Recent advances in the theory of transfer operators arising in statistical mechanics

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Abstract. We first make explicit the analytic properties of transfer operators due to Mayer and Ruelle that generalize the classical Perron-Frobenius operator. The purpose of this paper is to give and discuss two generalizations of them. We mostly focus on the analysis of what we call generalized Mayer-Ruelle operators depending on two complex parameters.

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1. Introduction

Statistical mechanics problems motivated the consideration of a class of functional operators - transfer operators - due to Mayer ([5], [6]) and Ruelle [9]. This class of operators including as a special case the Perron-Frobenius operator is the basic ingredient, well-developed in dynamical systems theory [1].

This paper surveys the main properties of Mayer-Ruelle operators and gives two generalizations of them.

The analytic properties of transfer operators associated to continued fractions have been investigated by Mayer and Roepstorff in a series of papers ([3], [4], [7], [8]) that provide the technical background for the functional analysis aspects of our paper.

2. The main properties of transfer operators

Let $D_1 = \{ z \in \mathbb{C} | |z - 1| < 3/2 \}$ and consider the collection $A_\infty(D_1)$ of all holomorphic functions in $D_1$ which are continuous in $\bar{D}_1$; $A_\infty(D_1)$ is a Banach space under the supremum norm

$$||f|| = \sup_{z \in \bar{D}_1} |f(z)|, \ f \in A_\infty(D_1).$$

The transfer operators of Mayer-Ruelle are defined by

$$R_s f(z) = \sum_{i \in \mathbb{N}_+} \frac{1}{(z+i)^s} f\left(\frac{1}{z+i}\right), \ z \in \bar{D}_1,$$
for $s$ a complex number satisfying $\text{Re } s > 1$ and $f \in A_\infty(D_1)$. It is easy to check that $R_s$ is a bounded linear operator on $A_\infty(D_1)$. $R_s$ is nuclear of order 0, and thus has a discrete spectrum.

For $s = 2$, $R_s$ has the same analytical expression as the Perron-Frobenius operator $P_\lambda = P$ of $\tau$ under $\lambda$

$$P f(x) = \sum_{i \in \mathbb{N}_+} \frac{1}{(x + i)^2} \left( \frac{1}{x + i} \right)^2, \quad f \in L^1, \ x \in [0, 1],$$

where $\lambda$ is the Lebesgue measure on $I$ and $\tau$ is the continued fraction transformation on $I$ defined as

$$\tau(x) = \begin{cases} \frac{1}{x} - \left[ \frac{1}{x} \right] & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

(here $[\cdot]: \mathbb{R} \to \mathbb{Z}$ is the greatest integer function).

In what follows we give without proofs the most important properties of the Mayer-Ruelle operator $R_s$ for $s > 1$, which generalize those of $P$. For proofs we refer the reader to Mayer ([5], [6]), Flajolet and Vallée [2].

**Theorem 1.** Let $s$ be real, strictly greater than 1. Then the following results hold.

(i) The operator $R_s: A_\infty(D_1) \to A_\infty(D_1)$ has a positive dominant eigenvalue $\lambda_s$ which is simple and strictly greater in absolute value than all other eigenvalues. The corresponding eigenfunction $g_s \in A_\infty(D_1)$ is strictly positive on $\bar{D}_1 \cap \mathbb{R} = \left[ -\frac{1}{2}, \frac{5}{2} \right]$.

(ii) The map $s \mapsto \lambda_s$ defines on $(1, \infty)$ a strictly decreasing and logconcave function with

$$\lim_{s \to 1} \lambda_s = \infty, \ \lambda_{s=2} = 1, \ \lim_{s \to \infty} \frac{\log \lambda_s}{s} = \log \frac{\sqrt{5} - 1}{2}.$$

Moreover, $\lambda_{s+u} \leq \left( \frac{\sqrt{5} - 1}{2} \right)^u \lambda_s$, $u \in \mathbb{R}_+$.

(iii) There exists a linear functional $l_s$ on $A_\infty(D_1)$ with $l_s(g_s) = 1$ and $l_s(f) > 0$ for any $f \in A_\infty(D_1)$ such that $f|_{[-1/2, 5/2]} > 0$. If $\Pi_{1s}$ denotes the projection defined as $\Pi_{1s} f = l_s(f) g_s$, $f \in A_\infty(D_1)$, then $R_s = \lambda_s \Pi_{1s} + T_{0s}$ with $\Pi_{1s} T_{0s} = T_{0s} \Pi_{1s} = 0$. Hence

$$R^n_s = \lambda_s^n \Pi_{1s} + T^n_{0s}, \ n \in \mathbb{N}_+.$$

(iv) The spectral radius $\rho_s$ of the linear operator $T_{0s}: A_\infty(D_1) \to A_\infty(D_1)$ is strictly smaller than $\lambda_s$, and for any $f \in A_\infty(D_1)$ such that $f|_{[-1/2, 5/2]} > 0$ we have

$$\frac{R^n_s f(z)}{\lambda_s^n l_s(f) g_s(z)} = 1 + O \left( \left( \frac{\rho_s}{\lambda_s} \right)^n \right)$$

as $n \to \infty$, where the constant implied in $O$ is independent of $z \in \bar{D}_1$. 

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(v) There exists \( \varepsilon = \varepsilon(s) > 0 \) such that for any \( t \in \mathbb{C} \) satisfying \( |s - t| \leq \varepsilon \) the dominant spectral properties of \( R_s : A_\infty(D_1) \to A_\infty(D_1) \) transfer to \( R_t : A_\infty(D_1) \to A_\infty(D_1) \): quantities \( \lambda_t, \rho_t, g_t, l_t \) (thus \( \Pi_{1t} \)) and \( T_{0t} \) can be defined to represent the dominant spectral objects associated with \( R_t \), and all of them are analytical with respect to \( t \). Moreover, let \( a \in (\rho_s/\lambda_s, 1) \). For any \( f \in A_\infty(D_1) \) such that \( f_{[-1/2,5/2]} > 0 \) we have

\[
\frac{R^n_s f(z)}{\lambda^n_t(f) g_t(z)} = 1 + O(a^n)
\]

as \( n \to \infty \), where the constant implied in \( O \) is independent of \( z \in D_1 \) and \( t \) satisfying \( |s - t| \leq \varepsilon \), but depends on \( a \), \( f \) and \( s \). Finally, \( \rho_s + iu < \rho_s \) for \( u \in [-\varepsilon, \varepsilon] \), \( u \neq 0 \).

The Mayer-Ruelle operators enjoy better properties when they operate on suitable Hilbert spaces.

Let \( \text{Re} s > 1 \). Consider the collection \( H^{(s)} \) of functions \( f \) which are holomorphic in the half-plane \( \text{Re} z > -\frac{1}{2} \), bounded in any half-plane \( \text{Re} z > -\frac{1}{2} + \varepsilon, \varepsilon > 0 \), and can be represented in the form

\[
f(z) = \int_{\mathbb{R}_+} e^{-zu}(s^{-1}/2)m'(du), \quad \text{Re} z > -\frac{1}{2},
\]

where \( m' \) is the measure on \( \mathcal{B}_{\mathbb{R}_+} \) with density

\[
\frac{dm'}{du} = \left\{ \begin{array}{ll}
1 & \text{if } u > 0 \\
0 & \text{if } u = 0,
\end{array} \right.
\]

for some \( \varphi \in L^2_{m'}(\mathbb{R}_+) \) the Hilbert space of \( m' \)-square integrable functions \( \varphi : \mathbb{R}_+ \to \mathbb{C} \) with inner product \( (\cdot, \cdot)_{m'} \) defined as

\[
(\varphi, \psi)_{m'} = \int_{\mathbb{R}_+} \varphi \psi^* dm', \quad \varphi, \psi \in L^2_{m'}(\mathbb{R}_+)
\]

and norm

\[
||\varphi||_{2,m'} = \left( \int_{\mathbb{R}_+} |\varphi|^2 dm' \right)^{\frac{1}{2}}, \quad \varphi \in L^2_{m'}(\mathbb{R}_+).
\]

Introducing the inner product

\[
(f_1, f_2)_{(s)} = (\varphi_1, \varphi_2)_{m'},
\]

where \( \varphi_i \) is associated with \( f_i, i = 1, 2 \), by (1), \( H^{(s)} \) is made a Hilbert space with norm

\[
|f|_{(s)} = ||\varphi||_{2,m'}, \quad f \in H^{(s)},
\]

where \( f \) and \( \varphi \) are again associated by (1).
Theorem 2. Let \( \text{Re} s > 1 \). Then the following results hold.

(i) The linear operator \( R_s \) takes boundedly \( H^s \) into itself.

(ii) For any \( f \in H^s \) we have
\[
R_s f(z) = \int_{\mathbb{R}_+} e^{-zu} K_s \varphi(u) u^{(s-1)/2} m'(du), \quad \text{Re} z > -\frac{1}{2},
\]
where \( K_s : L^2_{m'}(\mathbb{R}_+) \rightarrow L^2_{m'}(\mathbb{R}_+) \) is a symmetric integral operator defined as
\[
K_s \varphi(u) = \int_{\mathbb{R}_+} J_{s-1}(2\sqrt{uv}) \varphi(v) m'(dv), \quad \varphi \in L^2_{m'}(\mathbb{R}_+), \quad u \in \mathbb{R}_+.
\]
Here \( J_{s-1} \) is the Bessel function of order \( s - 1 \) defined as
\[
J_{s-1}(u) = \frac{(-1)^k}{k! \Gamma(k + s)} \left( \frac{u}{2} \right)^{2k}, \quad u \in \mathbb{R}_+.
\]

Hence \( R_s : H^s \rightarrow H^s \) can be diagonalized in an orthonormal basis of \( H^s \). Moreover, if \( s \in \mathbb{R} \), then \( R_s \) is self-adjoint and its spectrum is real.

(iii) The spectra of the operators \( R_s : A_\infty(D_1) \rightarrow A_\infty(D_1) \), \( R_s : H^s \rightarrow H^s \) and \( K_s : L^2_{m'}(\mathbb{R}_+) \rightarrow L^2_{m'}(\mathbb{R}_+) \) are identical. Hence, for \( s \in \mathbb{R} \), these spectra are all real.

3. A first generalization

For any subset \( M \) of \( \mathbb{N}_+ \) define
\[
R_M f(z) = \sum_{i \in M} \frac{1}{(z + i)^s} f \left( \frac{1}{z + i} \right), \quad z \in \mathbb{D}_1,
\]
whatever \( s \in \mathbb{C} \) with \( \text{Re} s > 1 \) and \( f \in A_\infty(D_1) \). Clearly, \( R_M \) is a bounded linear operator on \( A_\infty(D_1) \), hence a nuclear one of trace-class, which coincides with \( R_s \) when \( M = \mathbb{N}_+ \). Now, for an arbitrarily fixed \( k \in \mathbb{N}_+ \), let \( M_i, 1 \leq i \leq k \), be subsets of \( \mathbb{N}_+ \) and write \( M = (M_1, \ldots, M_k) \). Consider the linear operator \( R_M : A_\infty(D_1) \rightarrow A_\infty(D_1) \) defined as
\[
R_M f(z) = R_{M_k} \circ \cdots \circ R_{M_1},
\]
which is nuclear of trace-class, too.

The operators \( R_M \) for various \( M \) control the dynamics of continued fraction expansions of irrationals subject to periodical constraints. Their spectral properties are entirely similar to those of \( R_s \).

4. A second generalization

This generalization has been motivated by the study of the transformation
\[
z \mapsto \frac{1}{z} - \left[ \text{Re} \frac{1}{z} \right], \quad 0 \neq z \in \mathbb{C},
\]
which extends to the complex domain the continued fraction transformation \( \tau \) defined in Section 2. For a detailed account we refer the reader to [10].

Let \( D_2 = \{ z | |z - 1| < 5/4 \} \) and consider the collection \( B_\infty(D_2) \) of all functions \( F \) which are holomorphic in \( D_2^2 \) and continuous in \( \bar{D}_2^2 \). Under the supremum norm

\[
|F| = \sup_{(z,w) \in D_2^2} |F(z,w)|,
\]

\( B_\infty(D_2) \) is a Banach space. Then, for any \( (s,t) \in \mathbb{C}^2 \) with \( \text{Re}(s + t) > 1 \), a linear bounded operator \( R_{s,t} : B_\infty(D_2) \to B_\infty(D_2) \) is defined by

\[
R_{s,t}F(z,w) = \sum_{i \in \mathbb{N}_+} \frac{1}{(z + i)^s(w + i)^t} F\left(\frac{1}{z + i}, \frac{1}{w + i}\right)
\]

for any \( F \in B_\infty(D_2) \) and \( (z,w) \in D_2^2 \). The spectral properties of \( R_{s,t} \) which is positive and nuclear of trace-class, are strongly related to those of \( R_{s+t+2\ell, l} \), \( \ell \in \mathbb{N} \).

**Theorem 3.** For any \( (s,t) \in \mathbb{C}^2 \) with \( \text{Re}(s + t) > 1 \), \( \text{Re}(s) \geq 1 \) and \( \text{Re}(t) > -1 \) the following results hold.

(i) The operator \( R_{s,t} : B_\infty(D_2) \to B_\infty(D_2) \) has a unique dominant eigenvalue \( \lambda_{s,t} \) which is equal to the dominant eigenvalue \( \lambda_{s+t} \) of \( R_{s+t} \). The corresponding eigenfunction \( G_{s,t} \) of \( R_{s,t} \) is defined by

\[
G_{s,t}(z,w) = \int_0^1 \beta_{t,s}(y) g_{s+t}(z + (w - z)y) dy,
\]

where \( g_{s+t} \) is the eigenfunction of \( R_{s+t} \) and \( \beta_{t,s} \) is the classical density \( \beta \)

\[
\beta_{t,s}(y) = \frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)} y^{t-1}(1-y)^{s-1};
\]

moreover, \( G_{s,t} \) satisfies \( G_{s,t}(z, z) = g_{s+t}(z) \). The adjoint operator \( R_{s,t}^* \) has a dominant eigenfunction \( G_{s,t}^*(f) = g_{s+t}^*(f) \), for all \( F \in B_\infty(D_2) \) whose diagonal function is \( f \). If \( \Pi_{s,t} \) denotes the projection on the dominant eigensubspace, \( \Pi_{s,t} = g_{s+t}^* \otimes G_{s,t}, \) then \( R_{s,t} \) has the representation \( R_{s,t} = \lambda_{s+t} \Pi_{s,t} + T_{s,t} \), where \( \Pi_{s,t} \circ T_{s,t} = T_{s,t} \circ \Pi_{s,t} = 0 \). Hence, for any \( F \in B_\infty(D_2) \) we have

\[
R_{s,t}^n F(z,w) = \lambda_{s+t}^n g_{s+t}^*(f) G_{s,t}(z,w) + T_{s,t}^n F(z,w),
\]

for all \( (z,w) \in D_2^2 \) and \( n \in \mathbb{N}_+ \).

(ii) The spectral radius \( \rho_{s+t} \) of the linear operator \( T_{s+t} \) is strictly smaller than \( \lambda_{s+t} \).
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(iii) Let \( a \in (\nu_{s+t}, 1) \), where \( \nu_{s+t} = \frac{\rho_{s+t}}{\lambda_{s+t}} < 1 \). Moreover,

\[
\nu_{s+t} = \frac{1}{\lambda_{s+t}} \max(\lambda_{s+t+2}, \rho_{s+t}).
\]

For any \( F \in B_\infty(D_2) \) such that \( F\|_{[-1/4, 9/4]} > 0 \) we have

\[
\frac{R_{s+t}^n F(z, w)}{\lambda_{s+t}^n} = g_{s+t}^n(f)G_{s,t}(z, w)(1 + O(\|F\|a^n))
\]

as \( n \to \infty \), where the constant implied in \( O \) is independent of \( (z, w) \in D_2^2 \), but depends on \( a \).

**Proof.** Since, by Theorem 1 (ii), the map \( s \mapsto \lambda_s \) defines a strictly decreasing function of \( s \), it follows that \( \lambda_{s+t} \) is the dominant eigenvalue and \( \nu_{s+t} \) satisfies (4).

With the change of variable \( u = z + (w - z)y \) in (2), we get the expression of \( G_{s,t} \)

\[
G_{s,t}(z, w) = \frac{\Gamma(s + t)}{\Gamma(s)\Gamma(t)} \int_\gamma g_{s+t}(u) \frac{(u - z)^{t-1}(w - u)^{s-1}}{(w - z)^{s+t-1}} \, du,
\]

where \( \gamma \) is the interval \([z, w]\). Now, let \( \tilde{h}(z) \) be the holomorphic function that coincides with \( \sqrt{|h'(z)|} \) on \( D_2 \), for any homography of depth 1, \( h(z) = h_i(z) = \frac{1}{1 + z}, i \in \mathbb{N}_+ \).

If \( F \) is defined by (6), to obtain \( R_{s,t}F \) we evaluate the expression

\[
\frac{\tilde{h}(z)^s \tilde{h}(w)^t}{|h(w) - \tilde{h}(z)|^{s+t-1}} \int_\delta f(u) [u - h(z)]^{t-1} [h(w) - u]^{s-1} \, du,
\]

for any simple path \( \delta \) that links \( h(z) \) to \( h(w) \). Put \( \delta = h(\gamma) \), where \( \gamma \) is a simple path that links \( z \) to \( w \). Using the change of variable \( u = h(r) \) in (7) and relations \( du = h'(r)dr = -\tilde{h}(r)^2dr, h(a) - h(b) = -\tilde{h}(a)\tilde{h}(b)(a - b) \), for any \( a \) and \( b \), we can rewrite (7) as

\[
\frac{1}{(w - z)^{s+t-1}} \int_\gamma \tilde{h}(r)^{s+t} f \circ h(r)(r - z)^{t-1}(w - r)^{s-1} \, dr.
\]

If \( F \) is defined by (6), we get

\[
R_{s,t}F = \frac{\Gamma(s + t)}{\Gamma(s)\Gamma(t)} \int_\gamma R_{s+t}^n(f)(r) \frac{(r - z)^{t-1}(w - r)^{s-1}}{(w - z)^{s+t-1}} \, dr.
\]

If \( f \) is the eigenfunction of \( R_{s+t} \) relative to eigenvalue \( \lambda \), then \( F \) is the eigenfunction of \( R_{s,t} \) relative to \( \lambda \). Since \( R_{s+t}^nF(z, z) = R_{s+t}^n(f)(z) \), it is clear that \( G_{s,t}^* \) is expressible in terms of \( g_{s+t}^* \).

(iii) The proof is similar to that of (iv) in Theorem 1. \( \square \)
References


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