On a control system of trailer-truck jackknifing

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Abstract. The paper analyses a control differential system on a cylindrical hypersurface in R^6 or R^4. This system is a model for the position of a trailer relative to the cab which is pulling it, and therefore our results extend those of [2], [3].
§1 formulates the model problem and defines the notion of jackknifing. §2 describes the influence of the radius of curvature of control curve upon the jackknifing. §3 refers to trailer-truck jackknifing, explicit examples and MAPLE 6 simulations.

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1 Model of articulated bars moving

We consider two articulated bars BA and AC, like in the Fig.1, which are moving in the following conditions:

1) $B = \beta(t)$, $A = \alpha(t)$, $t \in I = [0, a]$, $\overline{BA} = \bar{\alpha}(t) - \bar{\beta}(t)$, where the curves $\alpha, \beta : I \to \mathbf{R}^3$ are C^∞;

2) $\dot{\bar{\beta}}(t)$ is collinear to $\overline{BA}$, $\dot{\bar{\alpha}}(t)$ is collinear to $\overline{AC}$;

3) $||\overline{BA}||$ = constant, and the point $\beta(0)$ is given.

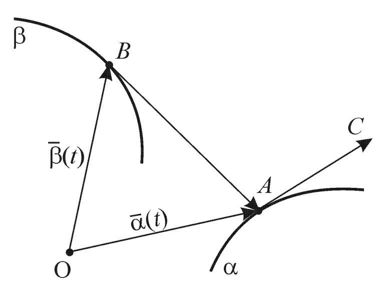

Fig. 1

Some problems require to model the trajectory $\beta : I \to R^3$ as follows. First, after calibration, we can accept

$$||\overline{BA}||^2 = ||\bar{\alpha}(t) - \bar{\beta}(t)||^2 = 1. \tag{1}$$

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This condition implies the relation

$$(\dot{\bar{\alpha}}(t), \bar{\alpha}(t) - \bar{\beta}(t)) = (\dot{\bar{\beta}}(t), \bar{\alpha}(t) - \bar{\beta}(t)). \tag{2}$$

Consequently the collinearity condition

$$\dot{\bar{\beta}}(t) = \lambda(t)(\bar{\alpha}(t) - \bar{\beta}(t))$$

is satisfied for

$$\lambda(t) = (\dot{\bar{\beta}}(t), \bar{\alpha}(t) - \bar{\beta}(t)) = (\dot{\bar{\alpha}}(t), \bar{\alpha}(t) - \bar{\beta}(t)).$$

In other words, for a given curve α, the curve β must be a solution of the control differential system

$$\dot{\bar{\beta}}(t) = (\dot{\bar{\alpha}}(t), \bar{\alpha}(t) - \bar{\beta}(t))(\bar{\alpha}(t) - \bar{\beta}(t))$$

on the cylindrical hypersurface

$$||\bar{\alpha} - \bar{\beta}||^2 = 1$$

in R^6. This control differential system is invariant with respect to the changing of the parameter t. It implies that the speed of β is at most the speed of α.

1.1. Theorem. *If the control system*

$$\dot{\bar{\beta}}(t) = (\dot{\bar{\alpha}}(t), \bar{\alpha}(t) - \bar{\beta}(t))(\bar{\alpha}(t) - \bar{\beta}(t))$$

holds, then the relations (1),(2) and the initial condition $||\bar{\alpha}(t_0) - \bar{\beta}(t_0)||^2 = 1$ *are equivalent.*

Proof. The solutions β of the differential system must satisfy the differential equation

$$\frac{1}{2}\frac{d}{dt}(\bar{\alpha} - \bar{\beta}, \bar{\alpha} - \bar{\beta}) = (\bar{\alpha} - \bar{\beta}, \dot{\bar{\alpha}} - \dot{\bar{\beta}}) = (\dot{\bar{\alpha}}, \bar{\alpha} - \bar{\beta})(1 - (\bar{\alpha} - \bar{\beta}, \bar{\alpha} - \bar{\beta})).$$

The equivalences are consequences of this equation.

This theorem reduces the problem

$$\dot{\bar{\beta}}(t) = (\dot{\bar{\alpha}}(t), \bar{\alpha}(t) - \bar{\beta}(t))(\bar{\alpha}(t) - \bar{\beta}(t)), \quad ||\bar{\alpha}(t) - \bar{\beta}(t)||^2 = 1$$

to the Cauchy problem

$$\dot{\bar{\beta}}(t) = (\dot{\bar{\alpha}}(t), \bar{\alpha}(t) - \bar{\beta}(t))(\bar{\alpha}(t) - \bar{\beta}(t)), \quad ||\bar{\alpha}(t_0) - \bar{\beta}(t_0)||^2 = 1.$$

Supposing $\bar{\alpha}(t) = x_1(t)\bar{i}_1 + x_2(t)\bar{i}_2 + x_3(t)\bar{i}_3$, $\bar{\beta}(t) = y_1(t)\bar{i}_1 + y_2(t)\bar{i}_2 + y_3(t)\bar{i}_3$, the control system and the cylindrical hypersurface can be written under the form

$$\dot{y}_1 = (x_1 - y_1)\sum_{i=1}^{3} \dot{x}_i(x_i - y_i)$$

$$\dot{y}_2 = (x_2 - y_2)\sum_{i=1}^{3} \dot{x}_i(x_i - y_i)$$

$$\dot{y}_3 = (x_3 - y_3)\sum_{i=1}^{3} \dot{x}_i(x_i - y_i)$$

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 1.$$

Remark. The relation

$$\sum_{i=1}^{3}(\dot{x}_i - \dot{y}_i)(x_i - y_i) = 0$$

describes a tangency (orthogonality) condition since

$$\sum_{i=1}^{3}\dot{x}_i(x_i - y_i) + \sum_{i=1}^{3}\dot{y}_i(-x_i + y_i) = 0$$

shows that $(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{y}_1, \dot{y}_2, \dot{y}_3)$ is tangent to the cylindrical hypersurface $(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 1$ as a vector orthogonal to the normal vector field

$$(x_1 - y_1, x_2 - y_2, x_3 - y_3, -x_1 + y_1, -x_2 + y_2, -x_3 + y_3).$$

The parametrization of the cylindrical hypersurface in R^6 obtained from

$$x_1 - y_1 = \cos\varphi\cos\theta, x_2 - y_2 = \cos\varphi\sin\theta, x_3 - y_3 = \sin\varphi,\ \theta \in [0, 2\pi], \varphi \in [0, \pi]$$

transforms the previous control differential system into the control differential system

$$\begin{aligned}
&\dot{\theta} = \frac{\dot{x}_2\cos\theta - \dot{x}_1\sin\theta}{\cos\varphi}, \quad \varphi \neq \frac{\pi}{2}\\
&\dot{\varphi} = \dot{x}_3\cos\varphi - \dot{x}_2\sin\varphi\sin\theta - \dot{x}_1\sin\varphi\cos\theta,\\
&\theta(t_0) = \theta_0, \quad \varphi(t_0) = \varphi_0.
\end{aligned}$$

For convenience, let $\bar{\gamma}(t) = \bar{\alpha}(t) - \bar{\beta}(t)$. The previous control differential system can be rewritten

$$\dot{\bar{\beta}}(t) = (\dot{\bar{\alpha}}(t), \bar{\gamma}(t))\bar{\gamma}(t),\ ||\bar{\gamma}(t)||^2 = 1. \tag{3}$$

The transmission angle τ between the vectors $\overline{BA}$ and $\overline{AC}$ is described by the function $f : I \to R,\ f(t) = (\dot{\bar{\alpha}}(t), \bar{\gamma}(t)) = ||\dot{\bar{\alpha}}(t)|| \cos\tau(t)$. The sign of the function f describes the jackknifing of the articulated bars in the following sense:

- if the point A is moving forward, we say that $\overline{AC}$ and $\overline{BA}$ are jackknifed if $f(t) < 0$; otherwise, we say they are unjackknifed (Fig.2); - if $\overline{AC}$ is backing up, the above situation is reversed. Namely, $\overline{AC}$ and $\overline{BA}$ are jackknifed if $f(t) > 0$, and unjackknifed otherwise (Fig.3.)

Of course, we can change the previous point of view at least in two ways:

1) the control is described by a vector field $\bar{X}$ which did not vanish anywhere (the curves α are field lines of $\bar{X}$):

$$\dot{\bar{\alpha}}(t) = \bar{X}(\alpha(t)), \dot{\bar{\beta}}(t) = (\bar{X}(\alpha(t)), \bar{\alpha}(t) - \bar{\beta}(t))(\bar{\alpha}(t) - \bar{\beta}(t)), ||\bar{\alpha}(t_0) - \bar{\beta}(t_0)||^2 = 1,$$

and $(\alpha(t), \beta(t)) \in R^6$;

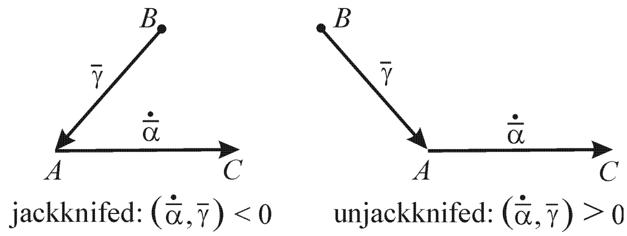


Fig. 2. Forward

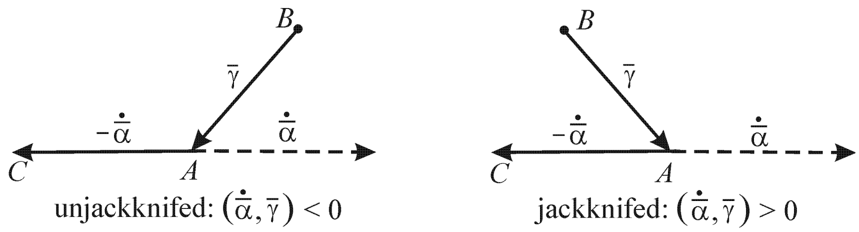


Fig. 3. Backward

2) the control differential system is replaced by a Pfaff system in R^6:

$$dy_i = (x_i - y_i)\sum_{j=1}^{3}(x_j - y_j)dx_j, \quad i = 1, 2, 3; \quad \sum_{j=1}^{3}(x_{j0} - y_{j0})^2 = 1.$$

Generalization. The linkage with two bars can be extended to a linkage with n bars. In this sense, our research is good for snake robots consisting of n modules (LEM = lunar excursion mobile).

2 Influence of curvature radius of control curve α upon the jackknifing

Let us give a very important property of the control system (3).

2.1. Theorem. *Suppose the radius of curvature* $\frac{1}{k}$ *of the control curve* α *is strictly greather than 1. If* $f(0) > 0$*, then* $f(t) > 0$ *for all* $t \in (0, a]$.

Alternatively: if the bar AC *is moving forward and the* $BA * AC$ *combination is not originally jackknifed, then it will remain unjackknifed; on the other hand, if* AC *is moving backward and the* $BA * AC$ *combination is originally jackknifed, then it will remain jackknifed.*

Proof. The function f is of class C^∞. We proceed by reductio ad absurdum. Suppose the conclusion is false: then there exists $t_1 > 0$ such that $f(t_1) = 0$, $f'(t_1) \leq 0$. By hypothesis $||\dot{\bar{\alpha}}(t)|| > 0$ (AC does not stop). From $f(t_1) = 0$, we obtain

$\dot{\bar{\alpha}}(t_1) \perp \bar{\gamma}(t_1)$, i.e., $\bar{\gamma}(t_1)$ belongs to the normal plane of the curve α at the point $\alpha(t_1)$.

On the other hand, taking the derivative and using Frenet formulas we find

$$f'(t) = (\ddot{\bar{\alpha}}(t), \bar{\gamma}(t)) + (\dot{\bar{\alpha}}(t), \dot{\bar{\gamma}}(t)) = (\dot{\bar{\alpha}}(t), \dot{\bar{\alpha}}(t) - \dot{\bar{\beta}}(t)) + +(\ddot{\bar{\alpha}}(t), \bar{\gamma}(t))$$
$$= ||\dot{\bar{\alpha}}(t)||^2 - (\dot{\bar{\alpha}}(t), \dot{\bar{\beta}}(t)) + \left(\frac{\dot{\bar{\alpha}}(t)}{||\dot{\bar{\alpha}}(t)||}\frac{d}{dt}||\dot{\bar{\alpha}}(t)|| + k||\dot{\bar{\alpha}}(t)||^2 \bar{N}, \bar{\gamma}(t)\right)$$
$$= ||\dot{\bar{\alpha}}(t)||^2 - (\dot{\bar{\alpha}}(t), (\dot{\bar{\alpha}}(t), \bar{\gamma}(t))\bar{\gamma}(t)) + k||\dot{\bar{\alpha}}(t)||^2(\bar{N}, \bar{\gamma}(t)) + \left(\frac{\dot{\bar{\alpha}}(t)}{||\dot{\bar{\alpha}}(t)||}, \bar{\gamma}(t)\right)\frac{d}{dt}||\dot{\bar{\alpha}}(t)||$$
$$= ||\dot{\bar{\alpha}}(t)||^2 - f^2(t) + k||\dot{\bar{\alpha}}(t)||^2(\bar{N}, \bar{\gamma}(t)) + \frac{f(t)}{||\dot{\bar{\alpha}}(t)||}\frac{d}{dt}||\dot{\bar{\alpha}}(t)||,$$

$$f'(t_1) = (1 + k(t_1)\cos\theta(t_1))||\dot{\bar{\alpha}}(t_1)||^2,$$

where $\theta(t_1) \in [0, \pi]$ is the angle between $\bar{N}(t_1)$ and $\bar{\gamma}(t_1)$. Since $-1 \leq \cos\theta(t_1) \leq 1$, it follows $1 - k(t_1) \leq 1 + k(t_1)\cos\theta(t_1) \leq 1 + k(t_1)$, and hence $f'(t_1) > 0$, which contradicts $f'(t_1) \leq 0$.

3 Trailer-truck jackknifing

To simplify the problem we consider the trailer-truck movement in R^2. Suppose that a cab is pulling a trailer which is 1 unit long. We represent the positions of the cab and trailer by two vectors (Fig.4):

- the vector $\bar{\alpha}(t)$ whose ends is at the trailer hitch on the cab,

- the vector $\bar{\beta}(t)$ whose ends is at the midpoint between the wheels of the trailer.

Given $\bar{\alpha}(t)$, we would like to be able to predict $\bar{\beta}(t)$. Particularly, we want to know

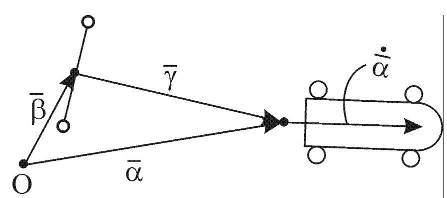


Fig. 4. Trailer-truck

if the truck-trailer will jackknife or we want to determine what conditions we must impose to $\bar{\alpha}(t)$ to prevent jackknifing (Figs. 2,3).

Let us accept that the trailer-track problem is described by the control differential system

$$\dot{y}_1 = (x_1 - y_1)\sum_{i=1}^{2}\dot{x}_i(x_i - y_i), \quad \dot{y}_2 = (x_2 - y_2)\sum_{i=1}^{2}\dot{x}_i(x_i - y_i), \quad \sum_{i=1}^{2}(x_i - y_i)^2 = 1.$$

The parametrization of the cylindrical hypersurface in R^4 obtained from

$$x_1 - y_1 = \cos\theta, \quad x_2 - y_2 = \sin\theta, \quad \theta \in [0, 2\pi] \tag{3}$$

converts the previous control differential system into the control differential equation

$$\dot{\theta} = \dot{x}_2 \cos\theta - \dot{x}_1 \sin\theta, \quad \theta(t_0) = \theta_0. \tag{4}$$

Examples with analytic solutions. Here we give our solutions for the examples in [3].

1) Consider the cab moving forward on the straight line $x_1 = t$, $x_2 = 0$. The differential equation (4) becomes

$$\dot{\theta} = -\sin\theta, \quad \theta(0) = \theta_0.$$

If $\theta_0 = 2\pi$, then $\theta(t) = 2\pi$, and the trailer trajectory is the straight line $y_1 = t-1$, $y_2 = 0$. If $\theta_0 = \pi$, then $\theta(t) = \pi$, and the trailer trajectory is the straight line $y_1 = t+1$, $y_2 = 0$.

Suppose $\theta_0 \neq \pi, 2\pi$. Then

$$\int \frac{d\theta}{\sin\theta} = -t + \ln C, \quad \tan\frac{\theta}{2} = Ce^{-t},$$

$$\sin\theta = \frac{2Ce^{-t}}{1 + C^2e^{-2t}}, \quad \cos\theta = \frac{1 - C^2e^{-2t}}{1 + C^2e^{-2t}}, \quad t \in \mathbb{R}$$

and the trailer trajectory is the curve

$$y_1 = t - \frac{1 - C^2e^{-2t}}{1 + C^2e^{-2t}}, \quad y_2 = -\frac{2Ce^{-t}}{1 + C^2e^{-2t}}, \quad t \in R, \quad C = \tan\frac{\theta_0}{2}.$$

Limit cases. For $C \to 0$, we obtain the stable limiting solution $y_1 = t-1$, $y_2 = 0$. For $C^2 \to \infty$, we find the unstable solution $y_1 = t+1$, $y_2 = 0$.

2) Consider the cab traveling along a circle of radius r, i.e., $x_1(t) = r\cos t$, $x_2(t) = r\sin t$, $t \in [0, 2\pi]$. The differential equation (4) becomes

$$\dot{\theta} = r\cos(\theta - t), \quad \theta(0) = \theta_0.$$

Using the substitution $\tan\dfrac{\theta - t}{2} = u$, we find $\dfrac{2du}{u^2(-r-1) + r - 1} = dt$. But

$$\int \frac{2du}{u^2(-r-1)+r-1} = \begin{cases} \dfrac{1}{\sqrt{r^2-1}} \ln \dfrac{(r+1)u + \sqrt{r^2-1}}{(r+1)u - \sqrt{r^2-1}} + C_1 \quad \text{for} \quad r > 1 \\ \dfrac{1}{u} + C_1 \quad \text{for} \quad r = 1 \\ \dfrac{1}{\sqrt{1-r^2}} \text{arctg} \dfrac{(-r-1)u}{\sqrt{1-r^2}} + C_1 \quad \text{for} \quad r < 1. \end{cases}$$

For the moment, assume $r > 1$. Then

$$\frac{(r+1)\tan\dfrac{\theta-t}{2}+\sqrt{r^2-1}}{(r+1)\tan\dfrac{\theta-t}{2}-\sqrt{r^2-1}} = Ce^{\sqrt{r^2-1}t},$$

$$\tan\frac{\theta-t}{2} = \frac{-\sqrt{r^2-1}\left(1+Ce^{\sqrt{r^2-1}t}\right)}{(r+1)\left(1-Ce^{\sqrt{r^2-1}t}\right)}$$

$$\theta = t + 2\text{arctan}\,\frac{-\sqrt{r^2-1}\left(1+Ce^{\sqrt{r^2-1}t}\right)}{(r+1)\left(1-Ce^{\sqrt{r^2-1}t}\right)},$$

and the trailer trajectory is the curve

$$y_1 = r\cos t - \cos\theta(t), \quad y_2 = r\sin t - \sin\theta(t).$$

Limit cases: For $C \to \infty$ or $C \to -\infty$, we obtain a stable limit cycle,

$$\theta = t + 2\text{arctan}\,\frac{\sqrt{r^2-1}}{r+1}, \quad \tan\alpha = \frac{\sqrt{r^2-1}}{r+1}, \quad \cos 2\alpha = \frac{1}{r},\ \sin 2\alpha = \frac{\sqrt{r^2-1}}{r}$$

$$y_1 = \sqrt{r^2-1}\cos\left(t+2\alpha-\frac{\pi}{2}\right), \quad y_2 = \sqrt{r^2-1}\sin\left(t+2\alpha-\frac{\pi}{2}\right).$$

For $C = 0$, we find the unstable (node) solution

$$\theta = t + \text{arctan}\frac{-\sqrt{r^2-1}}{r+1},$$

$$y_1 = \sqrt{r^2-1}\cos\left(t-2\alpha-\frac{\pi}{2}\right), \quad y_2 = \sqrt{r^2-1}\sin\left(t-2\alpha-\frac{\pi}{2}\right).$$

Though it is the same orbit as the previous one, the cab-trailer combination is in a jackknifed position initially and will remain in that position. However any deviation from that initial position will cause the cab-trailer to wonder farther away from the initial configuration.

Now we accept $r < 1$. Then

$$\text{arctan}\,\frac{(-r-1)\tan\dfrac{\theta-t}{2}}{\sqrt{1-r^2}} = \sqrt{1-r^2}t + C$$

$$\theta = t + 2\text{arctan}\,\left(\frac{\sqrt{1-r^2}}{-r-1}\tan\left(\sqrt{1-r^2}t\right)+C\right),$$

and the trailer trajectory is the curve $\begin{cases} y_1 = r\cos t - \cos\theta(t) \\ y_2 = r\sin t - \sin\theta(t). \end{cases}$

For $r < 1$, we have no limit case. Let us take $r = 1$. Then

$$\frac{1}{u} = t + C, \quad \tan\frac{\theta - t}{2} = \frac{1}{t} + C, \quad \theta = t + 2\mathrm{arctan}\,\frac{1}{t+C}$$

and the trailer trajectory is the curve

$$y_1 = \cos t - \cos\left(t + 2\mathrm{arctan}\,\frac{1}{t+C}\right), \quad y_2 = \sin t - \sin\left(t + 2\mathrm{arctan}\,\frac{1}{t+C}\right), \quad t \in [0, 2\pi].$$

We did not have limit cases.

Now we accept that the teoretical part of the trailer-truck problem is based on the control differential system

$$\dot{x}_1 = X_1(x_1, x_2), \quad \dot{x}_2 = X_2(x_1, x_2)$$

$$\dot{y}_1 = (x_1 - y_1)\sum_{i=1}^{2} X_i(x_1, x_2)(x_i - y_i), \quad \dot{y}_2 = (x_2 - y_2)\sum_{i=1}^{2} X_i(x_1, x_2)(x_i - y_i)$$

$$\sum_{i=1}^{2}(x_{i_0} - y_{i_0})^2 = 1.$$

The MAPLE 6 version of this differential system is obvious. Let us now illustrate the procedure for the following three cases:

1) $X_1(x_1, x_2) = -x_2$, $X_2(x_1, x_2) = x_1$ (trailer trajectory = circle);

2) $X_1(x_1, x_2) = 1$, $X_2(x_1, x_2) = 0$ (trailer trajectory = straight line);

3) $X_1(x_1, x_2) = x_2$, $X_2(x_1, x_2) = 1$ (trailer trajectory = parabola);

The output is included in the Figs. 5-12.

FIG.5. TRUCK TRAJECTORY (TRAILER TRAJECTORY=CIRCLE, R>1)

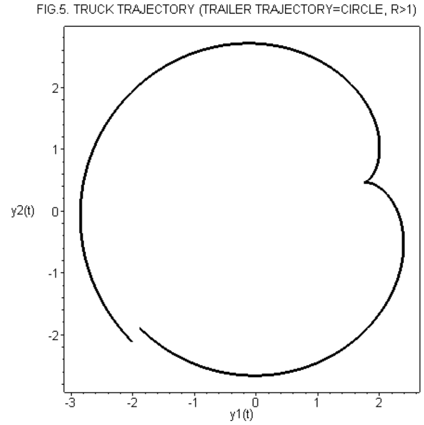


FIG.6. TRAILER TRAJECTORY=CIRCLE R>1;TRUCK TRAJECTORY

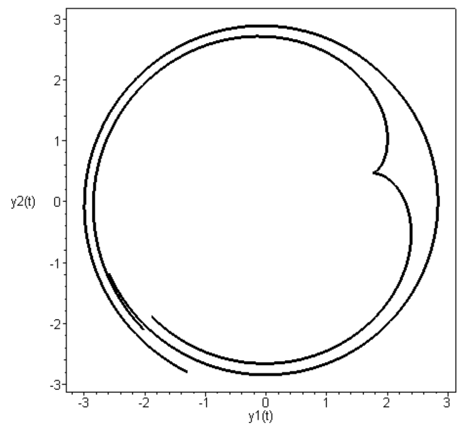


FIG.7. TRUCK TRAJECTORY (TRAILER TRAJECTORY=CIRCLE, R<1)

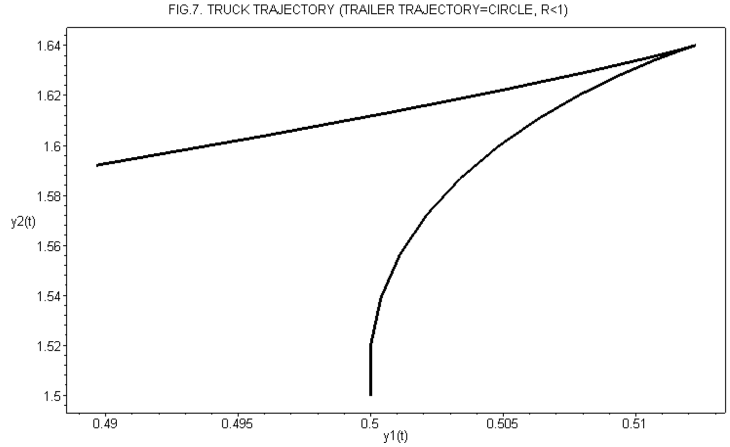


FIG.8. TRAILER TRAJECTORY=CIRCLE,R<1;TRUCK TRAJECTORY

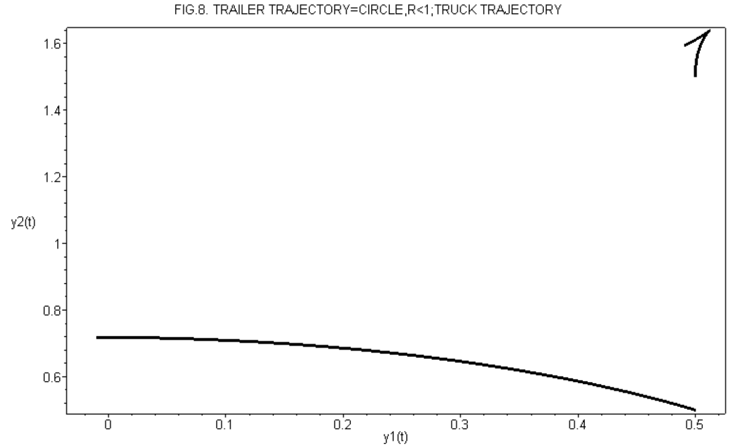


FIG.9. TRUCK TRAJECTORY (TRAILER TRAJECTORY=STRAIGHT LINE)

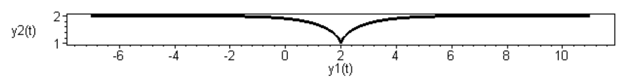


FIG.10. TRAILER TRAJECTORY=STRAIGHT LINE;TRUCK TRAJECTORY

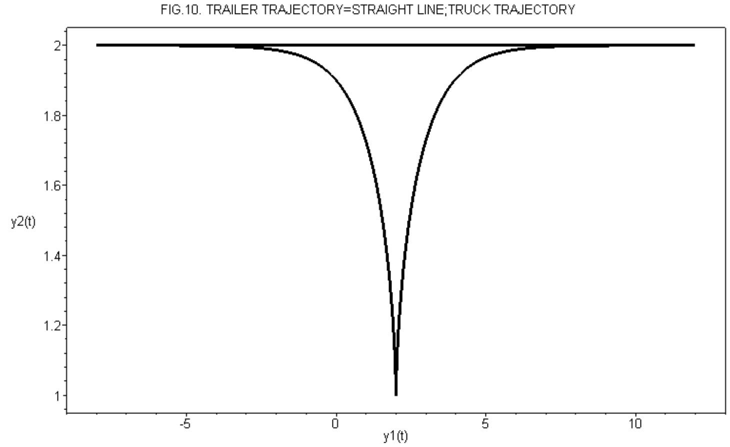
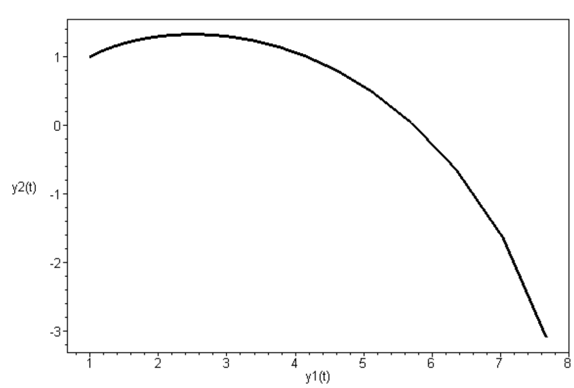
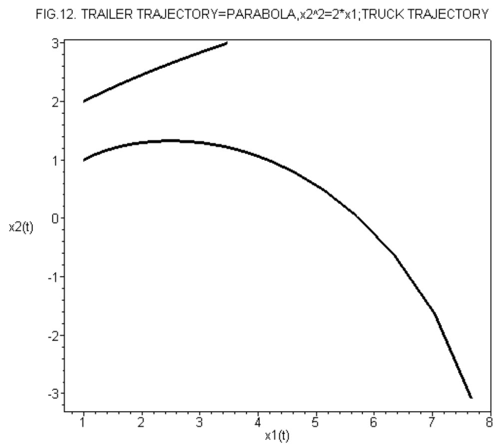


FIG.11. TRUCK TRAJECTORY (TRAILER TRAJECTORY=PARABOLA, x2^2=2*x1)





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