The exact expressions for the roots of Rayleigh wave equation

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Abstract. In this paper we address the following question: find the exact expression for the secular Rayleigh's wave equation in a homogenius and isotropic medium defined on a half space. Using Cardan's formula and taking advantage of Maple procedures we get an expression for these roots which are functions of the Poisson's ratio of the material under study. We show that above a critical value for this ratio, the structure of the solutions change and instead of having three real roots, the solutions will develop complex behavior.

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Key words: Rayleigh wave equation, surface wave, phase velocity, critical Poisson ratios, acoustics.

1 Introduction

Rayleigh surface waves are of particular importance in seismology, acoustic, geophysics and electronics applications. Resolution of Rayleigh wave equation has been the subject of intensive studies([4,5,6,8]), which was discovered as late as the 19^{th} century ([1] Rayleigh, 1887). The condition of propagation and the existence of Rayleigh waves has been predicted theoretically along times ago. In the following we will study the Rayleigh waves in a homogeneous and isotropic half-space with free boundary conditions. The solutions of the secular Rayleigh wave equation give the velocity of the waves in medium. The real roots determine the condition of the surface propagation however the physical meaning of the complex roots is still under study. In this paper we will solve the Rayleigh wave equation in function of the Poisson ratio of the medium and study the properties of the roots in order to understand their behaviour near the critical Poisson ratio. The section 2 is devoted to the mathematical solution and in the section 3 we present the displacement component in the medium.

2 The Rayleigh wave equation

We place ourselves in the ideal situation where we consider an isotropic, homogeneous and linearly elastic medium which fills the half-space defined by $y \geq 0$. We work in a Cartesian coordinate system $OXYZ$. The equation describing the displacement

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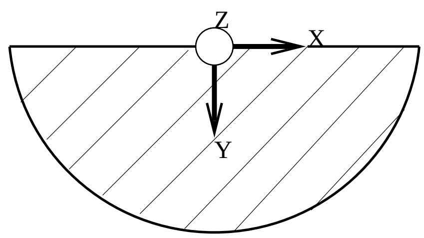
vector **U** of an infinitesimal element in a medium as function of time is a linear elastodynamic equation (we neglect in this equation the volume forces which are not relevant in this context):

$$\rho \, \frac{\partial^2 \mathbf{U}}{\partial t^2} - \mu \, \Delta \mathbf{U} - (\lambda + \mu) \, \nabla(\text{div } \mathbf{U}) = 0 \tag{2.1}$$

The symbols $\nabla = \overrightarrow{grad}$ and $\Delta = \nabla^2$ stand for gradient and Laplace operators respectively; ρ is the medium density (mass density), λ and μ are the Lame constants of the isotropic material.



Schematic representation of Rayleigh wave scattering

The displacement component vector can be presented as

$$\mathbf{U} = \mathbf{U}_l + \mathbf{U}_t = \nabla \phi + \overrightarrow{rot} \; \psi \tag{2.2}$$

With $\mathbf{U}_l = \nabla \phi = \overrightarrow{grad} \; \phi$, $\mathbf{U}_t = \overrightarrow{rot} \; \psi$, and ϕ, ψ are defined as the scalar and the vector potentials respectively.

Substituting (2.2) into (2.1) leads to the two independent equations

$$\rho \, \frac{\partial^2 \mathbf{U}_l}{\partial t^2} - (\lambda + 2\mu) \, \Delta \mathbf{U}_l = 0, \tag{2.3}$$

$$\rho \, \frac{\partial^2 \mathbf{U}_t}{\partial t^2} - \mu \, \Delta \mathbf{U}_t = 0. \tag{2.4}$$

The equation (2.3) describes the propagation of longitudinal waves, and (2.4) describes the propagation of transversal waves.

The Rayleigh wave propagates in the positive direction along the border of the half-space represented by the x-axis. Writing the explicit form of $\mathbf{U}_{l,t}$ we find that the potentials ϕ, ψ are solutions of the ordinary wave equations

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - {c_l}^2 \; \phi = 0; \tag{2.5}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - {c_t}^2 \; \psi = 0, \tag{2.6}$$

where c_l and c_t are wave numbers corresponding to longitudinal and transversal waves, defined by

$$c_l = \omega\sqrt{\frac{\rho}{\lambda+2\mu}}, \quad c_t = \omega\sqrt{\frac{\rho}{\mu}}.$$

We look for solutions propagating in the x-axis direction and with amplitude depending only on y:

$$\phi(x,y,t) = F(y)\ e^{\mathrm{i}(cx-\omega t)}, \quad \psi(x,y,t) = G(y)\ e^{\mathrm{i}(cx-\omega t)},$$

with c representing the phase velocities on the surface. Substituting the explicit expressions of ϕ and ψ into (2.5) and (2.6), we get

$$\frac{\partial^2 F(y)}{\partial y^2} = (c^2 - {c_l}^2)\ F(y), \tag{2.7}$$

$$\frac{\partial^2 G(y)}{\partial y^2} = (c^2 - {c_t}^2)\ G(y). \tag{2.8}$$

The above equations for $F(y),\ G(y)$ are linear differential equations. We will take the negative exponent solution, which is the physical solution in opposite the positive one which assume that the wave is increasing exponential in function of y - and this cannot be a realistic situation,

$$\begin{aligned} \phi(x,y,t) &= A_l\ e^{[-\sqrt{c^2-{c_l}^2}y+\mathrm{i}(cx-\omega t)]}, \\ \psi(x,y,t) &= A_t\ e^{[-\sqrt{c^2-{c_t}^2}y+\mathrm{i}(cx-\omega t)]}. \end{aligned}$$

With A_l, A_t arbitrary constants, we assume that $c^2 > c_l^2$ and $c^2 > c_t^2$ (or $c^2 > c_t^2 > c_l^2$), and $\mathrm{i}^2 = -1$.

From (2.2) we obtain

$$\begin{aligned} \mathbf{U} &= \mathbf{U}_l + \mathbf{U}_t = \overrightarrow{grad}\ \phi + \Delta\psi = \\ &= (\tfrac{\partial\phi}{\partial x} - \tfrac{\partial\psi}{\partial y})\ e_x + (\tfrac{\partial\phi}{\partial y} + \tfrac{\partial\psi}{\partial x})\ e_y, \end{aligned}$$

where $e_x,\ e_y$ are based vectors. The displacement components U_x, U_y are given by

$$U_x = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}, \qquad U_y = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x}.$$

Using Hook's law in elastic solids:

$$\sigma_{ij} = a_{ijkl}\ \epsilon_{kl}, \tag{2.9}$$

the stress components $\sigma_{xx},\ \sigma_{yy},\ \sigma_{xy}$ are given by

$$\begin{aligned}
\sigma_{xx} &= \lambda\ \left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+2\mu\ \left(\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\psi}{\partial x\partial y}\right)\\
&= (\lambda+2\mu)\ \left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)-2\mu\ \left(\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\psi}{\partial x\partial y}\right)\\
\sigma_{yy} &= \lambda\ \left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+2\mu\ \left(\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\psi}{\partial x\partial y}\right)\\
&= (\lambda+2\mu)\ \left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)-2\mu\ (\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\psi}{\partial x\partial y})\\
\sigma_{xy} &= \mu\ (2\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2})
\end{aligned}$$

Using the boundary conditions

$$\sigma_{yy}(x,y=0,z,t)=\sigma_{xy}(x,y=0,z,t)=0,$$

and substituting the expressions of ϕ and ψ in these conditions we get the following system containing the arbitrary constants A_l and A_t:

$$\begin{cases} c^2\frac{\lambda}{2\mu}\ A_l-(c^2-{c_l}^2)(1+\frac{\lambda}{2\mu})\ A_l+\mathrm{i}c\sqrt{c^2-{c_t}^2}\ A_t=0\\ 2\mathrm{i}c\sqrt{c^2-{c_l}^2}\ A_l+(2c^2-c_t^2)\ A_t=0.\end{cases} \tag{2.10}$$

The non-trivial solutions lead to the condition:

$$4c^2\sqrt{c^2-{c_l}^2}\sqrt{c^2-{c_t}^2}-(2c^2-{c_t}^2)^2=0. \tag{2.11}$$

The polynomial form of the above equation is

$$\eta^6-8\eta^4+8(3-2\xi^2)\eta^2-16(1-\xi^2)=0. \tag{2.12}$$

The above equation is called *the Rayleigh wave equation*, where $\eta=\frac{c_t}{c}$ and $\xi=\frac{c_l}{c_t}$, and c, c_l, c_t are defined as the wave numbers for phase velocities of surface, longitudinal and transversal waves respectively.

The velocity c at which Rayleigh waves propagate over an isotropic and elastic surface defined on the half-space $y\geq 0$ is the root of the equation (2.12).

After this change of variables we have

$$\eta^2=\theta=\left(\frac{c_t}{c}\right)^2,\qquad \xi^2=\alpha=\left(\frac{c_l}{c_t}\right)^2.$$

We obtain an equivalent equation of third degree in θ:

$$\theta^3-8\theta^2+8(3-2\alpha)\theta-16(1-\alpha)=0. \tag{2.13}$$

Using Cardan's formula and taking advantage of MAPLE procedures, we get a formula for the θ, where we have three solutions which can be pure real, pure imaginary or complex depending on the value of the Poisson ratio ν. The first root is given by

$$\begin{aligned}\theta_1 &= \tfrac{2}{3}\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})} \\ &- \tfrac{3}{2}\frac{(\frac{8}{9}-\frac{16}{3}\alpha)}{\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})}}+\tfrac{8}{3}.\end{aligned}$$

This root then describes the pure Rayleigh surface wave $\theta_R = \theta_1$. The phase velocity c of Rayleigh waves is obtained as

$$c = \frac{c_t}{\sqrt{\theta_R}}.$$

In a number of papers (e.g. [3]), an approximation of this root depending on Poisson's ratio ν is given using Bergmann's formula ([2]):

$$\theta_R = \theta_1 \approx \frac{0.87+1.12\nu}{1+\nu}. \tag{2.14}$$

In our approach we provide the exact expressions of these roots. It is straightforward to express θ or η as a function of Poisson's ratio ν via

$$\alpha = \xi^2 = \left(\frac{c_l}{c_t}\right)^2 = \frac{1-2\nu}{2(1-\nu)}.$$

Unfortunately, this has not a simple form as the approximate expression of θ_1 (2.14),

$$\begin{aligned}\theta_R = \theta_1 &= \tfrac{2}{3}[-17+\tfrac{45}{2}\tfrac{1-2\nu}{1-\nu}+\tfrac{3}{2}\sqrt{-\tfrac{-15+63\nu-48\nu^2+96\nu^3}{(-1+\nu)^3}}]^{\frac{1}{3}} \\ &- \tfrac{2}{3}\frac{\frac{8}{9}-\frac{8}{3}\frac{1-2\nu}{1-\nu}}{[-17+\frac{45}{2}\frac{1-2\nu}{1-\nu}+\frac{3}{2}\sqrt{-\frac{-15+63\nu-48\nu^2+96\nu^3}{(-1+\nu)^3}}]^{\frac{1}{3}}}+\tfrac{8}{3}.\end{aligned}$$

The Poisson coefficient is in the range $0 \le \nu \le 0.5$ for a physical material, while the real root $\theta_1 = \theta_R$ is discontinuous for $\alpha = \frac{1}{6}$ or $\nu = \frac{2}{5}$ - as we can see in the following expression and in Figure 2:

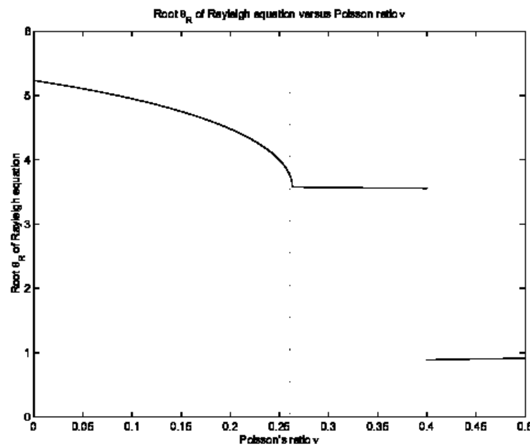
$$\theta_R(\nu = \frac{2}{5}) = \frac{2}{3}(4-\sqrt[3]{19}).$$

Substitution of the critical value $\alpha = \frac{1}{6}$ into the Rayleigh equation (12) yields to

$$\theta^3 - 8\theta^2 + \frac{64}{3}\theta - \frac{40}{3} = 0.$$

This has three roots, namely

$$\frac{2}{3}(4-\sqrt[3]{19}),\ \frac{1}{3}\sqrt[3]{19}(1-\mathrm{i}\sqrt{3})+\frac{8}{3},\ \frac{1}{3}\sqrt[3]{19}(1+\mathrm{i}\sqrt{3})+\frac{8}{3}.$$



Root θ_R of the Rayleigh equation versus Poisson's ratio ν.

The exact critical value of Poisson's ratio ν^* is defined by

$$\begin{aligned} \nu^* &= \frac{-11(77293+7296\sqrt{114})^{\frac{1}{3}}+(77293+7296\sqrt{114})^{\frac{2}{3}}-455}{85(77293+7296\sqrt{114})^{\frac{1}{3}}+(77293+7296\sqrt{114})^{\frac{2}{3}}-455} \\ &\approx 0.2630821. \end{aligned}$$

Similar expressions hold for the other roots; we present their analytical expressions. The complex roots of the Rayleigh wave equation are generally considered as insignificant. The formulas for the other roots are

$$\theta_2 = \theta_2^1 + \mathrm{i}\ \theta_2^2,$$

where

$$\begin{aligned} \theta_2^1 &= -\tfrac{1}{3}\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})} \\ &+ \tfrac{3}{4}\frac{(\frac{8}{9}-\frac{16}{3}\alpha)}{\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})}} + \tfrac{8}{3}, \\ \theta_2^2 &= \tfrac{\sqrt{3}}{2}[\tfrac{2}{3}\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})}+ \\ &+ \tfrac{3}{2}\frac{(\frac{8}{9}-\frac{16}{3}\alpha)}{\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})}}] \end{aligned}$$

and a simple value is again adopted for $\alpha = \frac{1}{6}$ or $\nu = \frac{2}{5}$:

$$\theta_2(\alpha = \frac{1}{6}) = \frac{1}{3}[8 + \sqrt[3]{19} + \mathrm{i}\sqrt{3}\sqrt[3]{19}].$$

The third root θ_3 is given by

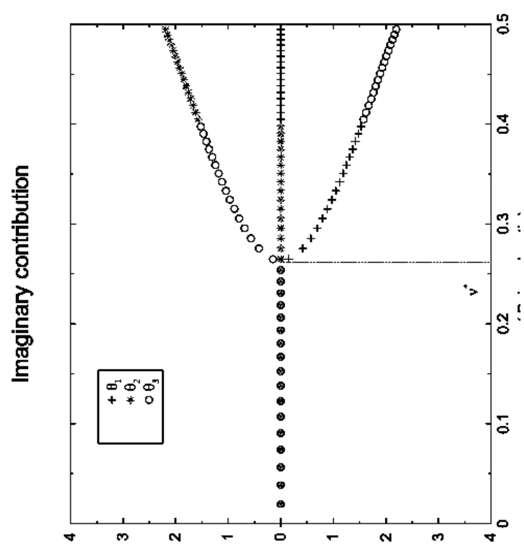
$$\theta_3 = \theta_3^1 + \mathrm{i}\ \theta_3^2,$$

with

$$\begin{aligned}\theta_3^1 &= -\tfrac{1}{3}\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})}\\ &+ \tfrac{3}{4}\frac{(\frac{8}{9}-\frac{16}{3}\alpha)}{\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})}}+\tfrac{8}{3},\\ \theta_3^2 &= -\tfrac{\sqrt{3}}{2}[\tfrac{2}{3}\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})}\\ &+ \tfrac{3}{2}\frac{(\frac{8}{9}-\frac{16}{3}\alpha)}{\sqrt[3]{(-17+45\alpha+3\sqrt{33-186\alpha+321\alpha^2-192\alpha^3})}}]\end{aligned}$$

and a simple value is again adopted for $\alpha = \frac{1}{6}$ or $\nu = \frac{2}{5}$:

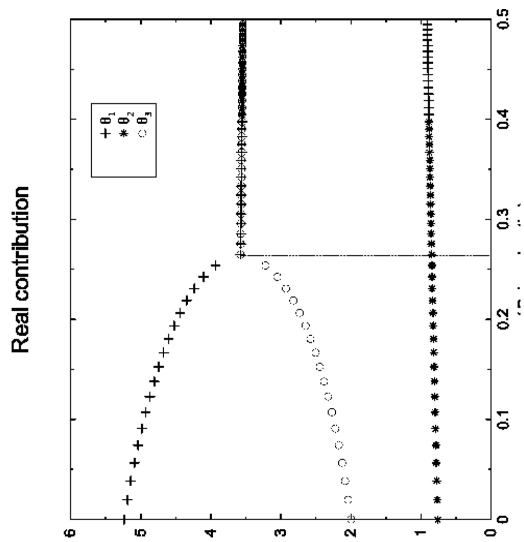
$$\theta_3(\alpha = \frac{1}{6}) = \frac{1}{3}[8 + \sqrt[3]{19} - \mathrm{i}\sqrt{3}\sqrt[3]{19}].$$



Root θ_R of the Rayleigh equation versus Poisson's ratio ν.

In conclusion we have found three solutions: for $0 \leq \nu < \nu^*$ all these roots are pure real; for $\nu^* \leq \nu \leq 0.4$ the first solution θ_2 remains real and however θ_1, θ_3 are complex; finally for $0.4 < \nu \leq 0.5$, we have the same conclusion as in the case before, by doing the permutation $\theta_1 \to \theta_2$, $\theta_2 \to \theta_3$, $\theta_3 \to \theta_1$, as seen in Figure 3.

We notice also that the critical value of the Poisson ratio ν^* determine the nature (real or complex) of the roots.



Real and imaginary parts for the roots of the Rayleigh wave equation.

3 Displacement components

Once we have the solution of the Rayleigh wave equation, we can get the expressions of the scalar and the vector potentials ϕ and ψ:

$$\begin{aligned}
\phi(x,y,t) &= A_l \; e^{[-\sqrt{c^2-c_l^2}y+\mathrm{i}(kx-\omega t)]} \\
\psi(x,y,t) &= -\frac{2ic\sqrt{c^2-c_l^2}}{2c^2-c_t^2} \; A_l \; e^{[-\sqrt{c^2-c_t^2}y+\mathrm{i}(cx-\omega t)]}.
\end{aligned}$$

Using the above equation and the system (2.10) we obtain the displacement components U_x, U_y:

$$U_x = A_l \; c \; [e^{-\sqrt{c^2-c_l^2}y} - \frac{2\sqrt{(c^2-c_l^2)(c^2-c_t^2)}}{c_t^2} \; e^{-\sqrt{c^2-c_t^2}y}] \; e^{i(cx-\omega t-\frac{\pi}{2})}$$

$$U_y = A_l \; \sqrt{c^2-c_l^2} \; [e^{-\sqrt{c^2-c_l^2}y} - \frac{2c^2}{2c^2-c_t^2} \; e^{-\sqrt{c^2-c_t^2}y}] \; e^{i(cx-\omega t)}.$$

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