On the Propagation of Transverse Acoustic Waves in the form of Harmonic Wavelets in Isotropic Media

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Abstract

In this paper we define the harmonic wavelet solutions of the fundamental equations of acoustic linear waves in isotropic media. The solution, given for the most general values of the physical constant, is parametrized by the scale (or level) factor of wavelets. In doing so, at each level some more details are added in order to give a finer model of the evolution.

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1 Introduction

The propagation of linear transverse acoustic waves in isotropic media, in presence of mechanical relaxation phœnomena [11, 6, 7, 14] is considered from the point of view of the wavelet theory. The velocity and the attenuation of the waves have been investigated in [11], where the authors show the relations between the *complex wave number* $\mathcal{K}$ and the *angular frequency* ω. This attenuation of the wave was considered [2] also as a localization in time, leading to a search for a suitable localized bases able to represent evolution. In this sense wave propagation is interpreted as a superposition of "small range" waves in the space of the physical variable (time). Moreover since the wave solution, classicaly, takes into account some complex functions, the basic "small" wave must be a complex function. This suggests us to apply as investigation tool of wave propagation, the harmonic wavelets [13], which are complex bases of wavelet functions. In other words, the wavelet solution of the acoustic waves in isotropic media is obtained as superposition of harmonic wavelets. Since each wavelet family, as well as the harmonic wavelets, depends on two parameters: the scale (level or resolution) factor and the translation factor, the wavelet solution can be investigated at various levels giving, as a function of the scale, various approximate solution. The approximation is not intended from numerical point of view but from modeling, in the sense that we propose a method to visualize, at each scale, the wave propagation. According the wavelet theory the lower scale gives the coarse approximation. The model will improve increasing the scale, where more details are present.

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In the following, we compute the harmonic wavelet solutions of the fundamental equations of the theory of acoustic waves in isotropic media, showing that the transverse waves, obtained in [11], are harmonic wavelet solutions at the coarsest level. Harmonic wavelets [13, 12] have an exact analytical expression allowing us to give the exact expression for the connection coefficients (see also [2]), so that, using the Galerkin-Petrov method, we obtain the harmonic wavelet solution of the acoustic waves in isotropic media.

The fundamental equations of the propagation of transverse acoustic waves in isotropic media reduce to a couple of equations in the displacement and on the scalar stress function. The wavelet solution, which in general depend on the physical constant will depend also on the level of resolution. At the level $N = 0$, the time harmonic solution coincides with the transverse (harmonic) wave solution [2, 11]. We will give also the general form of the solution at the reolution level $N = 1$, and in a special case, fixing the values of the many involved parameters the propagation is explicitly performed.

2 Preliminary remarks

The theory of acoustic wave propagation in isotropic continuous media, based on the non-equilibrium thermodynamics (see e.g. [9, 10, 11, 14]), is studied. We assume that, in the linear approach, the components of the strain tensor $\epsilon_{\alpha\beta}$ are related to the components of the displacement $u_\alpha = u_\alpha(x_\beta)$, $(\alpha, \beta = 1, 2, 3)$, by the equation

$$\epsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) . \tag{1}$$

and the stress field $\tau_{\alpha\beta}$ reduces to (see e.g. [2, 11])

$$\tau_{\alpha\beta} \overset{def}{=} \begin{pmatrix} -P_0 & 0 & \tau \\ 0 & -P_0 & 0 \\ \tau & 0 & -P_0 \end{pmatrix}$$

with $P_0 =$ constant and $\tau = \tau(x, t)$, $(x \equiv x_1)$.

We restrict ourselves to transverse propagation so that the displacement

$$u_3 = Af(\mathcal{K}x, \omega t), \quad u_1 = u_2 = 0 \quad (A, k, \omega \text{ complex constants}), \tag{2}$$

is a function of a single coordinate. There follows, from (1), that the only unvanishing components of the strain tensor are

$$\epsilon_{13} = \epsilon_{31} = \frac{1}{2}\frac{\partial f}{\partial x}$$

so that the fundamental equations of the transverse waves propagation in the isotropic medium are [2, 11]

$$\left\{ \begin{array}{rcl} \rho\dfrac{\partial^2 f}{\partial t^2} & = & \dfrac{\partial \tau}{\partial x} \\ R^{(\tau)}_{(d)0}\tau + \dfrac{\partial \tau}{\partial t} & = & \frac{1}{2}R^{(\epsilon)}_{(d)0}\dfrac{\partial f}{\partial x} + \dfrac{1}{2}R^{(\epsilon)}_{(d)1}\dfrac{\partial}{\partial t}\dfrac{\partial f}{\partial x} + \dfrac{1}{2}R^{(\epsilon)}_{(d)2})\dfrac{\partial^2}{\partial t^2}\dfrac{\partial f}{\partial x} \end{array} \right. , \tag{3}$$

being ρ the constant mass density. This system is a set of two coupled equations in the unknown functions $f(x,t)$ (transverse displacement) and $\tau(x,t)$ (scalar stress). The rehological (constant) coefficients $R^{(\tau)}_{(d)0}$, $R^{(\epsilon)}_{(d)0}$, $R^{(\epsilon)}_{(d)1}$, $R^{(\epsilon)}_{(d)2}$, which depend on the material, cannot be given arbitrarily, in fact, due to some stability considerations, the following inequalities [11]

$$\left\{ \begin{array}{l} R^{(\tau)}_{(d)0} \geq 0 \ , \ R^{(\epsilon)}_{(d)0} \geq 0 \ , \ R^{(\epsilon)}_{(d)1} \geq 0 \ , \ R^{(\epsilon)}_{(d)2} \geq 0 \ , \\ R^{(\epsilon)}_{(d)1} - R^{(\tau)}_{(d)0} R^{(\epsilon)}_{(d)2} \geq 0 \ , \quad R^{(\epsilon)}_{(d)1} R^{(\tau)}_{(d)0} - R^{(\epsilon)}_{(d)0} \geq 0 \ , \end{array} \right. \tag{4}$$

must hold.

It has already been shown [11] that the function

$$f(\mathcal{K}x, \omega t) = e^{i(\mathcal{K}x - \omega t)} \tag{5}$$

fulfills the system (3) and describes transverse waves. At the same time the fuctions $e^{i\omega t}$, depending on the frequency ω, are orthonormal bases functions, therefore as a generalization of (5) we will consider in the next sections some more general orthonormal functions which includes (5) as a special case.

3 Time harmonic wavelets

Based on a generalization of the so-called Shannon wavelets, the time harmonic wavelets are the complex valued functions [12, 13]

$$\Psi^n_k(t) \stackrel{def}{=} 2^{-n/2}\Psi(2^n t - k) = 2^{-n/2} \sum_{s=2^n}^{2^{n+1}-1} e^{-2\pi\ i\ s(t-k/2^n)} \tag{6}$$

with $n, k \in \mathrm{N} \cup 0$, or alternatively [13]

$$\Psi(2^n t - k) = \frac{e^{4\pi i(2^n t - k)} - e^{2\pi i(2^n t - k)}}{2\pi i(2^n t - k)} \ ,$$

so that the real part is a combination of "sinc" functions. They are defined in the time interval $(-\infty, +\infty)$ with slow decay in time (see Fig. 1), but their Fourier transforms $\widehat{\Psi}^n_k(\omega)$ are disjoint rectangle functions:

$$\widehat{\Psi}^n_k(\omega) = \left\{ \begin{array}{lcl} 1/(2^{n+1}\pi) & , & 2^{n+1}\pi < \omega < 2^{n+2}\pi \\ 0 & , & \text{elsewhere} \ , \end{array} \right.$$

with compact support at each frequency, so that they seems to be an efficient tool for separating frequencies (such as in the wave propagation). More in general, we can consider harmonic wavelets based on the interval $[0, 2^{-m})$, with fixed $m \in \mathrm{N} \cup \{0\}$,

$$\Psi^n_k(t) \stackrel{def}{=} 2^{-n/2} \sum_{s=2^n}^{2^{n+1}-1} e^{-2^{m+1}\pi\ i\ s(t-k/2^n)} \ ,$$

with period 2^{-m} and dyadic intervals $(k2^{-m}, (k+1)2^{-m}]$, $k \in \mathbf{Z}$. So that it might be possible to reconstruct functions defined in intervals shorter than 1.

Harmonic wavelets have an exact analytical expression and are infinitely differentiable functions, so that from (6), the first and second derivatives are:

$$\begin{cases} \dfrac{\mathrm{d}\Psi_k^n(t)}{\mathrm{d}t} = -2^{-n/2+1} \displaystyle\sum_{s=2^n}^{2^{n+1}-1} i\,\pi s\, e^{-2\pi\, i\, s(t-k/2^n)} \\ \\ \dfrac{\mathrm{d}^2\Psi_k^n(t)}{\mathrm{d}t^2} = -2^{-n/2+2} \displaystyle\sum_{s=2^n}^{2^{n+1}-1} \pi s^2 e^{-2\pi\, i\, s(t-k/2^n)} . \end{cases} \tag{7}$$

Fig. 1 exhibits real (thick line) and imaginary (thin line) part of the harmonic wavelets $\Psi_0^0(t)$, $\Psi_1^1(t)$, $\Psi_2^2(t)$, $\Psi_4^4(t)$ (first column, from top to bottom) and their corresponding first (second column) and second derivatives (third column).

Harmonic wavelets form an orhogonal set of independent periodic functions locally concentrated at the values $k/2^n, (k, n \in Z)$, with unit period, i.e. based on the unit interval $[0, 1]$.

3.1 Connection coefficients

Harmonic wavelets are periodic functions thus we restrict to the unit interval $[0, 1]$ and there we assume as scalar product

$$\langle f, g\rangle \stackrel{def}{=} \int_0^1 f(t)\overline{g(t)}\mathrm{d}t ,$$

where the bar stands for the complex conjugate.

From the definition (6) and the equations (7), it easily follows for the linear connection coefficients (see also [12]), applying the Plancherel-Fubini theorem

$$\begin{aligned} \gamma_{kh}^{nm} &\stackrel{def}{=} \langle \frac{\mathrm{d}}{\mathrm{d}t}\Psi_k^n(t), \overline{\Psi}_h^m(t)\rangle \\ &= -2^{-(n+m)/2+1} \sum_{s=2^n}^{2^{n+1}-1} \sum_{r=2^m}^{2^{m+1}-1} i\,\pi s \int_0^1 e^{2\pi\, i\, [(r-s)x-(h/2^m-k/2^n)]}\mathrm{d}t \end{aligned}$$

so that the unvanishing components of the connection coefficients are those for which $n = m$.

Since

$$\int_0^1 e^{2\pi\, i\, \xi}\mathrm{d}x = \delta_{\xi 0} ,$$

with δ Kronecker symbol, there follows

$$\gamma_{kh}^{nm} = -2^{-(n+m)/2+1} \sum_{s=2^n}^{2^{n+1}-1} \sum_{r=2^m}^{2^{m+1}-1} i\,\pi s\, e^{-2\pi\, i\, (h/2^m-k/2^n)}\delta_{rs} ,$$

and explicitly,

$$\gamma_{kh}^{nm} = \begin{cases} -2^{1-n}\,\pi\, i \displaystyle\sum_{s,r=2^n}^{2^{n+1}-1} s\, e^{-2^{1-n}\pi\, i\,(h-k)}\delta_{rs} & , \quad n = m\,, \\ 0 & , \quad n \neq m\,. \end{cases} \tag{8}$$

In particular, taking into account that $k = 0, \ldots, 2^n - 1$, $h = 0, \ldots, 2^m - 1$, we have

$$\gamma_{00}^{00} = -2\pi\, i\ ,$$

and up to $n = m = 3$, the (first order) connection coefficients are the matrices

$$\gamma_{kh}^{11} = (5\pi)\begin{pmatrix} -i & i \\ i & -i \end{pmatrix},$$

$$\gamma_{kh}^{22} = (11\pi)\begin{pmatrix} -i & 1 & i & -1 \\ -1 & -i & 1 & i \\ i & -1 & -i & 1 \\ 1 & i & -1 & -i \end{pmatrix},$$

$\gamma_{kh}^{33} = (23\pi)A$, where A is the matrix of lines

$$-i\,,\ -i\, e^{\frac{i}{4}\pi},\ 1,\ -i\, e^{\frac{3\,i}{4}\pi},\ i\,,\ \frac{-i}{e^{\frac{3\,i}{4}\pi}},\ -1,\ \frac{-i}{e^{\frac{i}{4}\pi}}$$

$$\frac{-i}{e^{\frac{i}{4}\pi}},\ -i\,,\ -i\, e^{\frac{i}{4}\pi},\ 1,\ -i\, e^{\frac{3\,i}{4}\pi},\ i\,,\ \frac{-i}{e^{\frac{3\,i}{4}\pi}},\ -1$$

$$-1,\ \frac{-i}{e^{\frac{i}{4}\pi}},\ -i\,,\ -i\, e^{\frac{i}{4}\pi}\ 1,\ -i\, e^{\frac{3\,i}{4}\pi},\ i\,,\ \frac{-i}{e^{\frac{3\,i}{4}\pi}}$$

$$\frac{-i}{e^{\frac{3\,i}{4}\pi}},\ -1,\ \frac{-i}{e^{\frac{i}{4}\pi}},\ -i\,,\ -i\, e^{\frac{i}{4}\pi}\ -i\, e^{\frac{3\,i}{4}\pi},\ i$$

$$i\,,\ \frac{-i}{e^{\frac{3\,i}{4}\pi}},\ -1,\ \frac{-i}{e^{\frac{i}{4}\pi}},\ -i\,,\ -i\, e^{\frac{i}{4}\pi},\ 1,\ -i\, e^{\frac{3\,i}{4}\pi}$$

$$-i\, e^{\frac{3\,i}{4}\pi},\ i\,,\ \frac{-i}{e^{\frac{3\,i}{4}\pi}},\ -1,\ \frac{-i}{e^{\frac{i}{4}\pi}},\ -i\,,\ -i\, e^{\frac{i}{4}\pi},\ 1$$

$$1,\ -i\, e^{\frac{3\,i}{4}\pi},\ i\,,\ \frac{-i}{e^{\frac{3\,i}{4}\pi}},\ -1,\ \frac{-i}{e^{\frac{i}{4}\pi}},\ -i\,,\ -i\, e^{\frac{i}{4}\pi}$$

$$-i\, e^{\frac{i}{4}\pi},\ 1,\ -i\, e^{\frac{3\,i}{4}\pi},\ i\,,\ \frac{-i}{e^{\frac{3\,i}{4}\pi}},\ -1,\ \frac{-i}{e^{\frac{i}{4}\pi}}\ -i\,.$$

With the exception of γ_{00}^{00}, all the connection coefficients matrices are singular, in the sense that $\det||\gamma_{kh}^{nn}|| = 0$.

Analogously, we have, for the connection coefficients of the second derivative,

$$\Gamma_{kh}^{nm} \stackrel{def}{=} \langle \frac{d^2}{dt^2}\Psi_k^n(t), \Psi_h^m(t)\rangle =$$

$$= -2^{-(n+m)/2+2} \sum_{s=2^n}^{2^{n+1}-1} \sum_{r=2^m}^{2^{m+1}-1} \pi s^2 e^{-2\pi\, i\, (h/2^m - k/2^n)} \delta_{rs}$$

and explicitly

$$\Gamma_{kh}^{nm} = \begin{cases} -2^{2-n}\pi \displaystyle\sum_{s,r=2^n}^{2^{n+1}-1} s^2 e^{-2^{1-n}\pi\, i\, (h-k)} \delta_{rs} & , \quad n = m \,, \\ 0 & , \quad n \neq m \,. \end{cases} \tag{9}$$

With a simple computation we obtain, in the interval $[0,1]$ the value

$$\Gamma_{00}^{00} = -4\pi \,,$$

and, at the first three scale levels, the singular matrices

$$\Gamma_{hk}^{11} = (26\pi) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\Gamma_{hk}^{22} = (126\pi) \begin{pmatrix} -1 & -i & 1 & i \\ i & -1 & -i & 1 \\ 1 & i & -1 & -i \\ -i & 1 & i & -1 \end{pmatrix}$$

$\Gamma_{hk}^{33} = (550\pi)B$, where B is the matrix of lines

$$-1,\ -e^{\frac{i}{4}\pi},\ -i\,,\ -e^{\frac{3\,i}{4}\pi},\ 1,\ -e^{\frac{-3\,i}{4}\pi},\ i\,,\ -e^{\frac{-i}{4}\pi}$$

$$-e^{\frac{-i}{4}\pi},\ -1,\ -e^{\frac{i}{4}\pi},\ -i\,,\ -e^{\frac{3\,i}{4}\pi},\ 1,\ -e^{\frac{-3\,i}{4}\pi},\ i$$

$$i\,,\ -e^{\frac{-i}{4}\pi},\ -1,\ -e^{\frac{i}{4}\pi},\ -i\,,\ -e^{\frac{3\,i}{4}\pi},\ 1,\ -e^{\frac{-3\,i}{4}\pi}$$

$$-e^{\frac{-3\,i}{4}\pi},\ i\,,\ -e^{\frac{-i}{4}\pi},\ -1,\ -e^{\frac{i}{4}\pi},\ -i\,,\ -e^{\frac{3\,i}{4}\pi},\ 1$$

$$1,\ -e^{\frac{-3\,i}{4}\pi},\ i\,,\ -e^{\frac{-i}{4}\pi},\ -1,\ -e^{\frac{i}{4}\pi},\ -i\,,\ -e^{\frac{3\,i}{4}\pi}$$

$$-e^{\frac{3\,i}{4}\pi},\ 1,\ -e^{\frac{-3\,i}{4}\pi},\ i\,,\ -e^{\frac{-i}{4}\pi},\ -1,\ -e^{\frac{i}{4}\pi},\ -i$$

$$-i\,,\ -e^{\frac{3\,i}{4}\pi},\ 1,\ -e^{\frac{-3\,i}{4}\pi},\ i\,,\ -e^{\frac{-i}{4}\pi},\ -1,\ -e^{\frac{i}{4}\pi}$$

$$-e^{\frac{i}{4}\pi},\ -i\,,\ -e^{\frac{3\,i}{4}\pi},\ 1,\ -e^{\frac{-3\,i}{4}\pi},\ i\,,\ -e^{\frac{-i}{4}\pi},\ -1.$$

It can be easily checked that for $h,\ k$ greater than $n = m$, the connection coefficients have the periodic properties

$$\gamma_{n+r\ n+s}^{nn} = \gamma_{rs}^{nn}\ ,\ \Gamma_{n+r\ n+s}^{nn} = \Gamma_{rs}^{nn}\ , \qquad (r, s > n)\,.$$

4 Time harmonic wavelet solutions

We restrict to the unit time interval [0, 1] and we assume as a wavelet generalization of the solution (5) the following function depending on the (scale) level of approximation $N \leq \infty$

$$u_3 = f(x,t) = \sum_{n=0}^{2^N-1} \sum_{k=0}^{2^n-1} \beta_k^n(x)\Psi_k^n(t) \tag{10}$$

with $\Psi_k^n(t)$ given by (6). There follows that when $N = 0$, we have from (6),

$$f(x,t) = A\beta_0^0(x)\Psi_0^0(t) = A\beta_0^0(x)e^{-2\,i\,\pi t}$$

so that (10) coincides with (5), in the particular case

$$\beta_0^0(x) = e^{2\,i\,\pi x}\;, \tag{11}$$

we finally recover equation (5) by a scaling of the variables: $f(x,t) \to f(\mathcal{K}x, \omega t)$. The level $N = 0$ of harmonic wavelets coincides with the classical harmonic solution. However at the next levels, essentially finer, the approximation scheme will add more detailed information on the wavelet solution representation.

We propose, as approximate solution of (3), at the resolution N, the time harmonic wavelet series:

$$\begin{cases} f(x,t) &= \displaystyle\sum_{n=0}^{2^N-1} \sum_{k=0}^{2^n-1} \left(\beta_k^n(x)\Psi_k^n(t) + \widetilde{\beta}\,_k^n(x)\overline{\Psi}_k^n(t)\right)\,, \\ \tau(x,t) &= \displaystyle\sum_{n=0}^{2^N-1} \sum_{k=0}^{2^n-1} \left(\eta_k^n(x)\Psi_k^n(t) + \widetilde{\eta}\,_k^n(x)\overline{\Psi}_k^n(t)\right)\,. \end{cases} \tag{12}$$

Taking into account the orthogonality property of the harmonic wavelets with the corresponding coniugate functions, system (3) becomes

$$\begin{cases} \rho\displaystyle\sum_{n,k} \beta_k^n(x)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Psi_k^n(t) = \sum_{n,k} \dot{\eta}_k^n(x)\Psi_k^n(t) \\ R_{(d)0}^{(\tau)}\displaystyle\sum_{n,k} \eta_k^n(x)\Psi_k^n(t) + \sum_{n,k} \eta_k^n(x)\frac{\mathrm{d}}{\mathrm{d}t}\Psi_k^n(t) = R_{(d)0}^{(\epsilon)}\sum_{n,k} \dot{\beta}_k^n(x)\Psi_k^n(t) + \\ +R_{(d)1}^{(\epsilon)}\displaystyle\sum_{n,k} \dot{\beta}_k^n(x)\frac{\mathrm{d}}{\mathrm{d}t}\Psi_k^n(t) + R_{(d)2}^{(\epsilon)})\sum_{n,k} \dot{\beta}_k^n(x)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Psi_k^n(t)\;, \end{cases}$$

in the two sets of unknown functions $\beta_k^n(x)$ and $\eta_k^n(x)$ with $n = 0, \ldots, 2^N - 1$, $k = 0, \ldots, 2^n - 1$ and, for short, the dot stands for the derivative with respect to x and $\displaystyle\sum_{n,k} = \sum_{n=0}^{2^N-1}\sum_{k=0}^{2^n-1}$. An equivalent system must be considered for the corresponding tilde and the orthonormality of wavelets it is (for t in the interval $[0,1]$)

$$\left\{\begin{array}{l}\rho\sum\limits_{m,h}\beta_h^m(x)\Gamma_{kh}^{nm}=\dot\eta_k^n(x)\\ R_{(d)0}^{(\tau)}\eta_k^n(x)+\sum\limits_{m,h}\eta_h^m(x)\gamma_{kh}^{nm}=R_{(d)0}^{(\epsilon)}\dot\beta_k^n(x)+\\ +R_{(d)1}^{(\epsilon)}\sum\limits_{m,h}\dot\beta_h^m(x)\gamma_{kh}^{nm}+R_{(d)2}^{(\epsilon)})\sum\limits_{m,h}\dot\beta_h^m(x)\Gamma_{kh}^{nm}\ .\end{array}\right. \tag{13}$$

Thus we obtain a linear (first order) ordinary differential system for $\beta_k^n(x)$, $\eta_k^n(x)$ (and analogously for $\widetilde{\beta}\,{}_k^n(x)$, $\widetilde{\eta}\,{}_k^n(x)$) whose general solution depends on the initial conditions and is parametrized by the physical constant parameters ρ, $R_{(d)0}^{(\tau)}$, $R_{(d)0}^{(\epsilon)}$, $R_{(d)1}^{(\epsilon)}$, $R_{(d)2}^{(\epsilon)}$ and by the geometrical constant coefficients γ_{kh}^{nm} , Γ_{kh}^{nm}.

4.1 Harmonic wavelet solution at the level $N=0$

At the lowest resolution level ($N=0\Rightarrow n=k=0$), it is $\gamma_{00}^{00}=-2\pi i$, $\Gamma_{00}^{00}=4\pi$ so that, if we denote by

$$a=4\pi\rho\ ,\ b=R_{(d)0}^{(\tau)}-2\pi i\ ,\ c=R_{(d)0}^{(\epsilon)}-2\pi iR_{(d)1}^{(\epsilon)}+4\pi R_{(d)2}^{(\epsilon)}\ , \tag{14}$$

we have, from (13), the following equations

$$\left\{\begin{array}{lcl}a\beta&=&\dfrac{\mathrm{d}\eta}{\mathrm{d}x}\ ,\qquad(\eta=\eta_0^0\ ,\beta=\beta_0^0)\\ b\eta&=&c\dfrac{\mathrm{d}\beta}{\mathrm{d}x}\ ,\end{array}\right. \tag{15}$$

with a solution given by

$$\beta(x)=Ae^{2\pi i\mathcal{K}x}\ ,\quad \eta(x)=A(2\pi i)\frac{c}{b}\mathcal{K}e^{2\pi i\mathcal{K}x}\ ,\qquad \mathcal{K}^2\stackrel{def}{=}-(2\pi)^{-1}\frac{a}{bc} \tag{16}$$

and the coefficients a, b, c are related to the the medium by (14).

Thus, assuming $\beta_k^n(x)=\widetilde{\beta}\,{}_k^n(x)$, $\eta_k^n(x)=\widetilde{\eta}\,{}_k^n(x)$, *the time harmonic wavelet solution* of (3) *at the level* $N=0$ and with the time scaling $t\to\omega t$, is

$$u_3(x,t)=Ae^{2\pi i(\mathcal{K}x\pm\omega t)}\ ,\quad \tau(x,t)=A(2\pi i)\frac{c}{b}\mathcal{K}e^{2\pi i(\mathcal{K}x\pm\omega t)}\ , \tag{17}$$

which *coincides with the harmonic wave solution* already described in [11, p.368].

4.2 Harmonic wavelet solution at the level $N=1$

At the resolution level $N=1\Rightarrow n=0,1,\ k=0,1,2$, it follows from (13)

$$\left\{\begin{array}{l}\rho\sum\limits_{m=0}^{1}\sum\limits_{h=0}^{2^m-1}\beta_h^m(x)\Gamma_{kh}^{nm}=\dot\eta_k^n(x)\\ R_{(d)0}^{(\tau)}\eta_k^n(x)+\sum\limits_{m=0}^{1}\sum\limits_{h=0}^{2^m-1}\eta_h^m(x)\gamma_{kh}^{nm}=R_{(d)0}^{(\epsilon)}\dot\beta_k^n(x)+\\ +R_{(d)1}^{(\epsilon)}\sum\limits_{m=0}^{1}\sum\limits_{h=0}^{2^m-1}\dot\beta_h^m(x)\gamma_{kh}^{nm}+R_{(d)2}^{(\epsilon)})\sum\limits_{m=0}^{1}\sum\limits_{h=0}^{2^m-1}\dot\beta_h^m(x)\Gamma_{kh}^{nm}\ .\end{array}\right. \tag{18}$$

Since, according to (8),(9) it is $\gamma^{nm}_{kh} = 0,\ n \neq m$ and $\Gamma^{nm}_{kh} = 0,\ n \neq m$, we can define the following matrices:

$$\gamma = \gamma^{00}_{kh} \oplus \gamma^{11}_{kh}\ , \quad \Gamma = \Gamma^{00}_{kh} \oplus \Gamma^{11}_{kh}$$

being $\oplus$ the direct sum of matrices,

$$\gamma = \begin{pmatrix} \gamma^{00}_{00} & 0 & 0 \\ 0 & \gamma^{11}_{00} & \gamma^{11}_{01} \\ 0 & \gamma^{11}_{10} & \gamma^{11}_{11} \end{pmatrix}\ , \quad \Gamma = \begin{pmatrix} \Gamma^{00}_{00} & 0 & 0 \\ 0 & \Gamma^{11}_{00} & \Gamma^{11}_{01} \\ 0 & \Gamma^{11}_{10} & \Gamma^{11}_{11} \end{pmatrix}\ ,$$

or explicitly:

$$\gamma = \begin{pmatrix} -2\pi i & 0 & 0 \\ 0 & -5\pi i & 5\pi i \\ 0 & 5\pi i & -5\pi i \end{pmatrix}\ , \quad \Gamma = \begin{pmatrix} -4\pi & 0 & 0 \\ 0 & -26\pi & 26\pi \\ 0 & 26\pi & -26\pi \end{pmatrix}\ .$$

The unknown functions might be represented by the components of the two vectors:

$$\beta \stackrel{def}{=} \begin{pmatrix} \beta^0_0 \\ \beta^1_0 \\ \beta^1_1 \end{pmatrix}\ , \quad \eta \stackrel{def}{=} \begin{pmatrix} \eta^0_0 \\ \eta^1_0 \\ \eta^1_1 \end{pmatrix}$$

so that system (18) can be written as

$$\begin{cases} \rho \Gamma \beta &= \dfrac{\mathrm{d}\eta}{\mathrm{d}x} \\ R^{(\tau)}_{(d)0}\eta + \gamma\eta &= R^{(\epsilon)}_{(d)0}\dfrac{\mathrm{d}\beta}{\mathrm{d}x} + R^{(\epsilon)}_{(d)1}\gamma\dfrac{\mathrm{d}\beta}{\mathrm{d}x} + R^{(\epsilon)}_{(d)2}\Gamma\dfrac{\mathrm{d}\beta}{\mathrm{d}x}\ , \end{cases}$$

or, using the identity matrix I, and defining

$$A \stackrel{def}{=} \left(R^{(\epsilon)}_{(d)0} I + R^{(\epsilon)}_{(d)1}\gamma + R^{(\epsilon)}_{(d)2}\Gamma\right)^{-1} \left(R^{(\tau)}_{(d)0} I + \gamma\right)$$

we have the linear system

$$\begin{cases} \dfrac{\mathrm{d}\eta}{\mathrm{d}x} &= \rho\Gamma\beta \\ \dfrac{\mathrm{d}\beta}{\mathrm{d}x} &= A\eta\ . \end{cases}$$

With the definitions

$$y \stackrel{def}{=} \begin{pmatrix} \eta \\ \beta \end{pmatrix}\ , \qquad M \stackrel{def}{=} \begin{pmatrix} 0 & \rho\Gamma \\ A & 0 \end{pmatrix}$$

the solution, for the wavelet coefficients, is given by the following Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} &= My \\ y(0) &= y_0\ . \end{cases} \tag{19}$$

A simple computation, gives the eigenvalues of the matrix M:

$$\lambda_1 = \lambda_2 = 0 \ , \ \lambda_3 = -2\sqrt{\pi}\ \mu \ , \ \lambda_4 = 2\sqrt{\pi}\ \mu \ , \ \lambda_5 = -2\sqrt{13\pi}\ \nu \ , \ \lambda_6 = 2\sqrt{13\pi}\ \nu$$

$$\mu \stackrel{def}{=} \left(\frac{\rho\left(2\pi + iR^{(\tau)}_{(d)0}\right)}{-2\pi R^{(\epsilon)}_{(d)1} + 4\pi i R^{(\epsilon)}_{(d)2} - iR^{(\epsilon)}_{(d)0}} \right)^{1/2}$$
$$\nu \stackrel{def}{=} \left(\frac{\rho\left(10\pi + iR^{(\tau)}_{(d)0}\right)}{-10\pi R^{(\epsilon)}_{(d)1} + 52\pi i R^{(\epsilon)}_{(d)2} - iR^{(\epsilon)}_{(d)0}} \right)^{1/2} . \tag{20}$$

In general, the eigenvalues $\lambda_{3,\dots,6}$ are complex values, and distinct when $\mu \neq \nu \neq 0$ (being $\rho \neq 0$); they are real if $\mu,\ \nu \in \mathrm{R}$, i.e., taking into account the constraints (4), it should be $R^{(\epsilon)}_{(d)2} = 0,\ R^{(\tau)}_{(d)0}R^{(\epsilon)}_{(d)1} - R^{(\epsilon)}_{(d)0} = 0$, so that $\mu = \nu = -\rho/R^{(\epsilon)}_{(d)1}$, like e.g. for the Hooke media [11, 14].

By using the Putzer method, the solution can be written as

$$\boldsymbol{y} = \left(r_1(x)\boldsymbol{I} + \sum_{m=2}^{6} r_m(x) P^{m-1}(\boldsymbol{M}) \right) \boldsymbol{y}_0$$

where

$$P^m(\boldsymbol{M}) \stackrel{def}{=} \prod_{k=1}^{m} (\boldsymbol{M} - \lambda_k \boldsymbol{I}) \ ,$$

and $r_1(x),\ r_2(x)\ , \dots\ , r_6(x)$ are scalar functions given by

$$\begin{cases} \dfrac{\mathrm{d}r_1}{\mathrm{d}x} &= \lambda_1 r_1 \\ r_1(0) &= 1 \end{cases} , \quad \begin{cases} \dfrac{\mathrm{d}r_2}{\mathrm{d}x} &= \lambda_2 r_2 + r_1 \\ r_2(0) &= 0 \end{cases} , \dots , \quad \begin{cases} \dfrac{\mathrm{d}r_6}{\mathrm{d}x} &= \lambda_6 r_6 + r_5 \\ r_6(0) &= 0 \ . \end{cases} \tag{21}$$

Fig. 2 exhibits a surface of the (real) part of the wavelet solution at the level $N = 1$ of the transverse wave u (left) and the corresponding evolution of the stress component τ (right), with $c_1 = 2,\ c_2 = 0,\ c_3 = 4,\ c_4 = 4,\ c_5 = 1,\ c_6 = 1,\ \rho = 1,\ k_m = 2\ , k_n = 10$, being the frequency normalized with respect the unit interval of x

According to (20), the solutions of (21) are

$$\begin{cases} r_1(x) &= 1\,, \\ r_2(x) &= x\,, \\ r_3(x) &= -\left(1-e^{-2\mu\sqrt{\pi}\,x}-2\mu\sqrt{\pi}\,x\right)/(4\mu^2\pi)\,, \\ r_4(x) &= -\left(e^{-2\mu\sqrt{\pi}\,x}-e^{2\mu\sqrt{\pi}\,x}+4\mu\sqrt{\pi}\,x\right)/(16\mu^3\pi^{\frac{3}{2}})\,, \\ r_5(x) &= \left[-2e^{-2\nu\sqrt{13\pi}\,x}\mu^3\sqrt{\pi}+13e^{2\mu\sqrt{\pi}\,x}\nu^2\left(\mu\sqrt{\pi}-\nu\sqrt{13\pi}\right)+\right. \\ &+\ 13e^{-2\mu\sqrt{\pi}\,x}\nu^2\left(\mu\sqrt{\pi}+\nu\sqrt{13\pi}\right)- \\ &-\ \left.2\mu\left(\mu^2-13\nu^2\right)\sqrt{\pi}\left(-1+2\nu\sqrt{13\pi}\,x\right)\right]/\left(416\mu^3\nu^2\left(\mu^2-13\nu^2\right)\pi^{5/2}\right) \\ r_6(x) &= \left[169\left(-e^{-2\mu\sqrt{\pi}\,x}+e^{2\mu\sqrt{\pi}\,x}\right)\nu^3\sqrt{\pi}-\right. \\ &-\ \left(-e^{-2\nu\sqrt{13\pi}\,x}+e^{2\nu\sqrt{13\pi}\,x}\right)\mu^3\sqrt{13\pi} \\ &+\ \left.52\mu\nu\left(\mu^2-13\nu^2\right)\pi x\right]/\left(10816\mu^3\nu^3\left(\mu^2-13\nu^2\right)\pi^3\right) \end{cases}$$

There follows, that the general solution of (19) is

$$\boldsymbol{y}=(\boldsymbol{I}+x\,\boldsymbol{M})\,\boldsymbol{y}_0+\left(\sum_{m=3}^{6} r_m(x)P^{m-1}(\boldsymbol{M})\right)\boldsymbol{y}_0\,.$$

where, taking into account (20),

$$P^2(\boldsymbol{M})=\boldsymbol{M}^2\,,\ P^3(\boldsymbol{M})=\boldsymbol{M}^2(\boldsymbol{M}+2\sqrt{\pi}\,\mu\boldsymbol{I})\,,\ P^4(\boldsymbol{M})=\boldsymbol{M}^2(\boldsymbol{M}^2-4\pi\mu^2\boldsymbol{I})\,,$$
$$P^5(\boldsymbol{M})=\boldsymbol{M}^2(\boldsymbol{M}^2-4\pi\mu^2\boldsymbol{I})(\boldsymbol{M}+2\sqrt{13\pi}\,\nu\boldsymbol{I})\,,$$

being

$$\boldsymbol{M}^2=\begin{pmatrix}\rho\,\boldsymbol{\Gamma}\,\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \rho\,\boldsymbol{A}\,\boldsymbol{\Gamma}\end{pmatrix}.$$

Assuming $\beta_k^n(x)=\widetilde{\beta}\,{}_k^n(x)$, $\eta_k^n(x)=\widetilde{\eta}\,{}_k^n(x)$, with a suitable scaling of the variables, the component $u_3(x,t)=u(x,t)$ of the velocity is

$$\begin{aligned} u(x,t) &= \beta_0^0(\mathcal{K}x)\left(\Psi_0^0(\omega t)+\overline{\Psi}_0^0(\omega t)\right)+\beta_0^1(\mathcal{K}x)\left(\Psi_0^1(\omega t)+\right.\\ &+\left.\overline{\Psi}_0^1(\omega t)\right)+\beta_1^1(\mathcal{K}x)\left(\Psi_1^1(\omega t)+\overline{\Psi}_1^1(\omega t)\right)\,, \end{aligned}$$

and analogously for τ we have,

$$\begin{aligned} \tau(x,t) &= \eta_0^0(\mathcal{K}x)\left(\Psi_0^0(\omega t)+\overline{\Psi}_0^0(\omega t)\right)+\eta_0^1(\mathcal{K}x)\left(\Psi_0^1(\omega t)+\right.\\ &+\left.\overline{\Psi}_0^1(\omega t)\right)+\eta_1^1(\mathcal{K}x)\left(\Psi_1^1(\omega t)+\overline{\Psi}_1^1(\omega t)\right)\,. \end{aligned}$$

Of course the solution depends on the physical parameters, expressed by μ and ν. For instance, assuming $\mu=k_m\ i$, $\nu=k_n\ i$, $(k_m, k_n\in\mathbf{R})$, and simplifying the dependence on ω, $\mathcal{K}$, the real part of the displacement is

$$u(x,t)=\frac{\sin 2\pi t\cos 6\pi t}{2\pi\,t(2t-1)}\left[c_1(4t-1)+(c_3+c_4)(2t-1)\cos 2k_m\sqrt{\pi}\,x+\right.$$

$$+ (c_5 + c_6) \cos 2\, k_n \sqrt{13\,\pi}\, x \Big] \ ,$$

being c_1 , c_2 , c_3 , c_4 , c_5, c_6 constant of integration, and the scalar stress component is

$$\tau(x,t) = \frac{\rho}{2\,\pi\, k_m\, k_n\, t(2\,t-1)} \left[2\,\sqrt{\pi} k_n (2t-1)(c_3 + c_4) \sin 2\,\sqrt{\pi}\, k_m\, x - \right.$$

$$\left. - \sqrt{13\,\pi}\, k_m (c_5 + c_6)(\sin 8\,\pi\, t - \sin 4\,\pi\, t) \sin 2\,\sqrt{13\,\pi}\, k_n\, x \right] \ .$$

As can be seen from Figure 2, the evolution in time, both of the displacement (wave propagation) and of the stress component, is represented by a superposition of wavelets, which add more details at each level of the approximation. This method allows us to modelling wave propagation by focusing on the role played by each level of approximation on the resulting evolution. Moreover the localization of wavelets would be an expedient tool for the analysis of the propagation of solitary waves. This approach could be considered, for instance, as a suitable tool for the investigation of an initial profile (solitary wave) localized in space and with a fast decay (say within $t < 1$), also in presence of some nonlinearities (see e.g. [1, 4, 5]).

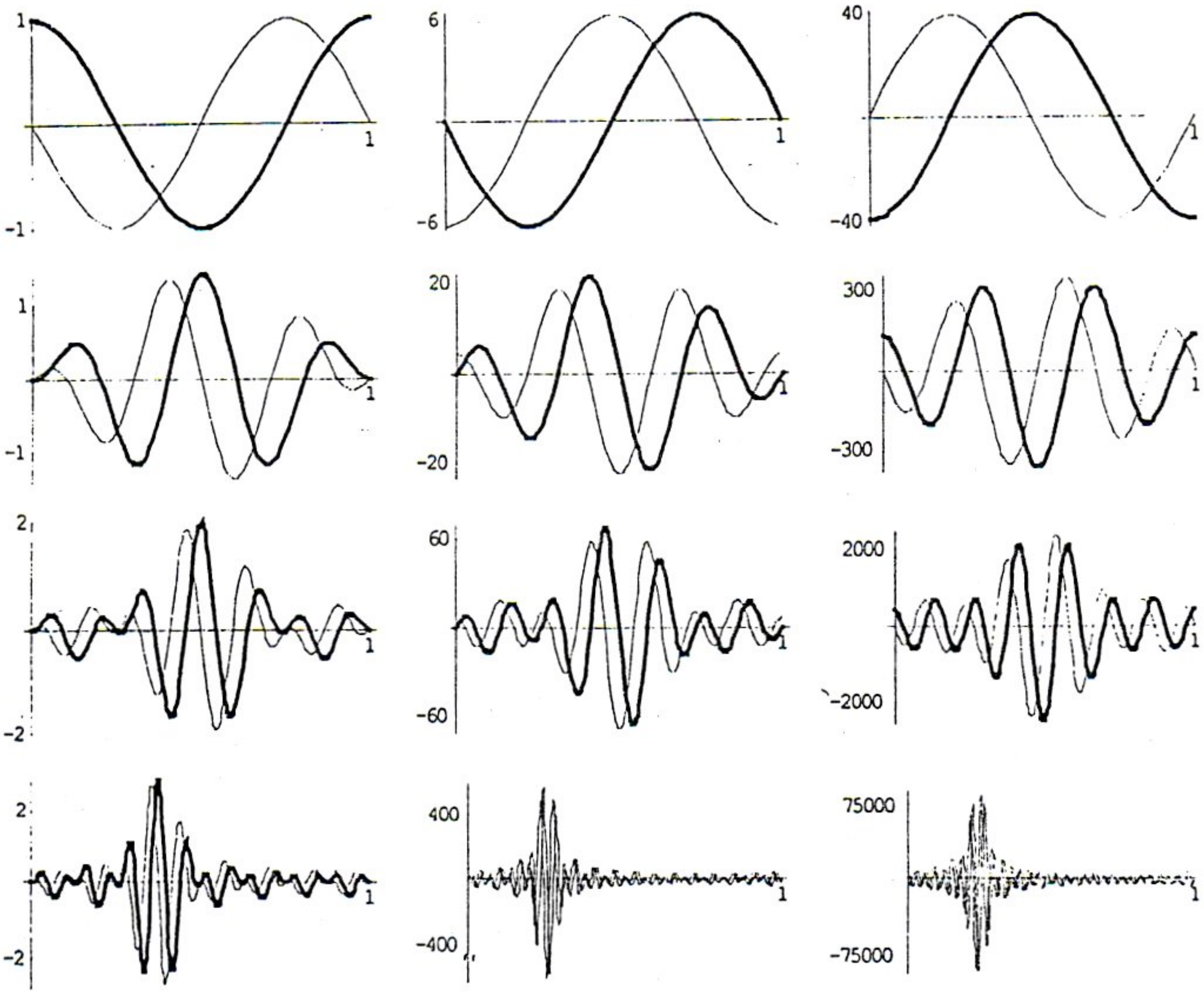

Fig. 1

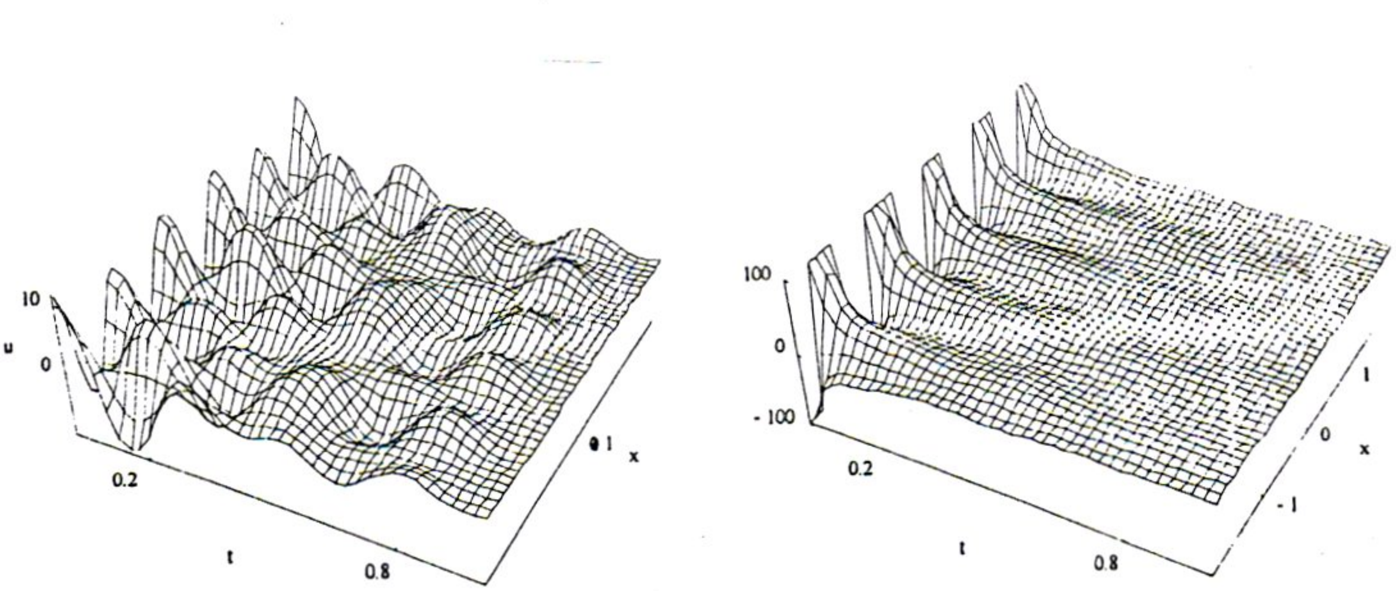

Fig. 2

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