \\ &- \int_{\Sigma_2} i * \widehat{\mathbf{t}} * \mathbf{u}\, da - \int_{\Sigma_4} i * \widehat{\mathbf{c}} * \mathbf{u}\, da - \frac{1}{T_0}\int_{\Sigma_6} i * \overline{\overline{h}} * \theta\, da \end{aligned} \tag{70}$$

for all $\mathbf{p} = \{\mathbf{u}, \boldsymbol{\varphi}, \theta, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}, \mathbf{g}, \mathbf{T}, \mathbf{C}, S, \mathbf{q}\} \in \mathcal{K}$. *Then*

$$\delta\, \Phi_t\{\mathbf{p}\} \;=\; 0 \qquad (0 \le t < +\infty) \tag{71}$$

at $\mathbf{p} \in \mathcal{K}$ *if and only if* $\mathbf{p}$ *is a solution of the mixed problem in* B *of the LTSMSST.*

Proof. Let $\mathbf{p} \in \mathcal{K}$ and $\widetilde{\mathbf{p}} \in \mathcal{A}$ be so that

$$\mathbf{p} + \lambda\, \widetilde{\mathbf{p}} \in \mathcal{K}, \qquad (\forall)\ \lambda \in \mathrm{I\!R}. \tag{72}$$

The condition (72) is true if and only if $\widetilde{\mathbf{p}}$ satisfies the constitutive equations, the geometrical equations, vanishing initial conditions and homogeneous boundary conditions on the sets $\overline{\Sigma}_1 \times I$, $\overline{\Sigma}_3 \times I$ and $\overline{\Sigma}_5 \times I$.

After a simple calculation, we find

$$\begin{aligned} \delta\, \Phi_t\{\mathbf{p}\} &= -\int_B \Big(i * \nabla \cdot \mathbf{T}^T + \widetilde{\mathbf{f}} - \rho\, \mathbf{u}\Big) * \widetilde{\mathbf{u}}\, dv - \\ &- \int_B \Big(i * \nabla \cdot \mathbf{C}^T + \widetilde{\mathbf{M}} - \rho\, \mathbf{J}[\boldsymbol{\varphi}]\Big) * \widetilde{\boldsymbol{\varphi}}\, dv - \\ &- \int_B i * \rho\, \Big(S + \frac{1}{\rho\, T_0}\nabla \cdot \overline{\mathbf{q}} - W\Big) * \widetilde{\theta}\, dv + \\ &+ \int_{\Sigma_2} i * (\mathbf{t} - \widehat{\mathbf{t}}) * \widetilde{\mathbf{u}}\, da + \int_{\Sigma_4} i * (\mathbf{c} - \widehat{\mathbf{c}}) * \widetilde{\boldsymbol{\varphi}}\, da + \\ &+ \frac{1}{T_0}\int_{\Sigma_6} i * (\overline{h} - \overline{\overline{h}}) * \widetilde{\theta}\, da, \end{aligned} \tag{73}$$

for any $t \in [0, +\infty)$, for any $\mathbf{p} \in \mathcal{K}$ and for any $\widetilde{\mathbf{p}} \in \mathcal{A}$ satisfying (72). The rest of the proof of this theorem is similar to the proof of the previous one. □

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